

Interrelationships among axioms for measures of non-compactness

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This talk is based on the paper:

J.Grzybowski, P.Kasprzak, P.Maćkowiak, *On the independence of axioms for measures of non-compactness*, arXiv:2607.06201, 2026

Axiomatic measure of non-compactness

Let E be a Banach space, $\mathcal{B}_E$ be the family of all non-empty and bounded subsets of E and $\mathcal{K}_E$ - the family of relatively compact members of $\mathcal{B}_E$.

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Definition (cf. Df. 3.1.3 in [1])

A function $\mu : \mathcal{B}_E \rightarrow [0, +\infty)$ is an *axiomatic measure of non-compactness*, if for any $\lambda \in [0, 1]$ and any $A, B \in \mathcal{B}_E$, it satisfies the following conditions

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- (A5) if the sets A_n are closed and $A_{n+1} \subseteq A_n \in \mathcal{B}_E$, $n \in \mathbb{N}$, with $\mu(A_n) \rightarrow 0$, then

$$\bigcap_{n=1}^{\infty} A_n \neq \emptyset.$$

Hausdorff continuity

Proposition (cf. Th. 3.2.2 in [1])

Let E be a Banach space, and let $v : \mathcal{B}_E \rightarrow [0, +\infty)$ be a set function satisfying axiom (A2) (monotonicity). Moreover, assume that there exists $\lambda \in (0, 1)$ such that

$$v(\lambda A + (1 - \lambda)B) \leq \lambda v(A) + (1 - \lambda)v(B) \text{ for all } A, B \in \mathcal{B}_E.$$

Then, v is continuous with respect to the Hausdorff distance, that is, if $(A_n)_{n \in \mathbb{N}} \subset \mathcal{B}_E$ converges in the Hausdorff distance to a set $A \in \mathcal{B}_E$, then $\lim_{n \rightarrow \infty} v(A_n) = v(A)$.

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Thus $\mu(A) = \mu(\text{cl}A)$, $A \in \mathcal{B}_E$.

Reduction of axioms

Proposition

Let $\nu: \mathcal{B}_E \rightarrow [0, +\infty)$ be a set function that satisfies axioms (A2) (monotonicity) and (A5) (intersection property). Then, $\ker \nu \subseteq \mathcal{K}_E$.

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Thus, axiom (A4) (convexity) is satisfied, if condition b) holds.

Independence of axioms

An axiom (A_i) is independent of the others, if there exists a set function that satisfies all the axioms A_1 - A_5 , except for i , $i \in \{1, \dots, 5\}$.

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- Axiom (A5) (intersection property) is independent in infinite-dimensional spaces.

Produce a closed infinite-dimensional subspace $F \subset E$ endowed with a Schauder basis and a special norm $\|\cdot\|_F$, then equip the whole linear space E with a new norm $\|\cdot\|_{\text{new}}$ constructed with help of the norm $\|\cdot\|_F$ and the original norm $\|\cdot\|_E$, finally, put $v(A) := \sup_{x,y \in A} \|x - y\|_{\text{new}}, A \in \mathcal{B}_E$ (see [2, 3])

Axiom (A3): convex hull property

Definition

A non-empty subset A of a Banach space has the *weak Carathéodory* property, if there exists $\lambda \in (0, 1]$ such that

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Theorem

Let $v : \mathcal{B}_E \rightarrow [0, +\infty)$ be a set function that satisfies axioms (A2) (monotonicity) and (A4) (convexity). Then $v(\text{conv}A) = v(A)$ for every set $A \in \mathcal{B}_E$ that has the weak Carathéodory property.

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Proposition

If E is finite-dimensional, then every non-empty and bounded set $A \subset E$ has the Carathéodory property. Therefore, axiom (A3) (convex hull property) is not independent in any finite-dimensional space.

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Question

Does every infinite-dimensional Banach space E admit a set function $v : \mathcal{B}_E \rightarrow [0, +\infty)$ that satisfies the axioms (A1), (A2), (A4), (A5), but not (A3)?

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References

- [1] J.Banaś, K.Goebel, *Measures of noncompactness in Banach spaces*, Lecture Notes in Pure and Applied Mathematics 60, Marcel Dekker, Inc., 1980.
- [2] X.Chen, L.Cheng, W.He, *On nullity and fullness of measures of noncompactness*, Sci. China Math. 53, 2023, 1577–1596.
- [3] L.Cheng, Q.Cheng, Q.Shen, K.Tu, W.Zhang, *A new approach to measures of noncompactness of Banach spaces*, Studia Math. 240, 2018, 21–45.