

Operation $*$ in the Baire space

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Let $(X, +)$ be an abelian group. For $A, B \subseteq X$ we write

$$A + B = \{a + b : a \in A, b \in B\}.$$

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Definition (Seredyński)

For a family $\mathcal{F} \subseteq \mathcal{P}(X)$, where X is a Polish group, let:

$$\mathcal{F}^* = \{A \subseteq X : \forall F \in \mathcal{F} A + F \neq X\}.$$

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Properties of operation $*$

For any $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}(X)$ we have:

- $\mathcal{G} \subseteq \mathcal{F}^* \implies \mathcal{F} \subseteq \mathcal{G}^*$,
- $\mathcal{F} \subseteq \mathcal{F}^{**}$,
- $\mathcal{G} \subseteq \mathcal{F} \implies \mathcal{F}^* \subseteq \mathcal{G}^*$,
- $\mathcal{F}^* = \mathcal{F}^{***}$,
- $\mathcal{F}^*$ is closed under taking subsets and translation invariant,
- $\mathcal{F} = \mathcal{F}^{**} \iff \exists A \mathcal{F} = A^*$.

Theorem (Galvin-Mycielski-Solovay)

A set $X \subseteq 2^\omega$ has strong measure zero if and only if for every meager set H it holds that $X + H \neq 2^\omega$.

So using operation $*$ we can write:

$$\mathcal{M}^* = \mathcal{SMZ}.$$

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Borel Conjecture

$$\mathcal{SMZ} = \mathcal{Ctbl}.$$

Corollary

$$\text{Borel Conjecture} \implies \mathcal{M} \neq \mathcal{M}^{**}.$$

What about the Baire space?

Theorem (Wohofsky)

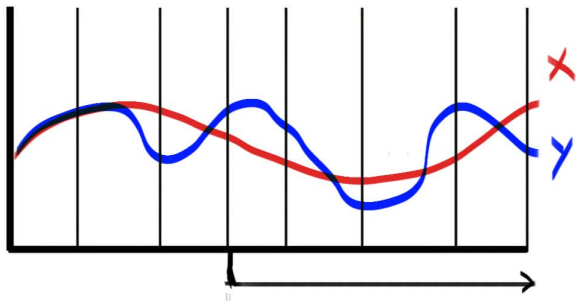
Galvin-Mycielski-Solovay Theorem doesn't hold in the Baire space. But we have $\mathcal{M}^* \subseteq \mathcal{SMZ}$.

Definition

For $x \in \omega^\omega$ and an interval partition $\bar{I}$ we define

$$M(x, \bar{I}) = \{y \in \omega^\omega : \forall^\infty n \ x \upharpoonright I_n \neq y \upharpoonright I_n\}$$

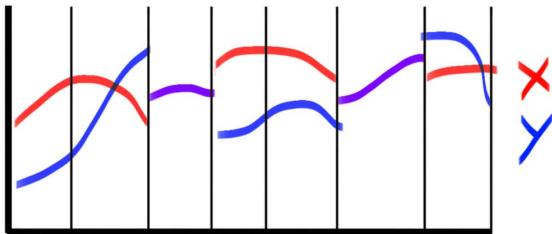
Let $\mathcal{M}_-$ be the σ -ideal generated by the sets $M(x, \bar{I})$ for $x \in \omega^\omega$ and $\bar{I} \in \mathbb{IP}$.



Definition

Set $X \subseteq \omega^\omega$ is $\mathcal{SMZ}^+$ if for every interval partition I there exists $y \in \omega^\omega$ such that:

$$\forall x \in X \quad \exists^\infty n \quad x \upharpoonright I_n = y \upharpoonright I_n$$



Theorem

For $X \subseteq \mathbb{Z}^\omega$, we have $X \in \mathcal{SMZ}^+$ if and only if for every $H \in \mathcal{M}_-$ we have $X + H \neq \mathbb{Z}^\omega$

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$$\mathcal{M}_- \subseteq \mathcal{M}$$

$$\mathcal{SMZ} \subseteq \mathcal{SMZ}^+$$

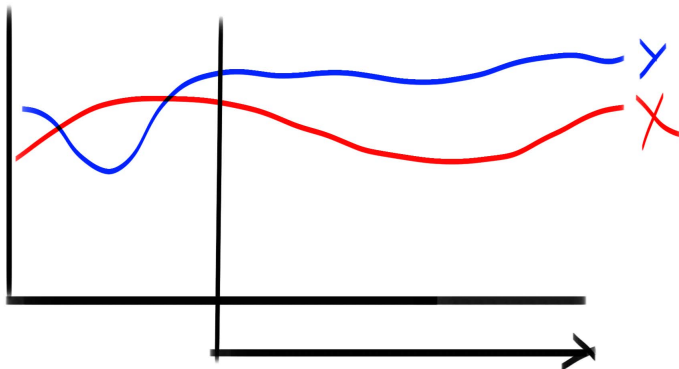
$$\mathcal{M}_-^* = \mathcal{SMZ}^+$$

Other ideals in the Baire space

Definition

Let $\mathcal{ED}$ be σ -ideal σ -generated by E_y where $y \in \omega^\omega$ and

