

# An approximation form of the Kuratowski Extension Theorem for Baire-alpha functions

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# Preliminaries

$\Omega$  - perfectly normal topological space,  $A \subset \Omega$ ,  $f: A \rightarrow \mathbb{R}$ ,

$\alpha$  - countable ordinal number,

- $\mathcal{B}_\alpha(A)$  – the space of all Baire-alpha functions defined on  $A$   
( $\mathcal{B}_0(A) = \mathcal{C}(A)$ ),
- $\mathcal{B}_\alpha^{bd}(A)$  – the family of all bounded functions from  $\mathcal{B}_\alpha(A)$ ,

We shall consider  $\mathcal{B}_\alpha^{bd}(A)$  (and its every linear subspace) with the norm

$$\|f\|_A = \sup_{x \in A} |f(x)|, \quad f \in \mathcal{B}_\alpha^{bd}(A).$$

## Definition

We say that  $f$

- **is a Baire-one function**, if it is a pointwise limit of a sequence of continuous functions on  $A$ ;
- **is a Baire-alpha function**, if it is a pointwise limit of a sequence

$$(f_n) \subset \bigcup_{\xi < \alpha} \mathcal{B}_\xi(A).$$

# Historical results

We consider the following

## Extension Problem

For given  $\alpha$ ,  $A$  and  $\Omega$  as above, does every element  $f \in \mathcal{B}_\alpha(A)$  have an extension  $\bar{f} \in \mathcal{B}_\alpha(\Omega)$ ?

## Tietze, 1915

The above question has a positive answer for  $\alpha = 0$ ,  $A$  closed and  $\Omega$  normal.

## Borsuk, 1933

Let  $A$  be a closed subset of a metric space  $\Omega$ . Then there is a linear extension operator from  $\mathcal{C}(A)$  into  $\mathcal{C}(\Omega)$ .

## Kuratowski, 1933

(K) If  $\Omega$  is a metric space and  $A$  is a  $G_\delta$  subset of  $\Omega$  then every element  $f \in \mathcal{B}_1(A)$  has an extension  $\bar{f} \in \mathcal{B}_1(\Omega)$ .

## Kalenda-Spurný, 2005

(K)  $+\sup f(A) = \sup \bar{f}(\Omega)$  and  $\inf f(A) = \inf \bar{f}(\Omega)$

## Shatery-Zafarani, 2004

The Extension Problem has a positive answer for  $\alpha$  finite,  $\Omega$  perfectly normal and a Borel subset  $A \subset \Omega$  of multiplicative class  $\alpha$  (in particular, for  $A \in G_\delta(\Omega)$ ).

The following two theorems were crucial to proof the main result concerning extensions of Baire-one functions.

## Lebesgue-Hausdorff

Let  $\Omega$  be a perfectly normal topological space and let a function  $f: \Omega \rightarrow \mathbb{R}$  be bounded. Then  $f \in \mathcal{B}_1(\Omega)$  if and only if  $f^{-1}(F) \in G_\delta(\Omega)$  for every closed subset  $F$  of  $\mathbb{R}$ .

## Separation Theorem (for two sets)

Let  $M, N$  be two (non-empty) disjoint  $G_\delta$ -subsets of a perfectly normal topological space  $\Omega$ . Then there exists a set  $H$  of both  $F_\sigma$ - and  $G_\delta$ -subset of  $\Omega$  such that  $M \subset H$  and  $N \cap H = \emptyset$ .

