

Topological nature of Formal Concept Analysis within the framework of multifunction theory

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Theory

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A pair $(A, B) \in \mathcal{P}(X) \times \mathcal{P}(Y)$ is called **concept** if

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In the unified form of these definitions, a pair of operators $\alpha : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$, $\beta : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ is used that satisfies the conditions

$$\alpha(\beta(\alpha(A))) = \alpha(A) \text{ and } \beta(\alpha(\beta(B))) = \beta(B).$$

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Shortly,

$$(A, B) \in \text{Def}(\alpha, \beta) = \{(A, B) : (A, B) = (\beta(B), \alpha(A))\}.$$

In this presentation, we will use multifunctions as a tool; the basic one is $F : X \rightarrow Y$ defined by $F(x) = \{y : x\mathcal{R}y\}$, which replaces the relation $\mathcal{R} \subset X \times Y$. In other words, $F(x)$ means the set of all attributes of x . Therefore, a formal context $(X, Y, \mathcal{R})$ takes the form (X, Y, F) .

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Throughout our investigation, we excluded some extreme cases by assuming that the following natural conditions are met:

1. $F(x) \neq \emptyset$ for all $x \in X$ (*any objects x has a attribute in Y*),
2. For all $y \in Y$, we have $\{x \in X : y \in F(x)\} \neq \emptyset$ (*any attributes y is possessed by at least one object in X*),
3. $F(x) \neq Y$ for all $x \in X$ (*there exists no object x possessing all attributes in Y*),
4. For all $y \in Y$, we have $\{x \in X : y \in F(x)\} \neq X$ (*there exists no attributes y shared by all objects in X*).

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3. $(F^-)^-(A) = F(A),$ where $A \subset X,$

Analogously for the multifunctions F_c and $F_c^-.$

Wille's concept

The first form of the formal concept, proposed by Wille, is defined by the operators $A \rightarrow A^*$ and $B \rightarrow B^*$, and can be presented by multifunctions because of the equalities $A^* = Y \setminus F_c(A)$ and $B^* = X \setminus F_c^-(B)$ [3].

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So, the following operators designate this type of concept.

$$\begin{aligned} Y \setminus F_c &: \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \\ X \setminus F_c^- &: \mathcal{P}(Y) \rightarrow \mathcal{P}(X). \end{aligned}$$

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The family of all Wille's concepts in (X, Y, F) is denoted in the operator form

$$\text{Def}(Y \setminus F_c, X \setminus F_c^-) = \{(A, B) : (A, B) = (X \setminus F_c^-(B), Y \setminus F_c(A))\}.$$

Wille's concept was developed by introducing seven additional versions of concepts, which were described in terms of multifunctions in [3] as follows

- (i) $(A, B) \in \text{Def}(F, F^+)$ if $(A, B) = (F^+(B), F(A))$ [1, 5, 6],
- (ii) $(A, B) \in \text{Def}(F_c, F_c^+)$ if $(A, B) = (F_c^+(B), F_c(A))$ [6],
- (iii) $(A, B) \in \text{Def}(Y \setminus F, X \setminus F^-)$ if $(A, B) = (X \setminus F^-(B), Y \setminus F(A))$ [7, 6],
- (iv) $(A, B) \in \text{Def}(Y \setminus F_c, X \setminus F_c^-)$ if $(A, B) = (X \setminus F_c^-(B), Y \setminus F_c(A))$ [2, 6],
- (v) $(A, B) \in \text{Def}((F^-)^+, F^-)$ if $(A, B) = (F^-(B), (F^-)^+(A))$ [5, 6],
- (vi) $(A, B) \in \text{Def}((F_c^-)^+, F_c^-)$ if $(A, B) = (F_c^-(B), (F_c^-)^+(A))$ [6],
- (vii) $(A, B) \in \text{Def}(Y \setminus (F^-)^+, X \setminus F^+)$ if $(A, B) = (X \setminus F^+(B), Y \setminus (F^-)^+(A))$ [6],
- (viii) $(A, B) \in \text{Def}(Y \setminus (F_c^-)^+, X \setminus F_c)$ if $(A, B) = (X \setminus F_c^+(B), Y \setminus (F_c^-)^+(A))$ [6].

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The main result states that all eight classic forms of concepts can be characterized equivalently by these topologies.

