

Uniformly convex functions and their generalizations

P. Shunmugaraj

Indian Institute of Technology Kanpur

and

Constantin Zălinescu

Octav Mayer Institute of Mathematics

Introduction

Uniform convexity (also called uniform rotundity) and its generalizations, along with their properties, are well known in the geometry of Banach spaces.

In this talk, we discuss certain counterparts of these notions in convex analysis.

Do such counterparts already exist in the literature?

Yes.

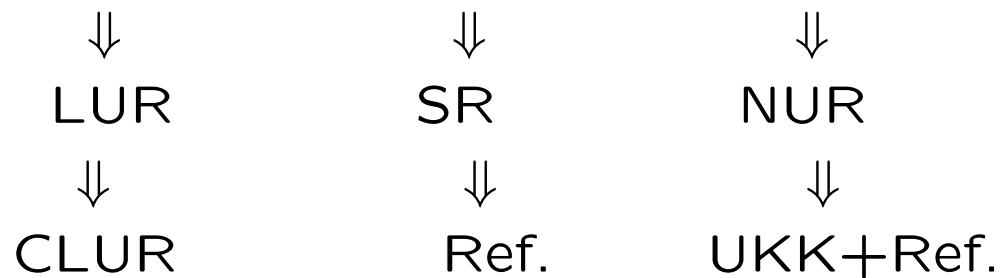
The well known *Brøndsted–Rockafellar theorem* and the *Šmulian theorem* in convex analysis can be viewed as counterparts of the *Bishop–Phelps–Bollobás* and *Šmulian theorem*, respectively, in the geometry of Banach spaces.

We refer to the following books for the results mentioned above, as well as for basic definitions and preliminary results in the geometry of Banach spaces and convex analysis:

1. J. M. Borwein and J. D. Vanderwerff, *Convex Functions: Constructions, Characterizations and Counter examples*, Cambridge university press, 2011.
2. C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, Singapore, 2002.

What is known in the Geometry of Banach spaces:

Uniformly Rotund (UR) Banach Spaces



where

LUR- Locally uniformly rotund

CLUR-Compactly LUR

SR-Strongly rotund

NUR-Nearly uniformly rotund

Ref.- Reflexivity

UKK-Uniformly Kadec-Klee

Can we find counterparts of these notions and their properties in convex analysis?

Yes, to some extent. However, not all of these notions and properties have counterparts in convex analysis.

At least, the notions and properties related to slices in the geometry of Banach spaces have natural counterparts in convex analysis.

Slices and Their Role in the Geometry of Banach Spaces:

$(X, \|\cdot\|)$ is a nontrivial real Banach space.

B_X - unit ball and S_X - unit sphere of X .

For $x \in S_X$, $x^* \in S_{X^*}$ and $\delta \in [0, 1]$, consider

$$S(X, x^*, \delta) := \{x \in B_X : x^*(x) \geq 1 - \delta\},$$

$$S(X^*, x, \delta) = \{x^* \in B_{X^*} : x^*(x) \geq 1 - \delta\};$$

of course,

$$S(X^{**}, x^*, \delta) = \{x^{**} \in B_{X^{**}} : x^{**}(x^*) \geq 1 - \delta\}$$

and $X \cap S(X^{**}, x^*, \delta) = S(X, x^*, \delta)$.

We now illustrate with one geometric property in Geometry of Banach Spaces and the corresponding counterpart in Convex Analysis.

The space X is said to be *uniformly rotund* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in B_X$,

$$\left\| \frac{1}{2}x + \frac{1}{2}y \right\| > 1 - \delta \Rightarrow \|x - y\| < \varepsilon.$$

- One can equivalently replace B_X by S_X .

Theorem (Šmulian): The following statements are equivalent.

- (i) X is uniformly convex.
- (ii) For every $\epsilon > 0$ there exists $\delta > 0$ such that $\text{diam}(S(X, x^*, \delta)) < \epsilon$ for all $x^* \in S_{X^*}$.
- (iii) For every $\epsilon > 0$ there exists $\delta > 0$ such that $\text{diam}(S(X^{**}, x^*, \delta)) < \epsilon$ for all $x^* \in S_{X^*}$.
- (iv) X^* is uniformly smooth.

We need the following notions:

For $f : X \rightarrow \overline{\mathbb{R}}$ ($:= \mathbb{R} \cup \{-\infty, +\infty\}$),

its *domain* is $\text{dom } f := \{x \in X : f(x) < \infty\}$;

f is proper if $\text{dom } f \neq \emptyset$ and $f(x) \neq -\infty$ for $x \in \text{dom } f$

The *conjugate* of $f : X \rightarrow \overline{\mathbb{R}}$ is the function

$$f^* : X^* \rightarrow \overline{\mathbb{R}}, \quad f^*(x^*) = \sup\{x^*(x) - f(x) : x \in \text{dom } f\}$$

(where $\sup \emptyset := -\infty$ and $\inf \emptyset := \infty$ ($:= +\infty$);)

For $\varepsilon \in \mathbb{R}_+ := [0, \infty[$, the ε -subdifferential of $f : X \rightarrow \overline{\mathbb{R}}$ at $x_0 \in X$ is defined by

$$\partial_\varepsilon f(x_0) := \{x^* \in X^* \mid \forall x \in X : x^*(x - x_0) \leq f(x) - f(x_0) + \varepsilon\}$$

if $f(x_0) \in \mathbb{R}$ and $\partial_\varepsilon f(x_0) := \emptyset$

We will consider f to be proper, convex and lower semi-continuous function.