# The first main result

## WS

Let  $\Omega$  be a perfectly normal topological space and let  $A$  be a  $G_\delta$ -subset of  $\Omega$ . Then every function  $f \in \mathcal{B}_1(A)$  has an extension  $\bar{f} \in \mathcal{B}_1(\Omega)$  such that  $\|f\|_\infty = \|\bar{f}\|_\infty$ . More exactly, if  $\sup_{x \in A} |f(x)| = c < \infty$ , then there exists a sequence  $(H_n)$  of both  $F_\sigma$ - and  $G_\delta$ -subsets of  $\Omega$ , such that the function

$$\bar{f}(x) := c \cdot \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+1} \left(\frac{1}{2} - \chi_{H_n}(x)\right) \quad (\text{uniform convergence on } \Omega)$$

extends  $f$  with  $\sup_{x \in \Omega} |\bar{f}(x)| = c$ .

The above result has a version for non-negative functions.

## WS

Let  $\Omega$  be a perfectly normal topological space and let  $A$  be a  $G_\delta$ -subset of  $\Omega$ . Moreover, let  $0 \leq f \in B_1^{bd}(A)$  with  $\sup_{x \in A} f(x) = 1$ . Then there is a sequence  $(G_n)$  of both  $F_\sigma$ - and  $G_\delta$ -subsets of  $\Omega$  such that the function

$$\bar{f}(x) := \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+1} \chi_{G_n}(x) \text{ (uniform convergence)}$$

is a (positive) extension of  $f$  with  $\sup_{x \in \Omega} \bar{f}(x) = 1$ .

From the above theorem we obtain the following

## Corollary

Let  $\Omega$  be a perfectly normal topological space and let  $\alpha$  be an ordinal number with  $\alpha < \omega_1$ . Every non-negative function  $f \in B_\alpha^{bd}(\Omega)$  is of the form

$$f(x) := c \cdot \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} \cdot \chi_{H_n}(x),$$

for a sequence  $(H_n)$  of Borel ambiguous class  $\alpha$  subsets of  $\Omega$ .

# The $(\delta)$ -property

## Lavrič, 1986

We say that a Riesz space  $E$  **has the  $(\delta)$ -property**, if for every  $x, y \in E$  such that  $x \wedge y = 0$ , there is an order projection  $P$  such that  $P(x) = x$  and  $P(y) = 0$ .

In 1994 Lipecki strengthened the results of Lavrič and showed [6, Theorem 2] that the  $(\delta)$ -property can be written as the following condition.

## Lipecki, 1994

Let  $E$  be a Riesz space and let  $x, y \in E_+$ . For every  $0 \leq y \leq x$  there are order projections  $P_i \in \mathcal{P}(E)$  such that

$$0 \leq y - \sum_{i=1}^k \alpha_i P_i(x) \leq \alpha_k x, \quad k = 1, 2, \dots, \quad (1)$$

where  $\alpha_i = \frac{1}{2^i}$ ,  $i = 1, 2, \dots$

# Order projections in $\mathcal{B}_\alpha(\Omega)$

## MW, WS

Let  $\Omega$  be a perfectly normal topological space,  $\alpha < \omega_1$  and let  $A$  be a Baire subset of  $\Omega$  of ambiguous class  $\alpha$ . Every order projection  $P: \mathcal{B}_\alpha(\Omega) \rightarrow \mathcal{B}_\alpha(\Omega)$  is of the form  $P(f) = \chi_A \cdot f$ .

# The main result

## Hypothesis: MW, WS

Let  $\Omega$  be a perfectly normal topological space and let  $\alpha \geq 1$  be an ordinal number  $< \omega_1$ . Then  $\mathcal{B}_\alpha(\Omega)$  has the  $(\delta)$ -property, i.e. for every  $f, g \in \mathcal{B}_\alpha(\Omega)$  with  $0 \leq f \leq g$ , there are Borel subsets  $A_i$  ( $i = 1, 2, \dots, n$ ) of ambiguous class  $\alpha$  in  $\Omega$  such that

$$0 \leq f - \sum_{i=1}^n \alpha_i \chi_{A_i} \leq g,$$

where  $(\alpha_i)$  is a sequence of positive numbers such that  $\sum_{i=1}^n \alpha_i = 1$ .