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(i) $(A, B) \in \text{Def}(F, F^+)$ i.e., $A = F^+(B) \wedge B = F(A)$,
is equivalent to $A = F^+(\text{int}_{\mathcal{T}_Y^c}(B)) \wedge B = F(\text{cl}_{\mathcal{T}_X^c}(A))$, where

$$\begin{aligned}\text{int}_{\mathcal{T}_Y^c}(B) &= \bigcup \{(F_c^-)^+ F_c^-(y) : (F_c^-)^+ F_c^-(y) \subset B\}, \\ \text{cl}_{\mathcal{T}_X^c}(A) &= \bigcup \{F^+ F(x) : x \in A\}.\end{aligned}$$

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(iii) $(A, B) \in \text{Def}(Y \setminus F, X \setminus F^-)$ i.e.,
 $A = X \setminus F^-(B) \wedge B = Y \setminus F(A)$ is equivalent to
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$$int_{T_X}(A) = \bigcup \{ F^+ F(x) : F^+ F(x) \subset A \}.$$

(vii) $(A, B) \in \text{Def}(Y \setminus (F^-)^+, X \setminus F^+)$ i.e.,
 $A = X \setminus F^+(B) \wedge B = Y \setminus (F^-)^+(A)$ is equivalent to
 $A = X \setminus F^+(\text{int}_{\mathcal{T}_Y^c}(B)) \wedge B = Y \setminus (F^-)^+(\text{int}_{\mathcal{T}_X^c}(A))$, where

$$\text{int}_{\mathcal{T}_Y^c}(B) = \bigcup \{(F_c^-)^+ F_c^-(y) : (F_c^-)^+ F_c^-(y) \subset B\},$$

$$\text{int}_{\mathcal{T}_X^c}(A) = \bigcup \{F_c^+ F_c^-(x) : F_c^+ F_c^-(x) \subset A\}.$$

(vii) $(A, B) \in \text{Def}(Y \setminus (F^-)^+, X \setminus F^+)$ i.e.,
 $A = X \setminus F^+(B) \wedge B = Y \setminus (F^-)^+(A)$ is equivalent to
 $A = X \setminus F^+(\text{int}_{\mathcal{T}_Y^c}(B)) \wedge B = Y \setminus (F^-)^+(\text{int}_{\mathcal{T}_X^c}(A))$, where

$$\text{int}_{\mathcal{T}_Y^c}(B) = \bigcup \{(F_c^-)^+ F_c^-(y) : (F_c^-)^+ F_c^-(y) \subset B\},$$

$$\text{int}_{\mathcal{T}_X^c}(A) = \bigcup \{F_c^+ F_c(x) : F_c^+ F_c(x) \subset A\}.$$

(viii) $(A, B) \in \text{Def}(Y \setminus (F_c^-)^+, X \setminus F_c^+)$ i.e.,
 $A = X \setminus F_c^+(B) \wedge B = Y \setminus (F_c^-)^+(A)$ is equivalent to
 $A = X \setminus F_c^+(\text{int}_{\mathcal{T}_Y}(B)) \wedge B = Y \setminus (F_c^-)^+(\text{int}_{\mathcal{T}_X}(A))$, where

$$\text{int}_{\mathcal{T}_Y}(B) = \bigcup \{(F^-)^+ F^-(y) : (F^-)^+ F^-(y) \subset B\},$$

$$\text{int}_{\mathcal{T}_X}(A) = \bigcup \{F^+ F(x) : F^+ F(x) \subset A\}.$$

Proof of (i)

$(A, B) \in \text{Def}(F, F^+)$ i.e., $A = F^+(B) \wedge B = F(A)$,
is equivalent to $A = F^+(\text{int}_{\mathcal{T}_Y^c}(B)) \wedge B = F(\text{cl}_{\mathcal{T}_X^c}(A))$.

$$\begin{aligned}
 F(cl_{T_X^c}(A)) &= F(\bigcup\{F^+F(x) : x \in A\}) & (1) \\
 &= \bigcup\{FF^+F(x) : x \in A\} & (2) \\
 &= \bigcup\{F(x) : x \in A\} & (3) \\
 &= F(A) = B
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad cl_{T_X^c}(A) &= \bigcup\{F^+F(x) : x \in A\}, \\
 (2) \quad F\left(\bigcup_{s \in S} A_s\right) &= \bigcup_{s \in S} F(A_s), [3] \\
 (3) \quad F(A) &= FF^+F(A), [3]
 \end{aligned}$$

$$\begin{aligned}
F^+(int_{\mathcal{T}_Y^c}(B)) &= F^+(Y \setminus cl_{\mathcal{T}_Y^c}(X \setminus B)) \\
&= X \setminus F^-(cl_{\mathcal{T}_Y^c}(X \setminus B)) & (4) \\
&= X \setminus F^-(\bigcup\{(F^-)^+F^-(y) : y \in Y \setminus B\}) & (5) \\
&= X \setminus \bigcup\{F^-(F^-)^+F^-(y) : y \in Y \setminus B\} & (6) \\
&= X \setminus \bigcup\{F^-(y) : y \in Y \setminus B\} & (7) \\
&= X \setminus F^-(Y \setminus B) = F^+(B) = A,
\end{aligned}$$

