

Packings in infinite dimensions

Tommaso Russo

tommaso.russo.math@protonmail.com

j./w. C.A. De Bernardi, Ş. Sezgek, and J. Somaglia

Packings in classical Banach spaces, arXiv:2602.12934

Tilings and coverings by balls in ℓ_1 , arXiv:2604.15092

FWF Austrian
Science Fund

40th International Summer Conference on Real Functions Theory

Stará Lesná, September 7–11, 2026



Kepler problem (1661)

What is the optimal way to stack cannonballs on a ship?



Kepler problem (1661)

What is the optimal way to stack cannonballs on a ship?

Gregory–Newton problem (1694)

Can a sphere in $\mathbb{R}^3$ touch 13 spheres of the same size?





Kepler problem (1661)

What is the optimal way to stack cannonballs on a ship?

Gregory–Newton problem (1694)

Can a sphere in $\mathbb{R}^3$ touch 13 spheres of the same size?

- ▶ **Gregory:** Yes.
- ▶ **Newton:** No, at most 12.





Kepler problem (1661)

What is the optimal way to stack cannonballs on a ship?

Gregory–Newton problem (1694)

Can a sphere in $\mathbb{R}^3$ touch 13 spheres of the same size?

- ▶ **Gregory:** Yes.
- ▶ **Newton:** No, at most 12.
- ▶ δ_n : the density of the optimal packing of spheres in $\mathbb{R}^n$.
- ▶ δ_n^* : with the extra requirement that the centers are a lattice (namely, a subgroup).



A few values



n	δ_n^*	Author	δ_n	Author
2	$\frac{\pi}{\sqrt{12}}$	Lagrange, 1773	$\frac{\pi}{\sqrt{12}}$	Thue, 1892
3	$\frac{\pi}{\sqrt{18}}$	Gauss, 1831	$\frac{\pi}{\sqrt{18}}$	Hales, 2005 <i>Annals</i>
4	$\frac{\pi^2}{16}$	Korkin, Zolotarev, 1872	???	
5	$\frac{\pi^2}{15\sqrt{2}}$	Korkin, Zolotarev, 1877	???	
6	$\frac{\pi^3}{48\sqrt{3}}$	Blichfeldt, 1925	???	
7	$\frac{\pi^3}{105}$	Blichfeldt, 1926	???	
8	$\frac{\pi^4}{384}$	Blichfeldt, 1934	$\frac{\pi^4}{384}$	Viazovska, 2017 <i>Annals</i>
24	$\frac{\pi^{12}}{12!}$	Cohn, Kumar, 2009 <i>Annals</i>	$\frac{\pi^{12}}{12!}$	Cohn et.al. 2017 <i>Annals</i>



- ▶ **Klartag (2026).** $\delta_n^* \geq cn^2 2^{-n}$.
- ▶ **Minkowski-Hlawka.** $\delta_n^* \geq c\zeta(n) 2^{-n}$.



- ▶ **Klartag (2026).** $\delta_n^* \geq cn^2 2^{-n}$.
 - ▶ **Minkowski-Hlawka.** $\delta_n^* \geq c\zeta(n) 2^{-n}$.
- ▶ **Kabatjanskii, Levenšteĭn (1978).** $\delta_n \lesssim 0.66^n$.



- ▶ **Klartag (2026).** $\delta_n^* \geq cn^2 2^{-n}$.
 - ▶ **Minkowski-Hlawka.** $\delta_n^* \geq c\zeta(n) 2^{-n}$.
- ▶ **Kabatjanskii, Levenšteĭn (1978).** $\delta_n \lesssim 0.66^n$.
- ▶ **Rogers (1964).** Conjecture that $\delta_n \neq \delta_n^*$ for some n .



- ▶ **Klartag (2026).** $\delta_n^* \geq cn^2 2^{-n}$.
 - ▶ **Minkowski-Hlawka.** $\delta_n^* \geq c\zeta(n)2^{-n}$.
- ▶ **Kabatjanskii, Levenšteĭn (1978).** $\delta_n \lesssim 0.66^n$.
- ▶ **Rogers (1964).** Conjecture that $\delta_n \neq \delta_n^*$ for some n .
- ▶ For a convex set C in $\mathbb{R}^n$, $\delta(C)$ is the optimal density of a packing by translates of C ; and similarly for $\delta^*(C)$.
- ▶ **Fáry–Courant–Rogers.** For a convex set C in the plane,

$$\delta(C) = \delta^*(C) \geq \frac{2}{3}$$

with equality if and only if C is a triangle.



