

Minimal usco maps and bornologies

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Outline

1 Basic definitions

2 Main Results

Basic definitions

Introduction

Definition

Let X, Y be a **Hausdorff** topological spaces.

- A multifunction F is **usc** at $x \in X$ iff for every open V such that $F(x) \subseteq V$ there is an open neighborhood U of x such that $F(U) \subseteq V$.
- A multifunction F is **usco**, iff for every $x \in X$ it is usc at x and $F(x)$ is non-empty compact.

Minimal Usco Maps

The system of usco multifunctions is partially ordered by $\subseteq$ (graph inclusion)

We say that F from X to Y is minimal usco iff it is usco and minimal with respect to $\subseteq$ in the family of all usco maps from X to Y .

Let $MU(X, Y)$ be the space of minimal usco maps from X to Y .

Note that it contains the class $C(X, Y)$ of continuous functions from X to Y

Example

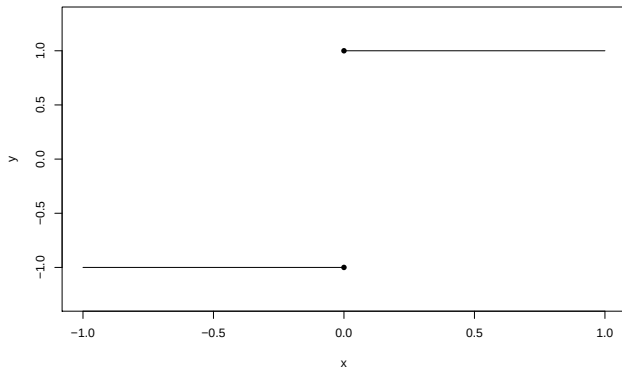


Figure: $F(x) = \operatorname{sgn} x, x \neq 0; F(0) = \{-1, 1\}$

Applications

- Recent book by [Holá et al., 2021]

Holá, L., Holý, D., and Moors, W. B. *USCO and Quasicontinuous Mappings*. (2021) De Gruyter.

- optimization [Čoban et al., 1989],
- differentiability of Lipschitz functions [Borwein, 1991], [Moors, 1995],
- selection theorems [Moors and Somasundaram, 2002],
[Holá and Holý, 2009],
- weak Asplund spaces [Fabian, 1997],
[Moors and Somasundaram, 2006].

Topology of Uniform Convergence on a Bornology

- Y is metric space ($Y = \mathbb{R}$)
- $\mathcal{B}$ is a **bornology** on X (an ideal that covers X)
- $F, G \in MU(X, Y)$ note $MU(X, \mathbb{R}) = MU(X)$
- $W[B; \varepsilon] = \{(F, G) : \forall x \in B : F(x) \subseteq S_\varepsilon(G(x)) \text{ and } G(x) \subseteq S_\varepsilon(F(x))\}$
- generate the uniformity $\mathcal{U}_{\mathcal{B}}$ of uniform convergence on $\mathcal{B}$
- corresponding topology is $\tau_{\mathcal{B}}$
- ... based on Hausdorff metric on $K(Y)$ - compact subsets of Y
- $\mathcal{B}$: $\mathcal{F}$ - pointwise convergence, $K(X)$ - uniform convergence on compacta, $\mathcal{P}(X)$ - uniform convergence

Previous Work

Previously, we have investigated when $MU(X, Y)$ is a

- 1 group / vector space

we need to define addition

[Holá and Novotný, 2020a, Holá and Novotný, 2020b]

- 2 completely metrizable / Fréchet space

[Holá and Novotný, 2022, Holá et al., 2023]

- 3 compactness and similar properties

[Holá and Holý, 2016, Bardyla et al., 2025]

Cardinal invariants on Tychonoff spaces

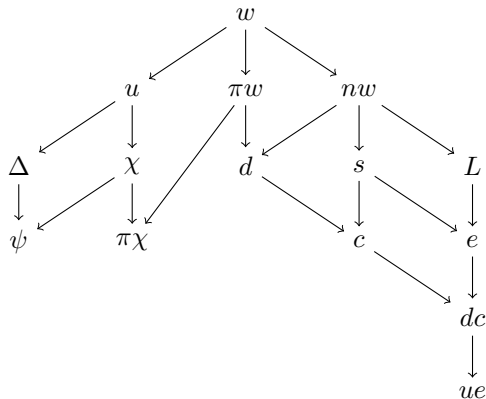


Figure: Symbol $a \rightarrow b$ means that $a(Z) \geq b(Z)$ for every Tychonoff Z .