$$(4) \quad F^+(Y \setminus B) = X \setminus F^-(B), [3]$$

$$(5) \quad cl_{\mathcal{T}_Y^c}(B) = \bigcup\{(F^-)^+F^-(y) : y \in B\},$$

$$(6) \quad F^-(\bigcup_{s \in S} A_s) = \bigcup_{s \in S} F^-(A_s), [3]$$

$$(7) \quad F^-(A) = F^-(F^-)^+F^-(A) [3].$$

$$F^+(int_{\mathcal{T}_Y^c}(B)) = F^+(Y \setminus cl_{\mathcal{T}_Y^c}(X \setminus B))$$

$$= X \setminus F^-(cl_{\mathcal{T}_Y^c}(X \setminus B)) \quad (4)$$

$$= X \setminus F^-(\bigcup\{(F^-)^+F^-(y) : y \in Y \setminus B\}) \quad (5)$$

$$= X \setminus \bigcup\{F^-(F^-)^+F^-(y) : y \in Y \setminus B\} \quad (6)$$

$$= X \setminus \bigcup\{F^-(y) : y \in Y \setminus B\} \quad (7)$$

$$= X \setminus F^-(Y \setminus B) = F^+(B) = A,$$

$$(4) F^+(Y \setminus B) = X \setminus F^-(B), [3]$$

$$(5) cl_{\mathcal{T}_Y^c}(B) = \bigcup\{(F^-)^+F^-(y) : y \in B\},$$

$$(6) F^-(\bigcup_{s \in S} A_s) = \bigcup_{s \in S} F^-(A_s), [3]$$

$$(7) F^-(A) = F^-(F^-)^+F^-(A) [3].$$

In every step of this proof, we used equivalences, so the inverse proof is obvious. The other points of this theorem can be proved in a similar way by using appropriate topologies.

Theorem

For a formal context (X, Y, F) , the multifunction F generates the topologies $\mathcal{T}_X$ and $\mathcal{T}_X^c$ in X , and topologies $\mathcal{T}_Y$ and $\mathcal{T}_Y^c$ in Y with interior and closure operators defined as follows

- (i) $\text{int}_{\mathcal{T}_X}(A) = \bigcup \{F^+F(x) : F^+F(x) \subset A\}$ and $\text{cl}_{\mathcal{T}_X}(A) = \bigcup \{F_c^+F_c(x) : x \in A\}$,
- (ii) $\text{int}_{\mathcal{T}_X^c}(A) = \bigcup \{F_c^+F_c(x) : F_c^+F_c(x) \subset A\}$ and $\text{cl}_{\mathcal{T}_X^c}(A) = \bigcup \{F^+F(x) : x \in A\}$.
- (iii) $\text{int}_{\mathcal{T}_Y}(B) = \bigcup \{(F^-)^+F^-(y) : (F^-)^+F^-(y) \subset B\}$ and $\text{cl}_{\mathcal{T}_Y}(B) = \bigcup \{(F_c^-)^+F_c^-(y) : y \in B\}$,
- (iv) $\text{int}_{\mathcal{T}_Y^c}(B) = \bigcup \{(F_c^-)^+F_c^-(y) : (F_c^-)^+F_c^-(y) \subset B\}$ and $\text{cl}_{\mathcal{T}_Y^c}(B) = \bigcup \{(F^-)^+F^-(y) : y \in B\}$.

General properties of multifunctions

The following properties of the multifunction were established in [3] and were used in the proof of the preceding theorem on topological definability.

1. $F(\bigcup_{s \in S} A_s) = \bigcup_{s \in S} F(A_s); \quad F_c(\bigcup_{s \in S} A_s) = \bigcup_{s \in S} F_c(A_s),$
2. $F^-(\bigcup_{s \in S} A_s) = \bigcup_{s \in S} F^-(A_s); \quad F_c^-(\bigcup_{s \in S} A_s) = \bigcup_{s \in S} F_c^-(A_s),$
3. $X \setminus F^+(B) = F^-(Y \setminus B); \quad X \setminus F_c^+(B) = F_c^-(Y \setminus B),$
4. $Y \setminus (F^-)^+(A) = F(X \setminus A); \quad Y \setminus (F_c^-)^+(A) = F_c(X \setminus A),$
5. $F(A) = FF^+F(A); \quad F_c(A) = F_cF_c^+F_c(A),$
6. $F^-(A) = F^-(F^-)^+F^-(A); \quad F_c^-(A) = F_c^-(F_c^-)^+F_c^-(A).$

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**Thank you for your
attention!**
