

Endpoint estimate of the rough maximal singular integral operator

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Hilbert Transform-The Begining

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where

$$L^{1,\infty}(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{C} : \text{measurable and } \|f\|_{1,\infty} < \infty\}$$

$$\|f\|_{1,\infty} = \sup_{\lambda > 0} \lambda |\{x \in \mathbb{R} : |f(x)| > \lambda\}|$$

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To prove the end point result the discovered Calderón-Zygmund decomposition.

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$$g(x) = \begin{cases} f(x) & \text{if } x \notin \bigcup_{j=1}^\infty Q_j = \Omega, \\ \frac{1}{|Q_j|} \int_{Q_j} f(x) dx & \text{if } x \in Q_j. \end{cases}$$

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Moreover, $\sum_j \|b_j\|_1 \lesssim \|f\|_1$

Endpoint Result

By using the **Calderón-Zygmund Decomposition** of a function f with a given height $\lambda > 0$, Calderón and Zygmund proved that

Theorem

T is weak $(1,1)$ i.e.

$$\left| \{x \in \mathbb{R}^d : |Tf(x)| > \lambda\} \right| \lesssim \frac{1}{\lambda} \|f\|_1.$$

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Theorem (Christ, Rubio deFrancia, 1989 Invent. Math)

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Rough Singular Integral

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Theorem (Seeger, 1996 JAMS)

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The long-standing open problem (since the 1980s–90s) is whether

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Honzik (2020, IMRN): If $\Omega \in L^\infty(\mathbb{S}^{n-1})$, then

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Theorem (Bhojak–Mohanty 2023 JFA)

Let $\Omega \in L \log L(\mathbb{S}^{d-1})$. Then

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Theorem (Xudong Lai, 2025 Arxiv)

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Variational Inequality

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Let $\{F_t : t \in I\}$ be a family of functions

$$V_q(F_t(x)) = \begin{cases} \sup_{N \in \mathbb{N}} \sup_{t_1 < t_2 < \dots < t_N} \left(\sum_{i=1}^{N-1} |F_{t_{i+1}}(x) - F_{t_i}(x)|^q \right)^{1/q}, & 1 \leq q < \infty, \\ \sup_{t_1, t_2 \in I} |F_{t_1}(x) - F_{t_2}(x)|, & q = \infty. \end{cases}$$

Weighted Estimates

For $1 < p < \infty$, we say that a locally integrable function $w \geq 0$ belongs to the Muckenhoupt A_p class if

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Thank You!