Main Results

Character

$$\chi(z, Z) = \aleph_0 + \min\{|\mathcal{G}| : \mathcal{G} \text{ is a base at } z\},$$

$$\chi(Z) = \sup\{\chi(z, Z) : z \in Z\},$$

$$\pi\chi(z, Z) = \aleph_0 + \min\{|\mathcal{G}| : \mathcal{G} \text{ is a local } \pi\text{-base at } z\},$$

$$\pi\chi(Z) = \sup\{\pi\chi(z, Z) : z \in Z\},$$

$$u(Z) = \aleph_0 + \min\{|\mathcal{U}_0| : \mathcal{U}_0 \text{ is a base of a compatible uniformity on } Z, \}$$

cofinality of $\mathcal{B}$ by

$$\mathcal{B}\text{cof}(X) = \aleph_0 + \min\{|\mathcal{B}_0| : \mathcal{B}_0 \text{ is base of (cofinal in) } \mathcal{B}\}.$$

Character

Theorem

Let X be a regular topological space and $\mathcal{B}$ be a bornology on X with a closed base and $F \in MU(X)$. Then for $\tau_{\mathcal{B}}$ on $MU(X)$ holds

$$\begin{aligned}\pi\chi(F, MU(X)) &= \chi(F, MU(X)) = \pi\chi(MU(X)) = \chi(MU(X)) = \\ &= u(MU(X)) = \mathcal{B}\text{cof}(X).\end{aligned}$$

Finding an appropriate base of $\mathcal{U}_{\mathcal{B}}$ consisting of $W[B; 1/n]$, where $B \in \mathcal{B}_0$ - a base of $\mathcal{B}$

Metrizability

Corollary

Let X be a regular topological space and $\mathcal{B}$ be a bornology on X with a closed base. The following are equivalent.

- 1 $\mathcal{B}\text{cof}(X) = \aleph_0$,
- 2 $(MU(X), \tau_{\mathcal{B}})$ is metrizable,
- 3 $(MU(X), \tau_{\mathcal{B}})$ is first countable.

Pseudocharacter

$$\begin{aligned}\psi(z, Z) &= \aleph_0 + \min \left\{ |\mathcal{G}| : \mathcal{G} \text{ is a family of open sets in } Z \text{ such that } \bigcap \mathcal{G} = \{z\} \right\}, \\ \psi(Z) &= \sup \{ \psi(z, Z) : z \in Z \}, \\ \Delta(Z) &= \aleph_0 + \min \left\{ |\mathcal{G}| : \mathcal{G} \text{ is a family of open sets in } Z^2 \text{ such that } \bigcap \mathcal{G} = \Delta_Z \right\},\end{aligned}$$

weak $\mathcal{B}$ -covering number of X :

$$\mathrm{wBc}(X) = \aleph_0 + \min \{ |\beta| : \beta \subseteq \mathcal{B}, \overline{\bigcup \beta} = X \}.$$

Pseudocharacter

Theorem

Let X be a regular topological space with a bornology $\mathcal{B}$ and $F \in MU(X)$. Then for $\tau_{\mathcal{B}}$ on $MU(X)$ holds

$$\psi(F, MU(X)) = \psi(MU(X)) = \Delta(MU(X)) = \text{w}\mathcal{B}\text{c}(X).$$

Finding an appropriate subfamily of $\mathcal{U}_{\mathcal{B}}$ consisting of $\text{Int } W[B; 1/n]$, where $B \in \beta$ - a weak cover.

Cardinal invariants on Tychonoff spaces

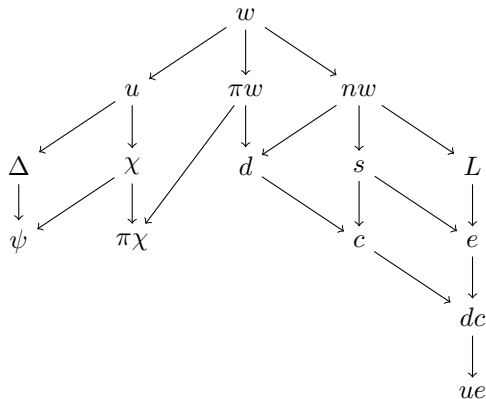


Figure: Symbol $a \rightarrow b$ means that $a(Z) \geq b(Z)$ for every Tychonoff Z .

