

The 1914 Kakeya Hypothesis

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Let $\sum a_n$ be a convergent series of real numbers. The *achievement set* (or the *set of subsums*) is defined by

$$E(a_n) := \left\{ x \in \mathbb{R} : \exists K \in \mathbb{N} \quad x = \sum_{n \in K} a_n \right\}.$$

A set $B \subset \mathbb{R}$ is *achievable* if there is a series $\sum a_n$ such that $B = E(a_n)$.
The n -th remainder

$$r_n := \sum_{i>n} a_i.$$

In particular, r_0 is the sum of the whole series.

S. Kakeya, On the partial sums of an infinite series, Tôhoku Sci. Rep. 3(1915) 159-164

Theorem

The achievement set of an absolutely convergent series is closed and bounded. If the series has infinitely many non-zero terms, then the achievement set is perfect.

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n -th reversed Takeya condition: $a_n \leq r_n$

$$RK(a_n) = \{n : a_n \leq r_n\}$$

Clearly, $K \sqcup RK = \mathbb{N}$.

S. Takeya, *On the set of partial sums of an infinite series*, Proc. Tokyo Math.-Phys. Soc., 2nd series, 7(1914), 250-251

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Takeya wrote sincerely: *that the relation $a_n \leq r_n$ fails only for an infinite number of values of n , seems to be the necessary and sufficient condition that the set $E(a_n)$ should be nowhere dense; but I have no proof of it.*

The 1914 Takeya Hypothesis

If $K(a_n)$ is infinite, then $E(a_n)$ is a Cantor set.

The Guthrie-Nymann classification

A.D. Weinstein, B.E. Shapiro, *On the structure of a set of $\overline{\alpha}$ -representable numbers*,
Izv. Vysš. Učebn. Zaved. Matematika. 24 (1980), 8–11.

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The Guthrie-Nymann Classification Theorem

The achievement set of any absolutely convergent series is of one of the following four types:

- (i) *a finite set;*
- (ii) *a multi-interval set;*
- (iii) *a Cantor set;*
- (iv) *a Cantorval.*

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a Cantorval = a bounded and regularly closed set whose boundary is a Cantor set

J.A. Guthrie, J.E. Nymann, *The topological structure of the set of subsums of an infinite series*, Colloq. Math. 55(1988) 323-327

Theorem

A nonempty compact set $P \subset \mathbb{R}$ is a Cantorval if and only if all endpoints of P -gaps and external P -gaps are limits of sequences of P -intervals and limits of sequences of P -gaps.

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Theorem

A nonempty compact set $P \subset \mathbb{R}$ is a Cantorval if and only if
(i) P -gaps, including the external ones, and P -intervals have no common endpoints
and
(ii) the union of all P -intervals is dense in P .

A. Bartoszewicz, M. Filipczak, F. Prus-Wiśniowski, *Topological and algebraic aspects of subsums of series*, Traditional and present-day topics in real analysis, 345–366, Faculty of Mathematics and Computer Science. University of Łódź, Łódź, 2013.

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$E(a_n)$ is a finite set if and only if the series $\sum a_n$ has finitely many non-zero terms.

Hence, it suffices to consider convergent series of positive and monotone terms.

The multigeometric series: $k_1 \geq k_2 \geq \dots \geq k_m > 0$, $q \in (0, 1)$

$$\begin{aligned} \sum(k_1, k_2, \dots, k_m; q) &:= \\ &:= k_1 + k_2 + \dots + k_m + k_1q + k_2q + \dots + k_mq + k_1q^2 + k_2q^2 + \dots \end{aligned}$$

For the Guthrie-Nymann Cantorval $E(3, 2; \frac{1}{4})$ we have

$$K(3, 2; \frac{1}{4}) = 2\mathbb{N} \quad \text{and} \quad RK(3, 2; \frac{1}{4}) = 2\mathbb{N} - 1$$

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The series $\sum(1, 1; \frac{1}{4})$ is semi-fast convergent and hence $E(1, 1; \frac{1}{4})$ is a Cantor set. Moreover,

$$K(1, 1; \frac{1}{4}) = 2\mathbb{N} \quad \text{and} \quad RK(1, 1; \frac{1}{4}) = 2\mathbb{N} - 1$$

Thus, the Keakeya conditions do not determine the topological type of $E(a_n)$ in the general case.

There is a pinch of truth in Kakeya's prediction:

Theorem

For any $S \subset \mathbb{N}$ with $\text{card } S = \text{card } S^c = \infty$, there is a series $\sum a_n$ such that $K(a_n) = S$ and $E(a_n)$ is a Cantor set.