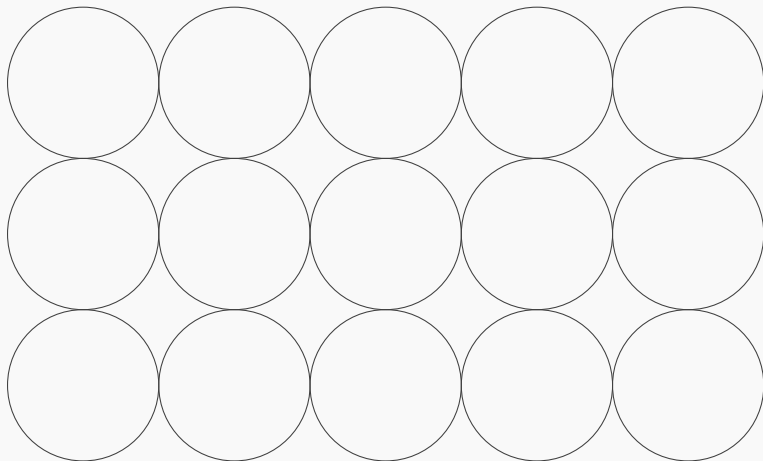
- ▶ **Klartag (2026).** $\delta_n^* \geq cn^2 2^{-n}$.
 - ▶ **Minkowski-Hlawka.** $\delta_n^* \geq c\zeta(n) 2^{-n}$.
- ▶ **Kabatjanskiĭ, Levenšteĭn (1978).** $\delta_n \lesssim 0.66^n$.
- ▶ **Rogers (1964).** Conjecture that $\delta_n \neq \delta_n^*$ for some n .
- ▶ For a convex set C in $\mathbb{R}^n$, $\delta(C)$ is the optimal density of a packing by translates of C ; and similarly for $\delta^*(C)$.
- ▶ **Fáry–Courant–Rogers.** For a convex set C in the plane,

$$\delta(C) = \delta^*(C) \geq \frac{2}{3}$$

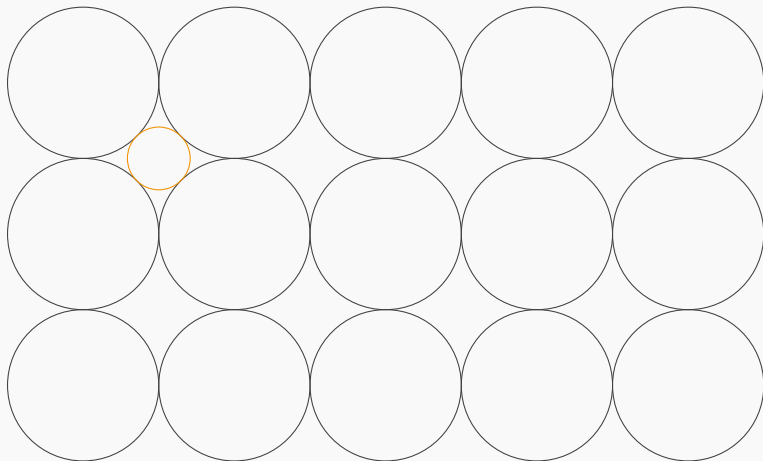
with equality if and only if C is a triangle.

- ▶ **The definition can't be extended to infinite dimensions.**
- ▶ **Rogers (1950).** An alternative, purely metric, notion of optimality.

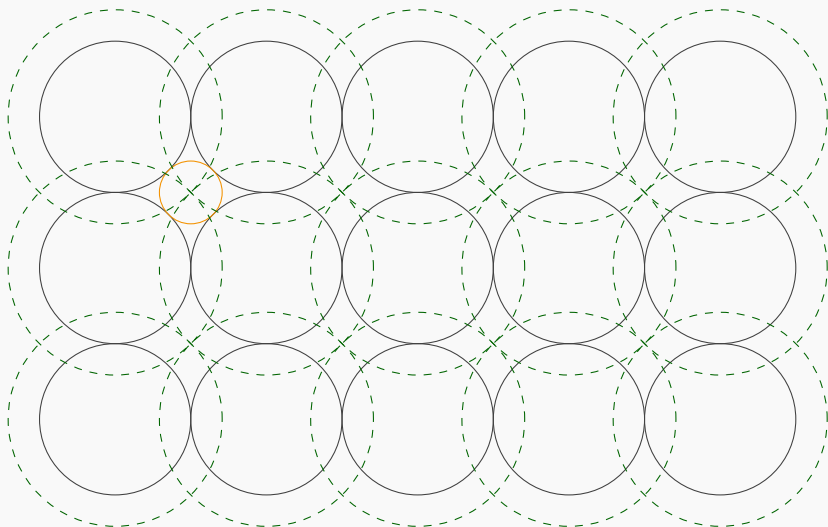
The simultaneous covering and packing constant

