

Cantorval as a set of subsums of a series - new sufficient condition and its consequences

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Definition

Let $(x_n) \subset \mathbb{R}$ be a sequence such that the series $\sum_n x_n$ is absolutely convergent. The set

$$E(x_n) = \left\{ \sum_{n \in A} x_n : A \subset \mathbb{N} \right\}$$

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We say that a nonempty perfect set $E \subset \mathbb{R}$ is a Cantorval if it is not an interval and both endpoints of all gaps are accumulation points of other gaps and accumulation points of nontrivial components of E .

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Definition

Let $A \subset \mathbb{N}$. The asymptotic density of A is defined as the following limit (if it exists):

$$d(A) := \lim_{n \rightarrow \infty} \frac{|A \cap \{1, \dots, n\}|}{n}.$$

Theorem [Kakeya]

Let (x_n) be a nonincreasing sequence of positive numbers such that the series $\sum_n x_n$ is convergent.

- i) $E(x_n)$ is an interval if and only if $x_n \leq r_n$ for all $n \in \mathbb{N}$.
- ii) $E(x_n)$ is a finite union of intervals if and only if there is $k \in \mathbb{N}$ such that $x_n \leq r_n$ for all $n \geq k$.
- iii) If there is $k \in \mathbb{N}$ such that $x_n > r_n$ for all $n \geq k$, then $E(x_n)$ is a Cantor set.

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- iii) If there is $k \in \mathbb{N}$ such that $x_n > r_n$ for all $n \geq k$, then $E(x_n)$ is a Cantor set.

Theorem [Guthrie, Nymann]

Let $\sum_n x_n$ be an absolutely convergent series in $\mathbb{R}$. Then $E(x_n)$ has one of the following forms:

- a finite set;
- a finite union of closed intervals;
- a Cantor set;
- a Cantorval.

Theorem [Marchwicki, Miska]

Let $K \subset \mathbb{N}$ be infinite. There exists a nonincreasing sequence of positive numbers (x_n) such that $x_n > r_n$ if and only if $n \in K$ and $E(x_n)$ is a Cantor set.

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Theorem [Marchwicki, Miska], [Prus-Wiśniowski, Ptak]

For any $d \in [0, 1]$ there exists a nonincreasing sequence of positive numbers (x_n) such that $E(x_n)$ is a Cantorval and

$$d(\{n \in \mathbb{N} : x_n \leq r_n\}) = d.$$

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Theorem [Marchwicki, Miska], [Prus-Wiśniowski, Ptak]

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Problem

Let $K \subset \mathbb{N}$ be infinite such that K^c is also infinite. Does there exist a nonincreasing sequence of positive numbers (x_n) such that $x_n > r_n$ if and only if $n \in K$ and $E(x_n)$ is a Cantorval.

Known results [Banakh, Bartoszewicz, Filipczak, Szymonik, Kenyon, Nitecki, Głąb, Marchwicki, Kula]

Let $x = (k_1, \dots, k_m; q)$ be a multigeometric sequence satisfying some assumptions. There are $0 < q_1 < q_2 < q_3 < 1$ such that:

- i) if $q < q_1$, then $E(x)$ is a Cantor set;
- ii) $E(x)$ is an interval (a finite union of intervals) if and only if $q \geq q_3$;
- iii) if $q \in [q_2, q_3)$, then $E(x)$ is a Cantorval;
- iv) if $q = q_1$, then in some cases $E(x)$ is a Cantorval and in other cases $E(x)$ is a Cantor set;
- v) for almost all $q \in (q_1, q_2)$ the set $E(x)$ has positive Lebesgue measure, but there is a decreasing sequence (p_n) converging to q_1 such that for $q = p_n$ we have that $E(x)$ is a Cantor set of Lebesgue measure zero.

Basic assumptions and notation

Let $x = (x_n)$ be a decreasing sequence of positive terms such that $\sum_n x_n$ is convergent, $x_n > r_n$ infinitely many times, $x_n < r_n$ infinitely many times and $x_1 < r_1$. We are now going to inductively construct two sequences (k_n) and (m_n) . Let

$k_1 = \min\{i \in \mathbb{N} : x_i > r_i\}$ and let $m_1 = \min\{i \in \mathbb{N} : x_{k_1+i} \leq r_{k_1+i}\}$.