The function f is said to be *uniformly convex* if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x, y \in \text{dom } f$

$$\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) \leq \delta \Rightarrow \|x - y\| \leq \varepsilon.$$

If $f := \| \cdot \|$ and $x, y \in S_X$, this definition reduces to the usual definition of uniform convexity of X .

Theorem 1. *The following assertions are equivalent.*

(i) *f is uniformly convex.*

(ii) *For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\text{diam}(\partial_\delta f^*(x^*) \cap X) \leq \varepsilon$ whenever $x^* \in \text{dom } f^*$.*

(iii) *For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\text{diam}(\partial_\delta f^*(x^*)) \leq \varepsilon$ whenever $x^* \in \text{dom } f^*$.*

(iv) *$\text{dom } f^* = X^*$, ∂f^* is single valued and ∇f^* is uniformly continuous on X^* .*

Nearly uniformly convex functions:

Nearly uniformly convex space:

The space X is said to be *nearly uniformly convex (NUC)* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\alpha(S(X, x^*, \delta)) < \varepsilon$ for all $x^* \in S_{X^*}$.

Here $\alpha(A)$ denotes the Hausdorff index of (or measure of) non-compactness.

It can be observed that if X is NUC then $S(X, x^*, 0)$ is nonempty compact for every $x^* \in S_X$ and hence X is reflexive.

Definition 1. Let $A^* \subseteq \text{dom } f^*$ be a nonempty set. We say that the function f is:

(i) nearly uniformly convex (NUC) with respect to (w.r.t.) A^* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\alpha(\partial_\delta f^*(x^*) \cap X) < \varepsilon$ for all $x^* \in A^*$;

(ii) nearly uniformly convex (NUC) if f is NUC w.r.t. $\text{dom } f^*$;

Examples:

1. The space X is NUC if and only if the function ι_{B_X} is NUC w.r.t. S_{X^*} . Observe that if $f = \iota_{B_X}$, then $f^* = \|\cdot\|_{X^*}$, and so $\partial_\delta f^*(x^*) \cap X = S(X, x^*, \delta)$ for every $x^* \in S_{X^*}$.

2. Every uniformly convex function is nearly uniformly convex.

3. Consider the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$g(x) := \begin{cases} 0 & \text{if } |x| \leq 1, \\ x^2 - 1 & \text{if } 1 < |x| \leq 2, \\ \infty & \text{if } |x| > 2. \end{cases}$$

One can verify that g is nearly uniformly convex as $\alpha(\partial_\delta^X g^*(x^*)) = 0$ for any $\delta > 0$ and $x^* \in \text{dom } g^*$, but g is not uniformly convex as it is not strictly convex.

Definition 2. Let $\emptyset \neq A \subseteq \text{dom } f$. The function f is said to be nearly uniformly smooth (NUS) on A if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\alpha(\partial_\delta f(x)) < \varepsilon$ for all $x \in A$.

Definition 2 is analogous to the known definition of nearly uniformly smooth Banach spaces, which is given below.

Definition 3. The space X is said to be nearly uniformly smooth (NUS) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\alpha(S(X^*, x, \delta)) < \varepsilon$ for every $x \in S_X$.

Remark: The space X is nearly uniformly smooth if and only if $f := \|\cdot\|$ is nearly uniformly smooth on S_X .

Theorem 2. *Let $\emptyset \neq A^* \subseteq \text{dom } f^*$. The following statements are equivalent:*

- (i) f is nearly uniformly convex w.r.t. A^* ;*
- (ii) f^* is nearly uniformly smooth on A^* .*

corollary:

The following statements are equivalent:

- (i) X is nearly uniformly convex;*
- (ii) X^* is nearly uniformly smooth.*

References:

1. J. M. Borwein and J. D. Vanderwerff, *Convex Functions: Constructions, Characterizations and Counterexamples*, Cambridge university press, 2011.
2. A. K. Chakrabarty, P. Shunmugaraj and C. Zălinescu, *Continuity properties for the subdifferential and ε -subdifferential of a convex function and its conjugate*, J. Convex Anal., **14** (2007), 479–514.
5. P. Shunmugaraj and C. Zălinescu, *Compactly locally uniformly convex functions*, J. Convex Anal., **31**(4) (2024), 1245–1272.

6. P. Shunmugaraj and C. Zălinescu, *Uniformly strongly convex Banach spaces*, Mediterr. J. Math., **22**(6) (2025),
7. C. Zălinescu, *On uniformly convex functions*, J. Math. Anal. Appl., **95** (1983), 344–374.
8. C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, Singapore, 2002.

Thanks