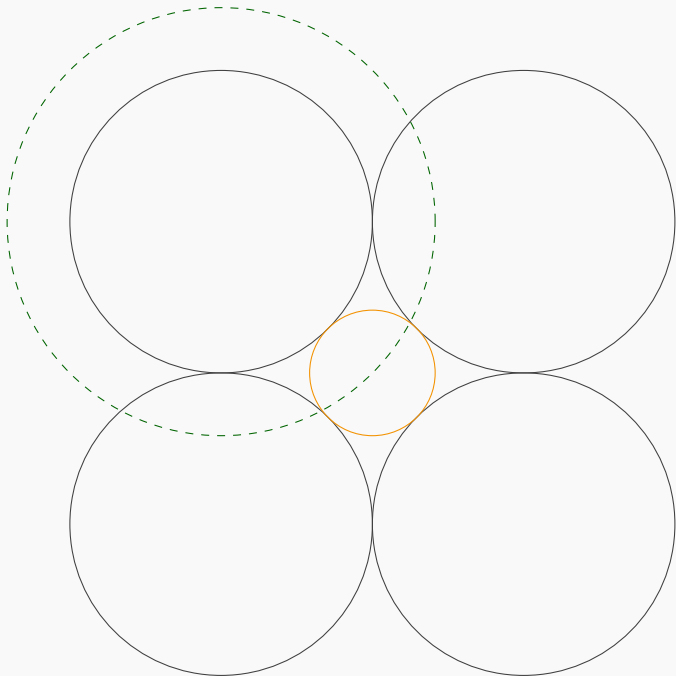


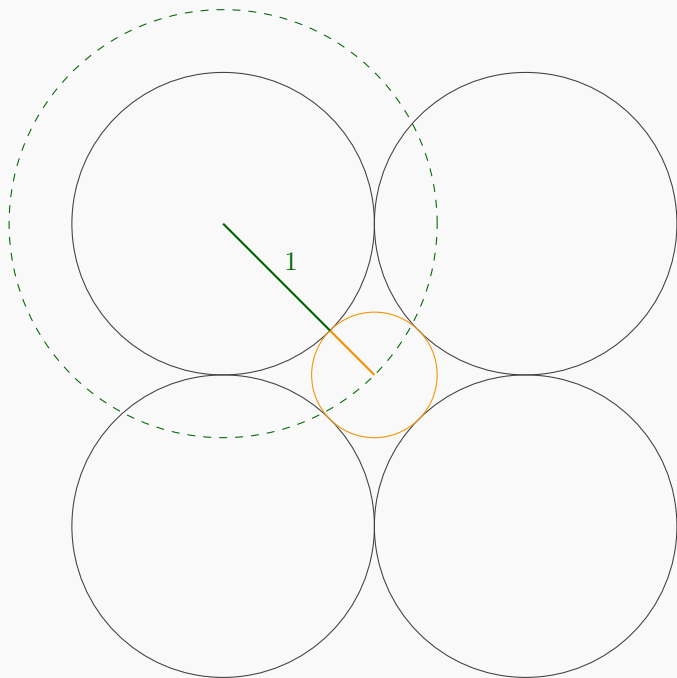
The simultaneous covering and packing constant



The simultaneous covering and packing constant







The simultaneous covering and packing constant



- ▶ Compute the radius of the largest non-overlapping ball.
- ▶ How much do we have to inflate the balls to cover the space?
- ▶ First is second -1 .

The simultaneous covering and packing constant



- ▶ Compute the radius of the largest non-overlapping ball.
- ▶ How much do we have to inflate the balls to cover the space?
- ▶ First is second -1.
- ▶ The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ measures this (the second quantity).