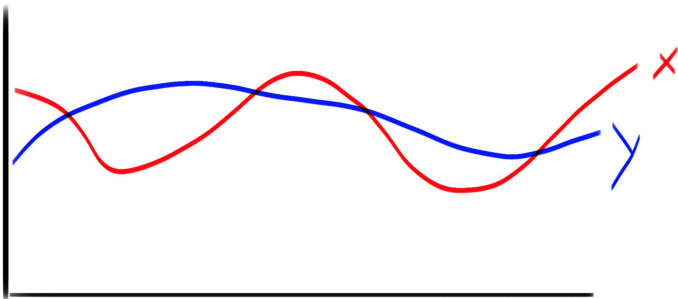
$$E_y = \{x \in \omega^\omega : \forall_n^\infty x(n) \neq y(n)\}.$$



Definition

Let $\mathcal{IE}$ be σ -ideal σ -generated by I_y where $y \in \omega^\omega$ and

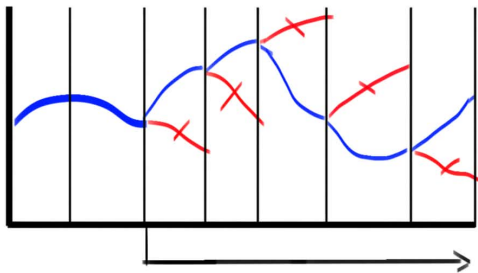
$$I_y = \{x \in \omega^\omega : \exists^\infty n \ x(n) = y(n)\}$$



Theorem

A set $X \subseteq \omega^\omega$ is uniformly meager if and only if there exists interval partition $\bar{I} \in \mathbb{IP}$ and a function $\psi : \bigcup_n \omega^{\min(I_n)} \rightarrow \bigcup_n \omega^{I_n}$ such that $\text{dom}(\psi(\sigma)) = I_n$, for $\sigma \in \omega^{\min(I_n)}$ and

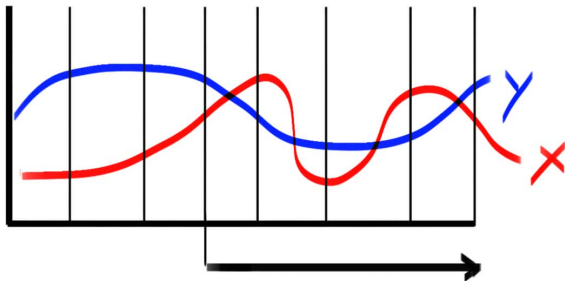
$$X \subseteq U(\psi, \bar{I}) = \{x \in \omega^\omega : \forall^\infty n \ x \upharpoonright I_n \neq \psi(x \upharpoonright \min(I_n))\}.$$



Definition

Let $\mathcal{H}$ be σ -ideal generated by $H_{I,y}$ where $f \in \omega^\omega$, I is an interval partition and:

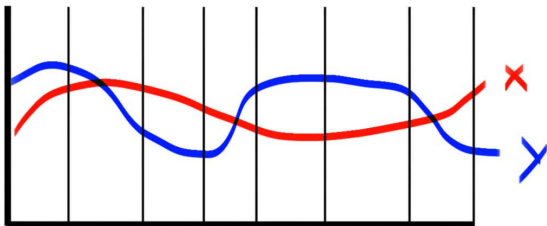
$$H_{I,y} = \{x \in \omega^\omega : \forall^\infty n \exists m \in I_n \ x(m) = y(m)\}.$$



Definition

Set $X \in \mathcal{G}$ if and only if for every interval partition I there exists $y \in \omega^\omega$ such that:

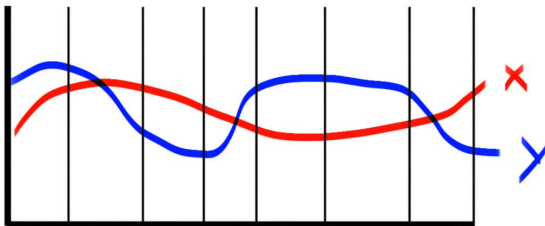
$$\forall x \in X \quad \exists^\infty n \quad \forall m \in I_n \quad x(m) \neq y(m).$$