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Thank You very much for  
Your attention :-)

# The $(\delta)$ -property: preliminaries

- An ordered set  $(M, \leq)$  is called a **lattice** if any two elements  $x, y \in M$  have a least upper bound  $x \vee y = \sup(x, y)$  and a greatest lower bound  $x \wedge y = \inf(x, y)$ ; Note that  $|x| = \sup(x, -x)$ ;
- A real vector space  $E$  is called a **linear lattice** (or a **Riesz space**), if it is a lattice determined by a partial order  $\leq$  satisfying the following condition:  
 $(\star)$  for every  $x, y \in E$  the condition  $x \leq y$  implies that  $x + z \leq y + z$  for all  $z \in E$  and  $ax \leq ay$  for all  $a \geq 0$ ;
- A subspace  $I$  of a Riesz space  $E$  is an **ideal** if for every  $x \in E$  and  $y \in I$  from  $|x| \leq |y|$  it follows that  $x \in I$ .

- An ideal  $B$  of  $E$  is called a **band** if  $\sup(A) \in B$  for every subset  $A \subset B$  with  $\sup(A) \in E$ ;
- The ideal (band) generated by a singleton  $\{x\}$  is called a **principal** ideal (**principal** band);
- A band  $B$  of  $E$  is called a **projection band** if there is a linear mapping  $P: E \rightarrow B$  such that  $0 \leq P(x) \leq x$  for all  $x \in E_+$ .

# Principal projection property

## Definition

If every principal band is a projection band then  $E$  is said to have the **principal projection property** (PPP).

# The $(\delta)$ -property: preliminaries

Two elements  $x$  and  $y$  of a Riesz space  $E$  are **disjoint** ( $x \perp y$ ) in  $E$  if  $|x| \wedge |y| = 0$ .

## Definition

For  $A$ , a nonempty subset of  $E$ , its **disjoint complement**  $A^d$  is defined as

$$A^d = \{x \in E : x \perp y \text{ for every } y \in A\}.$$

The **algebraic sum** of  $A$  and  $A^d$  is a set

$$A \oplus A^d = \{x + y : x \in A \text{ and } y \in A^d\}.$$

# Order projections in Riesz spaces

Let  $B$  be a projection band in a Riesz space  $E$ . Then  $E = B \oplus B^d$  and every element  $x \in E$  has a unique decomposition  $x = x_1 + x_2$ , where  $x_1 \in B$  and  $x_2 \in B^d$ .

## Definition

The projection  $P_B: E \rightarrow E$  defined by the formula

$$P_B(x) = x_1,$$

is called an **order projection determined by  $B$** . The set of all order projections on  $E$  is a Boolean algebra denoted by  $\mathcal{P}(E)$ .

# The relation between $\sigma$ -DC, (PPP) and the $(\delta)$ -property

## Theorem

$$\begin{aligned} \sigma\text{-DC} &\Rightarrow (PPP) \Rightarrow (\delta)\text{-property} \Rightarrow \\ &\Rightarrow \text{spectral Freudenthal property} \end{aligned}$$

## Example

Classical functional spaces like  $L^p$ ,  $c_0$ ,  $l_\infty$  or Orlicz spaces  $L^\varphi$  are Dedekind complete, hence possess the  $(\delta)$ -property. The space  $\mathcal{C}(K)$  of continuous mappings defined on  $K$  has the  $(\delta)$ -property provided that  $K$  is a zero-dimensional  $F$ -space.

# Order projections in functional spaces

Let  $\mathbb{R}^K$  be the lattice of all functions  $K \rightarrow \mathbb{R}$  ordered pointwise.

## Definition

Let  $\text{supp}(f) = \{x \in K : f(x) \neq 0\}$  be the **support** of  $f \in \mathbb{R}^K$ . We say that  $f, g \in \mathbb{R}^K$  are **disjoint** if  $\text{supp}(f) \cap \text{supp}(g) = \emptyset$ .