Uniform Extent

$c(Z) = \aleph_0 + \sup\{|\mathcal{U}| : \mathcal{U} \text{ is a pairwise disjoint family of nonempty open sets in } Z\},$

$dc(Z) = \aleph_0 + \sup\{|\mathcal{U}| : \mathcal{U} \text{ is a discrete family of nonempty open sets in } Z\},$

$e(Z) = \aleph_0 + \sup\{|E| : E \text{ is a closed discrete set in } Z\},$

$ue(Z) = \aleph_0 + \sup\{|E| : E \text{ is a uniformly discrete set in } Z\},$

$E \subseteq Z$ is *uniformly discrete*, if for every $x, y \in E$, $x \neq y$, we have that $U(x) \cap U(y) = \emptyset$, for some compatible uniformity $\mathcal{U}$ and some $U \in \mathcal{U}$.

Uniform Extent

Theorem

Let X be a regular topological space and $\mathcal{B} \neq \mathcal{F}$ be a bornology on X . If $\mathcal{B}\text{cof}(X) \leq \mathfrak{c}$, then for $\tau_{\mathcal{B}}$ on $MU(X)$ holds

$$\begin{aligned} ue(MU(X)) &= dc(MU(X)) = c(MU(X)) = e(MU(X)) = \\ &= L(MU(X)) = s(MU(X)) = d(MU(X)) = \\ &= nw(MU(X)) = \pi w(MU(X)) = w(MU(X)). \end{aligned}$$

$$w = u * ue$$

proving that $ue \geq u$

Special Cases I

Theorem

Let X be a Baire space, then $|MU(X)| \leq 2^{w(X)}$.

Corollary

Let X be a regular Baire space with a countable base. Let $\mathcal{B} \neq \mathcal{F}$ be a bornology with a closed base, then for $\tau_{\mathcal{B}}$ on $MU(X)$ holds

$$\begin{aligned} ue(MU(X)) &= dc(MU(X)) = c(MU(X)) = e(MU(X)) = \\ &= L(MU(X)) = s(MU(X)) = d(MU(X)) = \\ &= nw(MU(X)) = \pi w(MU(X)) = w(MU(X)) = \mathfrak{c}. \end{aligned}$$

Special Cases II

Theorem

Let X be a metrizable Baire space, then

$$ue(MU(X), \tau_U) = 2^{w(X)}.$$

Corollary

Let X be a metrizable Baire space. Let $\mathcal{B} = \mathcal{P}(X)$, then for $\tau_{\mathcal{B}} = \tau_U$ on $MU(X)$ holds

$$\begin{aligned} ue(MU(X)) &= dc(MU(X)) = c(MU(X)) = e(MU(X)) = \\ &= L(MU(X)) = s(MU(X)) = d(MU(X)) = \\ &= nw(MU(X)) = \pi w(MU(X)) = w(MU(X)) = 2^{w(X)}. \end{aligned}$$

Example

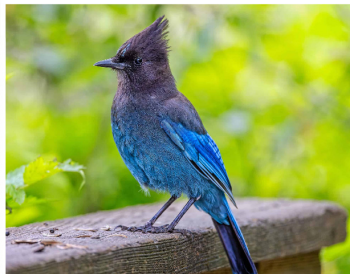
Example

Let $X = I \times \mathbb{R}$, where I is discrete with $|I| = 2^{\mathfrak{c}}$, and $\mathcal{B} = K(X)$, then for $\tau_{\mathcal{B}} = \tau_{UC}$ on $MU(X)$ holds

$$\mathfrak{c} = ue(MU(X)) = dc(MU(X)) = c(MU(X)) = d(MU(X)) < < 2^{\mathfrak{c}} = s(MU(X)) = nw(MU(X)) = \pi w(MU(X)) = w(MU(X)).$$

$(MU(X), \tau_{UC})$ is homeomorphic to $(MU(\mathbb{R}), \tau_{UC})^I$

Cardinal variants



Appendix

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