J. Marchwicki, J. Miska, *On Kakeya conditions for achievement sets*, Result in Math. (2021) 76:181; //doi.org./10.1007/s00025-021-01479-2

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Open Problem

Does for every $S \subset \mathbb{N}$ with $\text{card } S = \text{card } S^c = \infty$ exist a series $\sum a_n$ such that $E(a_n)$ is a Cantorval and $K(a_n) = S$?

Theorem

For any $0 < \alpha \leq \beta \leq 1$ there is $\sum a_n$ such that $E(a_n)$ is a Cantorval and $\underline{dRK}(a_n) = \alpha$ and $\overline{dRK}(a_n) = \beta$.

They asked if it is possible to get $dRK(a_n) = 0$.

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Theorem

For every sequence (m_n) of positive integers convergent to ∞ there is an achievable Cantorval $E(a_n)$ such that

$$\lim_{n \rightarrow \infty} \frac{\text{card}\{i \leq n : a_i \leq r_i\}}{m_n} = 0.$$

In particular, choosing $m_n := n$, we get the classic asymptotic density and the positive answer to the previous question.

Prus-Wiśniowski, F., Ptak J. *Achievable Cantorvals almost without reversed Kakeya conditions*, submitted

A generalization of asymptotic density

For $A \subset \mathbb{N}$ and $n \in \mathbb{N}$ let us denote $A_{\leq n} := \{k \in A : k \leq n\}$.
Then, given an infinite set $A \subset \mathbb{N}$ and another subset B of $\mathbb{N}$, we can define

$$d_A B := \lim_{n \rightarrow \infty} \frac{\text{card } B_{\leq n}}{\text{card } A_{\leq n}}.$$

In particular, $d = d_{\mathbb{N}}$.

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$$\forall A \ d_A A = 1$$

$\mathcal{I}_A := \{B : d_A B = 0\}$ is an ideal of sets.

$$\bigcap_A \mathcal{I}_A = \text{Fin}$$

A set $B \subset \mathbb{N}$ is finite iff $d_A B = 0$ for every infinite A .

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None of the generalized densities provides a necessary condition in the form: if $E(a_n)$ is a Cantorval, then $d_A RK(a_n) > 0$.

The generalized Marchwicki-Miska Cantorvals

The Marchwicki-Miska series are defined by consecutive groups of terms, each one built on the same scheme. Choose any positive integer n and define $n + 2$ positive integers as follows:

$$b_1^n := 2^{n+1}, b_2^n := 2^n + 1, b_3^n := 2^n, b_4^n := 2^{n-1}, \dots, b_{n+1}^n := 2^2, b_{n+2}^n := 2^1.$$

What do the subsums of those integers look like?

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What do the subsums of those integers look like?

Then choose any sequence (n_k) of positive integers and for each k build the appropriate group of integers as above but multiplied additionally by a scaling factor

$$q_k := \frac{1}{3^k} \prod_{j=1}^{k-1} \frac{1}{2^{n_j}}.$$

The resulting generalized Marchwicki-Miska series yields a Cantorval as the set of subsums.

Boundaries of Cantorvals

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The following discovery was announced during the 5th Workshop on Postmodern Real Analysis in Będlewo, Poland.

Theorem (P.Nowakowski, FPW)

The boundary of any achievable Cantorval generated by a multigeometric series is of Lebesgue measure zero.

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Theorem

The boundary of any Marchwicky-Miska Cantorval is of Lebesgue measure zero.

Given an $\varepsilon > 0$, a finite nonempty set $B \subset \mathbb{R}$ will be called ε -*tight* either if $\text{card } B = 1$ or if distance between any two consecutive elements of B (in the natural order inherited from $\mathbb{R}$) does not exceed ε .

An ε -tight subset C of a finite set B will be called *maximal* if it is maximal with respect to inclusion. The set of all maximal ε -tight subsets of B will be denoted by $\mathcal{M}_\varepsilon(B)$. Every nonempty finite subset of real numbers has a unique decomposition into finitely many disjoint maximal ε -tight subsets.

Finally, given a finite nonempty set $B \subset \mathbb{R}$, we define

$$\Delta_\varepsilon B := \max\{\text{diam } C : C \in \mathcal{M}_\varepsilon(B)\}.$$

Theorem

Let $\sum x_n$ be a convergent series of non-negative and non-increasing terms. Then the following conditions are equivalent:

- (i) the achievement set $E(x_n)$ contains an interval;*
- (ii) $\lim_{n \rightarrow \infty} \Delta_{r_n} F_n > 0$;*
- (iii) $\lim_{k \rightarrow \infty} \Delta_{r_{n_k}} F_{n_k} > 0$ for some increasing sequence (n_k) of indices.*

J. Marchwicki, P. Nowakowski, F. Prus-Wiśniowski, *Algebraic sums of achievable sets involving Cantorvals*, arXiv: 2309.01589v1[math. CA], 4 Sep 2023.