The simultaneous covering and packing constant



- ▶ Compute the radius of the largest non-overlapping ball.
- ▶ How much do we have to inflate the balls to cover the space?
- ▶ First is second -1.
- ▶ The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ measures this (the second quantity).
- ▶ **Folklore.** $\gamma(\mathbb{R}^2) = \gamma^*(\mathbb{R}^2) = \frac{2}{\sqrt{3}}$.

The simultaneous covering and packing constant



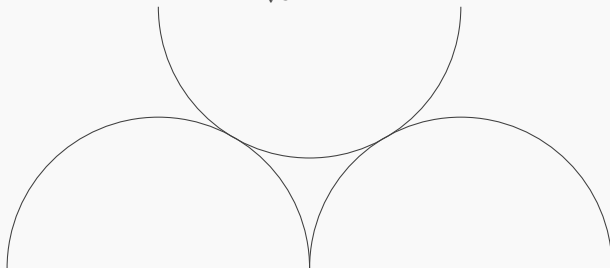
- ▶ Compute the radius of the largest non-overlapping ball.
- ▶ How much do we have to inflate the balls to cover the space?
- ▶ First is second -1.
- ▶ The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ measures this (the second quantity).
- ▶ **Folklore.** $\gamma(\mathbb{R}^2) = \gamma^*(\mathbb{R}^2) = \frac{2}{\sqrt{3}}$.



The simultaneous covering and packing constant



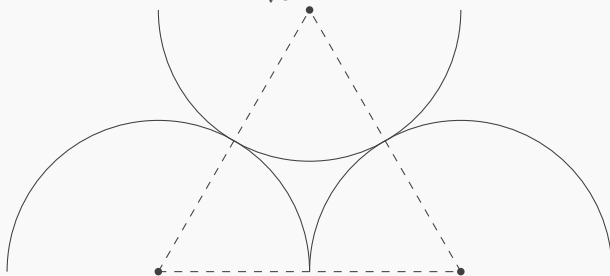
- ▶ Compute the radius of the largest non-overlapping ball.
- ▶ How much do we have to inflate the balls to cover the space?
- ▶ First is second -1.
- ▶ The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ measures this (the second quantity).
- ▶ **Folklore.** $\gamma(\mathbb{R}^2) = \gamma^*(\mathbb{R}^2) = \frac{2}{\sqrt{3}}$.



The simultaneous covering and packing constant



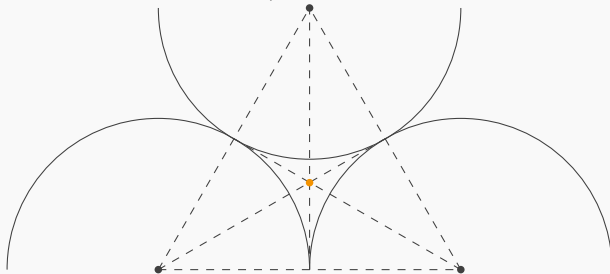
- ▶ Compute the radius of the largest non-overlapping ball.
- ▶ How much do we have to inflate the balls to cover the space?
- ▶ First is second -1.
- ▶ The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ measures this (the second quantity).
- ▶ **Folklore.** $\gamma(\mathbb{R}^2) = \gamma^*(\mathbb{R}^2) = \frac{2}{\sqrt{3}}$.



The simultaneous covering and packing constant



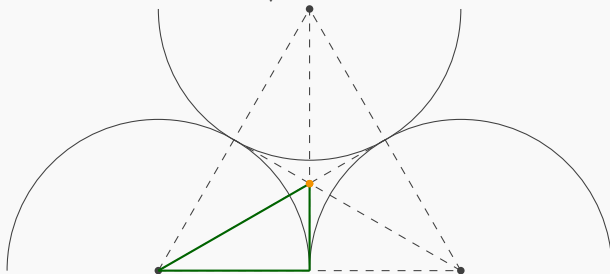
- ▶ Compute the radius of the largest non-overlapping ball.
- ▶ How much do we have to inflate the balls to cover the space?
- ▶ First is second -1.
- ▶ The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ measures this (the second quantity).
- ▶ **Folklore.** $\gamma(\mathbb{R}^2) = \gamma^*(\mathbb{R}^2) = \frac{2}{\sqrt{3}}$.