If we have defined k_n and m_n for some $n \in \mathbb{N}$, then let

$k_{n+1} = \min\{i > k_n + m_n : x_i > r_i\}$ and let

$m_{n+1} = \min\{i \in \mathbb{N} : x_{k_{n+1}+i} \leq r_{k_{n+1}+i}\}$. Put

$K_n := \{k_n, \dots, k_n + m_n - 1\}$ and $K := \bigcup_{n=1}^{\infty} K_n$.

Theorem 1

Let (x_n) be as in Basic assumptions. For $i \notin K$ let $\delta_i = r_i - x_i$. For $i < k_1 - 1$ let $M_i^1 := \delta_i$. Assume that for some $n \in \mathbb{N}$ we have defined a finite sequence $(M_i^n)_i$. Assume that for all i one of the following conditions $(*)$ or $(**)$ is satisfied and that $(*)'$ holds:

$$(*) \quad \begin{cases} M_i^n - x_{k_n} > 0 \\ r_{k_n} + r_{k_n+m_n-1} - M_i^n > 0 \\ \forall j \in \{1, \dots, m_n-1\} \quad M_i^n - 2x_{k_n+j} - \sum_{l=1}^{j-1} x_{k_n+l} > 0 \end{cases}$$

$$(**) \quad M_i^n - 2x_{k_n} > 0$$

$$(*)' \quad \begin{cases} \delta_{k_n-1} - \sum_{j=1}^{m_n-1} x_{k_n+j} > 0 \\ \forall j \in \{1, \dots, m_n-1\} \quad 2r_{k_n+j} - \delta_{k_n-1} + \sum_{i=1}^{j-1} x_{k_n+i} > 0 \end{cases}$$

We define a finite sequence (M_i^{n+1}) as a sequence consisting of left hand sides of all satisfied inequalities from conditions $(*)$, $(*)'$ or $(**)$ and all numbers δ_i for $k_n + m_n \leq i < k_{n+1} - 1$. Then $E(x_n)$ is a Cantorval.

Example 1

Consider the set $E(1, 2\sqrt{2} - 2; q)$, where $q = \frac{2-\sqrt{2}}{2}$. We will show that this set is a Cantorval. First, observe that $k_n = 2n$. Indeed,

$$r_2 - x_2 = q((2\sqrt{2} - 1) \cdot \frac{q}{1 - q} - 2\sqrt{2} + 2) \approx -0,07q < 0,$$

so $r_2 < x_2$. Moreover, $r_{2n} = r_2 \cdot q^{n-1}$ and $x_{2n} = x_2 \cdot q^{n-1}$, so $r_{2n} < x_{2n}$ for $n \in \mathbb{N}$. On the other hand,

$$r_1 - x_1 = q((2\sqrt{2} - 1) \cdot \frac{q}{1 - q} + 2\sqrt{2} - 2 - 1) \approx 0,59q > 0,$$

so $r_1 > x_1$. Moreover, $r_{2n-1} = r_1 \cdot q^{n-1}$ and $x_{2n-1} = x_1 \cdot q^{n-1}$, so $r_{2n-1} > x_{2n-1}$ for $n \in \mathbb{N}$.

We will show that the assumptions of Theorem 1 are satisfied.

There is no $i \notin K$, $i < k_1 - 1 = 1$, so there is no M_1^1 and $\delta_1 > 0$, so $(*)'$ is satisfied.

Example 1

$$\delta_1 - 2x_4 = r_1 - x_1 - 2x_2q \approx 0, 1q > 0,$$

so it satisfies (**).

For $M = \delta_1 - 2x_4$ we have

$$M - x_6 = r_1 - x_1 - 2x_2q - x_2q^2 \approx 0, 03q^{n-2} > 0.$$

After calculations we also get $2r_6 - M = M - x_6 > 0$. So, the condition (*) is satisfied. We also get

$$M - x_6 = 2r_6 - M = \delta_3 - 2x_6,$$

so it also satisfies (*). As the consequence we get that all numbers M_i^4 are equal to $M - x_6$ which satisfies (*) or δ_5 which satisfies (**). Therefore, the assumptions of Theorem 1 are satisfied, so $E(x_n)$ is a Cantorval.