## Zaanen, 1975

If  $f$  and  $g$  are disjoint in  $\mathbb{R}^K$ , the order projection  $P: \mathbb{R}^K \rightarrow \mathbb{R}^K$  that separates them is of the form

$$P(u) = u \cdot \chi_{\text{supp}(f)}.$$

# The main result - conclusion

Since  $\mathcal{B}_\alpha(\Omega)$  (for  $\Omega$  perfectly normal) is an  $f$ -algebra with multiplicative identity  $1_\Omega$  and the  $(\delta)$ -property, we get the following

## Corollary

*Let  $\Omega$  be a perfectly normal topological space and let  $\alpha$  be an ordinal number such that  $1 \leq \alpha < \omega_1$ . Every function  $f \in \mathcal{B}_\alpha(\Omega)$  **has a sign**: there exists a Baire subset  $A \subset \Omega$  of ambiguous class  $\alpha$  such that for  $u = \chi_A - (1_\Omega - \chi_A)$  there is  $|f| = uf$  and  $f = u|f|$ .*

# Some remarks on the $\sigma$ -DC property of $\mathcal{B}_\alpha(\Omega)$ and the Baire order of $C(\Omega)$

Let  $\Omega$  be the Hausdorff topological space and let  $\Phi_0 = \mathcal{C}(\Omega)$ . Moreover, let  $\Phi_1 = \mathcal{B}_1(\Omega)$  and  $\Phi_k = \mathcal{B}_k(\Omega)$  for every  $k \in \mathbb{N}$ .

## Definition

The *Baire order of a family of functions*  $\Phi$  is the first ordinal  $\alpha$  such that  $\Phi_\alpha = \Phi_{\alpha+1}$ .

## MW, WS; 2010

If  $\Omega$  is a  $P$  space (=every  $G_\delta$ -subset of  $\Omega$  is open) then the Baire order of  $\mathcal{C}(\Omega)$  is 0.

It turns out that not only for the domain being a P-space the Baire order of  $\mathcal{C}(\Omega)$  may be finite.

### Szpilrajn, 1930

If  $\aleph_1 = 2^{\aleph_0}$  then there is an uncountable number set  $\Omega$  of cardinality  $\aleph_1$  such that the Baire order of  $\mathcal{C}(\Omega)$  is 1.

### Sierpiński, 1930

If  $\aleph_1 = 2^{\aleph_0}$  then there is a number set  $\Omega$  of cardinality  $\aleph_1$  such that the Baire order of  $\mathcal{C}(\Omega)$  is 2.

There are also spaces  $\Omega$  for which the Baire order of  $\mathcal{C}(\Omega)$  is uncountable.

### Theorem

If compact  $\Omega$  contains a perfect subset then  $\mathcal{B}_\alpha(\Omega) \subsetneq \mathcal{B}_{\alpha+1}(\Omega)$  for every  $\alpha < \omega_1$  and thus the Baire order of  $\mathcal{C}(\Omega)$  is  $\omega_1$ .

In the last part we give some examples showing that the  $(\delta)$ -property is weaker, in general, than the  $\sigma$ -DC property.

### Mauldin, 1974

$B_1[0, 1]$  is not  $\sigma$ -DC, yet it is a  $(\delta)$ -space. More generally, if compact  $\Omega$  contains a perfect subset then  $B_1(\Omega)$  is not  $\sigma$ -DC.

### MW, WS; 2010

If  $\Omega$  is a  $P$ -space then  $B_1(\Omega) = \mathcal{C}(\Omega)$ , and hence  $B_1(\Omega)$  is  $\sigma$ -DC.

### WS, MW

Let  $0 \leq \alpha < \omega_1$ . If  $B_\alpha(\Omega) = B_{\alpha+1}(\Omega)$  for some  $\alpha < \omega_1$ , then  $B_\alpha(\Omega)$  is  $\sigma$ -DC.