The simultaneous covering and packing constant



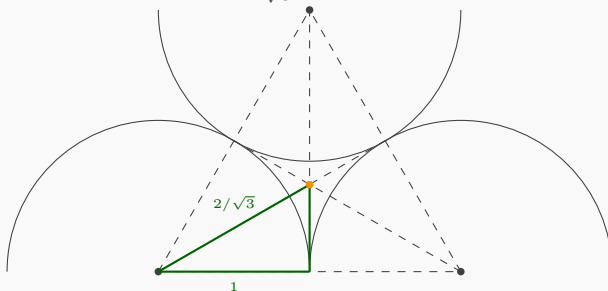
- ▶ Compute the radius of the largest non-overlapping ball.
- ▶ How much do we have to inflate the balls to cover the space?
- ▶ First is second -1.
- ▶ The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ measures this (the second quantity).
- ▶ **Folklore.** $\gamma(\mathbb{R}^2) = \gamma^*(\mathbb{R}^2) = \frac{2}{\sqrt{3}}$.



The simultaneous covering and packing constant



- ▶ Compute the radius of the largest non-overlapping ball.
- ▶ How much do we have to inflate the balls to cover the space?
- ▶ First is second -1.
- ▶ The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ measures this (the second quantity).
- ▶ **Folklore.** $\gamma(\mathbb{R}^2) = \gamma^*(\mathbb{R}^2) = \frac{2}{\sqrt{3}}$.



The simultaneous covering and packing constant



- ▶ Compute the radius of the largest non-overlapping ball.
- ▶ How much do we have to inflate the balls to cover the space?
- ▶ First is second -1.
- ▶ The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ measures this (the second quantity).
- ▶ **Folklore.** $\gamma(\mathbb{R}^2) = \gamma^*(\mathbb{R}^2) = \frac{2}{\sqrt{3}}$.
- ▶ **If $\mathcal{X}$ is finite dimensional:**

The simultaneous covering and packing constant



- ▶ Compute the radius of the largest non-overlapping ball.
- ▶ How much do we have to inflate the balls to cover the space?
- ▶ First is second -1.
- ▶ The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ measures this (the second quantity).
- ▶ **Folklore.** $\gamma(\mathbb{R}^2) = \gamma^*(\mathbb{R}^2) = \frac{2}{\sqrt{3}}$.
- ▶ **If $\mathcal{X}$ is finite dimensional:**
 - ▶ **Folklore.** $\gamma(\mathcal{X}) < 2$.



- ▶ Compute the radius of the largest non-overlapping ball.
- ▶ How much do we have to inflate the balls to cover the space?
- ▶ First is second -1.
- ▶ The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ measures this (the second quantity).
- ▶ **Folklore.** $\gamma(\mathbb{R}^2) = \gamma^*(\mathbb{R}^2) = \frac{2}{\sqrt{3}}$.
- ▶ **If $\mathcal{X}$ is finite dimensional:**
 - ▶ **Folklore.** $\gamma(\mathcal{X}) < 2$.
 - ▶ **Venkov–McMullen.** $\gamma(\mathcal{X}) = 1$ if and only if $\gamma^*(\mathcal{X}) = 1$.
 - ▶ **Rogers.** Is $\gamma(\mathcal{X}) = \gamma^*(\mathcal{X})$?

The simultaneous covering and packing constant



- ▶ Compute the radius of the largest non-overlapping ball.
- ▶ How much do we have to inflate the balls to cover the space?
- ▶ First is second -1.
- ▶ The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ measures this (the second quantity).
- ▶ **Folklore.** $\gamma(\mathbb{R}^2) = \gamma^*(\mathbb{R}^2) = \frac{2}{\sqrt{3}}$.
- ▶ **If $\mathcal{X}$ is finite dimensional:**
 - ▶ **Folklore.** $\gamma(\mathcal{X}) < 2$.
 - ▶ **Venkov–McMullen.** $\gamma(\mathcal{X}) = 1$ if and only if $\gamma^*(\mathcal{X}) = 1$.
 - ▶ **Rogers.** Is $\gamma(\mathcal{X}) = \gamma^*(\mathcal{X})$?
 - ▶ **Mahler (1946).** $\gamma(\mathcal{X}) = 1$ if and only if $\mathcal{X}$ has a (lattice) packing that is also a covering (aka, a *tiling*).