Theorem 2

Let $(k_n), (m_n)$ be sequences of natural numbers such that $k_1 > 1$ and $k_{n+1} > k_n + m_n$. Put $k_0 := 0, m_0 := 1$,
 $K_n := \{k_n, \dots, k_n + m_n - 1\}$, $K = \bigcup_{n=1}^{\infty} K_n$. Let $x_1 > 0$ and for $i > 1$

$$x_i := \begin{cases} \frac{1}{2}x_{i-1} & \text{if } i-1, i, i+1 \notin K \text{ or } i-1, i \in K \\ \frac{2^{m_n+1}}{2^{m_n+1}}x_{i-1} & \text{if } i = k_n - 1 > k_{n-1} + m_{n-1} \\ \frac{2^{m_n+1}}{2^{m_n+2}}x_{i-1} & \text{if } i = k_n - 1 = k_{n-1} + m_{n-1} \\ \frac{2^{m_n}}{2^{m_n+1}}x_{i-1} & \text{if } i = k_n \\ \frac{1}{4}x_{i-1} & \text{if } i, i+1 \notin K, i-1 \in K. \end{cases}$$

Then the sequence (x_n) is such that $x_n > r_n$ if and only if $n \in K$ and $E(x_n)$ is a Cantorval.

Example 2

Let $k_n = 2n$ and $m_n = 1$ for all $n \in \mathbb{N}$. We will use Theorem 2. For $n \in \mathbb{N}$ we have $x_{2n} = \frac{2}{3}x_{2n-1}$ and $x_{2n+1} = \frac{3}{8}x_{2n}$. In particular, if we put $x_1 = \frac{3}{4}$, we get $x_2 = \frac{1}{2}$, $x_{2n} = x_2 \cdot (\frac{2}{3} \cdot \frac{3}{8})^{n-1} = \frac{1}{2} \cdot \frac{1}{4^{n-1}}$ and $x_{2n+1} = x_1 \cdot (\frac{2}{3} \cdot \frac{3}{8})^n = \frac{3}{4} \cdot \frac{1}{4^n}$, and thus $E(x_n)$ is the well-known Guthrie-Nymann Cantorval $E(3, 2; \frac{1}{4})$.

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Example 3

Let $k_n = 3n$ and $m_n = 1$ for all $n \in \mathbb{N}$. We will again use Theorem 2. For $n \in \mathbb{N}$ we have $x_{3n} = \frac{2}{3}x_{3n-1}$, $x_{3n-1} = \frac{3}{4}x_{3n-2}$ and $x_{3n+1} = \frac{1}{4}x_{3n}$. In particular, if we put $x_1 = \frac{1}{2}$, we get $x_2 = \frac{3}{8}$, $x_3 = \frac{1}{4}$, $x_{3n} = x_3 \cdot (\frac{2}{3} \cdot \frac{3}{4} \cdot \frac{1}{4})^{n-1} = \frac{1}{4} \cdot \frac{1}{8^{n-1}}$ and $x_{3n-1} = x_2 \cdot (\frac{1}{8})^{n-1} = \frac{3}{8} \cdot \frac{1}{8^{n-1}}$ and $x_{3n+1} = x_1 \cdot \frac{1}{8^n}$, and thus $E(x_n)$ is the known Cantorval $E(4, 3, 2; \frac{1}{8})$.

Example 4

It is known that $E(4, 3, 2; q)$ is a Cantorval for $q \in [\frac{1}{6}, \frac{2}{11})$. It also follows from Theorem 1.

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Example 5

Consider the set $E(4, 3, 2; q)$, where $q = \frac{\sqrt{57}-5}{16}$. Then $q \in (\frac{1}{8}, \frac{1}{6})$ which is the "mysterious" interval. Moreover, by Theorem 1, $E(4, 3, 2; q)$ is a Cantorval.

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