Definition

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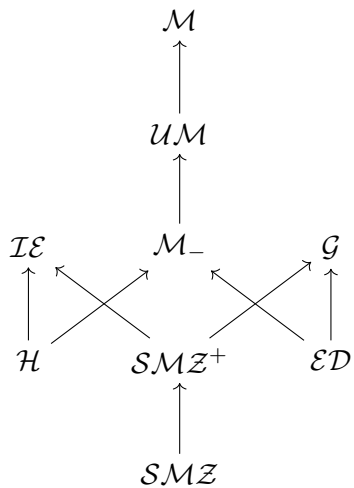
$$\forall x \in X \quad \exists^\infty n \quad \forall m \in I_n \quad x(m) \neq y(m).$$



Fact

In Cantor space we have :

- $\mathcal{M} = \mathcal{M}_- = \mathcal{H}$,
- $\mathcal{SMZ} = \mathcal{SMZ}^+ = \mathcal{G}$.



Operation $*$ in the Baire space

Theorem

In the Baire space we have:

- $\mathcal{IE}^* = \mathcal{ED}$,
- $\mathcal{ED}^* = \mathcal{IE}$,
- $\mathcal{H}^* = \mathcal{G}$,
- $\mathcal{M}_-^* = \mathcal{SMZ}^+$,
- $\mathcal{SMZ} \subseteq \mathcal{UM}^*$.

Operation $*$ in the Baire space

Theorem

In the Baire space we have:

- $\mathcal{IE}^* = \mathcal{ED}$,
- $\mathcal{ED}^* = \mathcal{IE}$,
- $\mathcal{H}^* = \mathcal{G}$,
- $\mathcal{M}_-^* = \mathcal{SMZ}^+$,
- $\mathcal{SMZ} \subseteq \mathcal{UM}^*$.

Corollary

- $\mathcal{IE} = \mathcal{IE}^{**}$
- $\mathcal{ED} = \mathcal{ED}^{**}$
- $\mathcal{SMZ}^+ = (\mathcal{SMZ}^+)^{**}$
- $\mathcal{G} = \mathcal{G}^{**}$

Theorem

If $\mathcal{I} \subseteq \mathcal{P}(\mathbb{Z}^\omega)$ is a translation and reflection invariant proper σ -ideal with $\text{cof}(\mathcal{I}) \leq \mathfrak{c}$, $\text{add}(\mathcal{I}) = \mathfrak{c}$ and any $A \notin \mathcal{I}$, then

$$(\mathcal{I} \cup \{A\})^* \neq \mathcal{I}^*.$$

Theorem

If $\mathcal{J} \subseteq \mathcal{P}(\mathbb{Z}^\omega)$ is a translation and reflection invariant proper σ -ideal with $\text{cof}(\mathcal{J}) \leq \mathfrak{c}$, $\text{add}(\mathcal{J}) = \mathfrak{c}$ and any $A \notin \mathcal{J}$, then

$$(\mathcal{J} \cup \{A\})^* \neq \mathcal{J}^*.$$

Corrolary

- Assuming $\text{add}(\mathcal{M}_-) = \mathfrak{c}$ we get $\mathcal{SMZ}^+ \neq \mathcal{SMZ}$.
- Assuming $\text{add}(\mathcal{H}) = \mathfrak{c}$ we get $\mathcal{G} \neq \mathcal{ED}$.

Cardinal invariants

Cardinal invariants of $\mathcal{M}_-$

- $add(\mathcal{M}_-) = add(\mathcal{M})$
- $cov(\mathcal{M}_-) = cov(\mathcal{M})$
- $cof(\mathcal{M}_-) = cof(\mathcal{M})$
- $non(\mathcal{M}_-) = non(\mathcal{M})$
- $cov_t(\mathcal{M}_-) = cov(\mathcal{M})$

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Cardinal invariants of $\mathcal{H}_-$

- $add(\mathcal{H}) = add(\mathcal{M})$
- $cov(\mathcal{H}) = cof(\mathcal{M})$
- $cof(\mathcal{H}) = cof(\mathcal{M})$
- $non(\mathcal{H}) = add(\mathcal{M})$

Fact

For any σ -ideal $\mathcal{I}$ we have $\text{cov}_t(\mathcal{I}) = \text{non}(\mathcal{I}^*)$.

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Corollary

$$\begin{aligned} \text{cov}(\mathcal{M}) = \text{cov}_t(\mathcal{M}_-) = \text{non}(\mathcal{M}_-^*) = \text{non}(\mathcal{SMZ}^+) \geq \text{non}(\mathcal{SMZ}) \geq \\ \text{non}(\mathcal{M}^*) = \text{cov}_t(\mathcal{M}) \geq \text{cov}(\mathcal{M}) \end{aligned}$$

Thank you for your attention!