The simultaneous covering and packing constant



- ▶ Compute the radius of the largest non-overlapping ball.
- ▶ How much do we have to inflate the balls to cover the space?
- ▶ First is second -1.
- ▶ The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ measures this (the second quantity).
- ▶ **Folklore.** $\gamma(\mathbb{R}^2) = \gamma^*(\mathbb{R}^2) = \frac{2}{\sqrt{3}}$.
- ▶ **If $\mathcal{X}$ is finite dimensional:**
 - ▶ **Folklore.** $\gamma(\mathcal{X}) < 2$.
 - ▶ **Venkov–McMullen.** $\gamma(\mathcal{X}) = 1$ if and only if $\gamma^*(\mathcal{X}) = 1$.
 - ▶ **Rogers.** Is $\gamma(\mathcal{X}) = \gamma^*(\mathcal{X})$?
 - ▶ **Mahler (1946).** $\gamma(\mathcal{X}) = 1$ if and only if $\mathcal{X}$ has a (lattice) packing that is also a covering (aka, a *tiling*).
- ▶ **What about infinite-dimensional spaces?**

Now, to infinite dimensions



- The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ is

$$\gamma(\mathcal{X}) := \inf\{r > 0: \exists \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense}\}.$$



- ▶ The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ is

$$\gamma(\mathcal{X}) := \inf\{r > 0: \exists \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense}\}.$$

- ▶ The **lattice simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ is

$$\gamma^*(\mathcal{X}) := \inf\{r > 0: \exists \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense **subgroup**}\}.$$

Now, to infinite dimensions



$$\gamma(\mathcal{X}) := \inf\{r > 0: \exists \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense}\}.$$

$$\gamma^*(\mathcal{X}) := \inf\{r > 0: \exists \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense **subgroup**}\}.$$

Now, to infinite dimensions



$$\gamma(\mathcal{X}) := \inf\{r > 0: \exists \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense}\}.$$

$$\gamma^*(\mathcal{X}) := \inf\{r > 0: \exists \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense subgroup}\}.$$

- ▶ If $\mathcal{X}$ has a packing that also covers, $\gamma(\mathcal{X}) = 1$.
- ▶ Taking any maximal packing $\leadsto \gamma(\mathcal{X}) \leq 2$.
- ▶ **De Bernardi, R., Somaglia (TAMS, 2026+).** $\gamma^*(\mathcal{X}) \leq 2$.

Now, to infinite dimensions



$$\gamma(\mathcal{X}) := \inf\{r > 0: \exists \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense}\}.$$

$$\gamma^*(\mathcal{X}) := \inf\{r > 0: \exists \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense subgroup}\}.$$

- ▶ If $\mathcal{X}$ has a packing that also covers, $\gamma(\mathcal{X}) = 1$.
- ▶ Taking any maximal packing $\leadsto \gamma(\mathcal{X}) \leq 2$.
- ▶ **De Bernardi, R., Somaglia (TAMS, 2026+).** $\gamma^*(\mathcal{X}) \leq 2$.
- ▶ To sum it up

$$\text{(good)} \quad 1 \leq \gamma(\mathcal{X}) \leq \gamma^*(\mathcal{X}) \leq 2 \quad \text{(bad)}$$

Now, to infinite dimensions



$$\gamma(\mathcal{X}) := \inf\{r > 0: \exists \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense}\}.$$

$$\gamma^*(\mathcal{X}) := \inf\{r > 0: \exists \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense **subgroup**}\}.$$

- ▶ If $\mathcal{X}$ has a packing that also covers, $\gamma(\mathcal{X}) = 1$.
- ▶ Taking any maximal packing $\leadsto \gamma(\mathcal{X}) \leq 2$.
- ▶ **De Bernardi, R., Somaglia (TAMS, 2026+).** $\gamma^*(\mathcal{X}) \leq 2$.
- ▶ To sum it up

$$\textbf{(good)} \quad 1 \leq \gamma(\mathcal{X}) \leq \gamma^*(\mathcal{X}) \leq 2 \quad \textbf{(bad)}$$

- ▶ **Is it true that $\gamma(\mathcal{X}) < 2$ for all spaces?**
 - ▶ **Are we smarter than Zorn?**

Now, to infinite dimensions



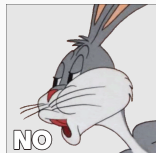
$$\gamma(\mathcal{X}) := \inf\{r > 0: \exists \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense}\}.$$

$$\gamma^*(\mathcal{X}) := \inf\{r > 0: \exists \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense subgroup}\}.$$

- ▶ If $\mathcal{X}$ has a packing that also covers, $\gamma(\mathcal{X}) = 1$.
- ▶ Taking any maximal packing $\leadsto \gamma(\mathcal{X}) \leq 2$.
- ▶ **De Bernardi, R., Somaglia (TAMS, 2026+).** $\gamma^*(\mathcal{X}) \leq 2$.
- ▶ To sum it up

$$\text{(good)} \quad 1 \leq \gamma(\mathcal{X}) \leq \gamma^*(\mathcal{X}) \leq 2 \quad \text{(bad)}$$

- ▶ **Is it true that $\gamma(\mathcal{X}) < 2$ for all spaces?**
 - ▶ **Are we smarter than Zorn?**





- ▶ **There are reflexive (resp. octahedral) spaces with $\gamma(\mathcal{X}) = 2$.**
- ▶ Take $p \in [1, \infty)$, $p_k \rightarrow \infty$. Then

$$\mathcal{X}_p = \left(\bigoplus_{k=1}^{\infty} \ell_{p_k}(\omega_k) \right)_{\ell_p}$$

satisfies $\gamma(\mathcal{X}_p) = 2$.



- ▶ **There are reflexive (resp. octahedral) spaces with $\gamma(\mathcal{X}) = 2$.**

- ▶ Take $p \in [1, \infty)$, $p_k \rightarrow \infty$. Then

$$\mathcal{X}_p = \left(\bigoplus_{k=1}^{\infty} \ell_{p_k}(\omega_k) \right)_{\ell_p}$$

satisfies $\gamma(\mathcal{X}_p) = 2$.

- ▶ **$\gamma(\ell_1) = 1$, but ℓ_1 does not admit any tiling by balls.**

- ▶ So, $\gamma(\mathcal{X}) = 1$ is weaker than the existence of a tiling.



- ▶ **There are reflexive (resp. octahedral) spaces with $\gamma(\mathcal{X}) = 2$.**

- ▶ Take $p \in [1, \infty)$, $p_k \rightarrow \infty$. Then

$$\mathcal{X}_p = \left(\bigoplus_{k=1}^{\infty} \ell_{p_k}(\omega_k) \right)_{\ell_p}$$

satisfies $\gamma(\mathcal{X}_p) = 2$.

- ▶ **$\gamma(\ell_1) = 1$, but ℓ_1 does not admit any tiling by balls.**

- ▶ So, $\gamma(\mathcal{X}) = 1$ is weaker than the existence of a tiling.

- ▶ For $1 \leq p < \infty$ and every infinite κ

$$\gamma(\ell_p(\kappa)) = \gamma^*(\ell_p(\kappa)) = \frac{2}{2^{1/p}}.$$

- ▶ If $\mathcal{K}$ is a zero-dimensional or metrisable compact space,

$$\gamma(\mathcal{C}(\mathcal{K})) = \gamma^*(\mathcal{C}(\mathcal{K})) = 1.$$



- ▶ Is there a **separable** Banach space $\mathcal{X}$ with $\gamma(\mathcal{X}) = 2$?
 - ▶ Or a **super-reflexive** one?
- ▶ Is there a normed space $\mathcal{X}$ with $\gamma(\mathcal{X}) \neq \gamma^*(\mathcal{X})$?



- ▶ Is there a **separable** Banach space $\mathcal{X}$ with $\gamma(\mathcal{X}) = 2$?
 - ▶ Or a **super-reflexive** one?
- ▶ Is there a normed space $\mathcal{X}$ with $\gamma(\mathcal{X}) \neq \gamma^*(\mathcal{X})$?
- ▶ What are the exact values of $\gamma(\ell_1 \oplus_2 \mathbb{R})$ and $\gamma^*(\ell_1 \oplus_2 \mathbb{R})$?
- ▶ **And many more...**



- ▶ Is there a **separable** Banach space $\mathcal{X}$ with $\gamma(\mathcal{X}) = 2$?
 - ▶ Or a **super-reflexive** one?
- ▶ Is there a normed space $\mathcal{X}$ with $\gamma(\mathcal{X}) \neq \gamma^*(\mathcal{X})$?
- ▶ What are the exact values of $\gamma(\ell_1 \oplus_2 \mathbb{R})$ and $\gamma^*(\ell_1 \oplus_2 \mathbb{R})$?
- ▶ **And many more...**

Thank you for your attention!

