

Infinite convolutions of measures applied to achievement sets

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Convolution of measures

If $\mu, \nu: \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$ are probability measures, then the *convolution* $\mu * \nu$ is defined by the formula

$$(\mu * \nu)(B) = \int_{\mathbb{R}^2} \mathbf{1}_B(x + y) d(\mu \otimes \nu)(x, y),$$

where $\mu \otimes \nu$ is the product measure.

Remark. If X and Y are random variables of distributions μ and ν respectively, then $\mu * \nu$ is the distribution of $X + Y$.

Weak limit of measures

Let X be a metric space. The Borel measure $\mu: \mathcal{B}(X) \rightarrow [0, \infty]$ is a *weak limit* of a sequence (μ_n) of measures $\mu_n: \mathcal{B}(X) \rightarrow [0, \infty]$, if for any continuous and bounded function $f: X \rightarrow \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \int_X f \, d\mu_n = \int_X f \, d\mu.$$

Remark. If μ_n are distributions with cumulative distribution functions F_n , then (μ_n) converges weakly to μ with cumulative distribution function F if and only if $F_n(x) \rightarrow F(x)$ at every continuity point of F .

Infinite convolutions

If $\nu_n: \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$ are probability measures for $n \in \mathbb{N}$, then the *infinite convolution* of (ν_n) is the weak limit (if it exists) of finite convolutions $\mu_n = \nu_1 * \cdots * \nu_n$.

Example

Let $\Sigma \subseteq \mathbb{R}$ be a finite set and $q \in (0, 1)$. We define a probability Borel measure ν_n by the formula

$$\nu_n(B) = \frac{|B \cap (q^n \Sigma)|}{|\Sigma|}.$$

Fact

The sequence (ν_n) has infinite convolution μ_∞ , which is supported on

$$E(\Sigma, q) := \left\{ \sum_{i=1}^{\infty} \sigma_i q^i : (\sigma_i) \in \Sigma^{\mathbb{N}} \right\}.$$

In a Banach space, if $\sum_n x_n$ is an unconditionally convergent series, then the set

$$A(x_n) := \left\{ \sum_{n \in B} x_n : B \subseteq \mathbb{N} \right\}$$

is well defined and called the *achievement set* of (x_n) .

S. Kakaya. *On the set of partial sums of an infinite series*. Tokyo Math. Ges. 2nd series, 7 (1914), pp. 250–251.

Theorem (Keakeya)

Let $\sum_{n \in \mathbb{N}} x_n$ be a convergent sequence of positive terms in non-increasing order and $r_n = \sum_{k > n} x_k$ for all $n \in \mathbb{N}$. Then

- ① $A(x_n)$ is an interval if and only if $x_n \leq r_n$ for all $n \in \mathbb{N}$;
- ② $A(x_n)$ is a finite union of intervals if and only if $x_n \leq r_n$ for all but finitely many $n \in \mathbb{N}$;
- ③ $A(x_n)$ is a Cantor set, if $x_n > r_n$ for all but finitely many $n \in \mathbb{N}$.

S. Keakeya. *On the set of partial sums of an infinite series*. Tokyo Math. Ges. 2nd series, 7 (1914), s. 250–251.

Theorem (Guthrie–Nymann)

The achievement set of absolutely convergent series on the real line has one of the following forms:

- ① *a finite set;*
- ② *a finite union of closed intervals;*
- ③ *a homeomorphic copy of the Cantor set;*
- ④ *M-Cantorval.*

J. A. Guthrie, J. E. Nymann. *The topological structure of the set of subsums of an infinite series.* Colloq. Math. 55 no. 2 (1988), pp. 323–327.

- A set $C \subseteq \mathbb{R}$ is called an *M-Cantorval* (or simply a *Cantorval*), whenever it is compact, regularly closed and endpoints of any maximal interval $I \subseteq C$ are accumulation points of $C \setminus I$.

Fact

All M-Cantorvals are homeomorphic.

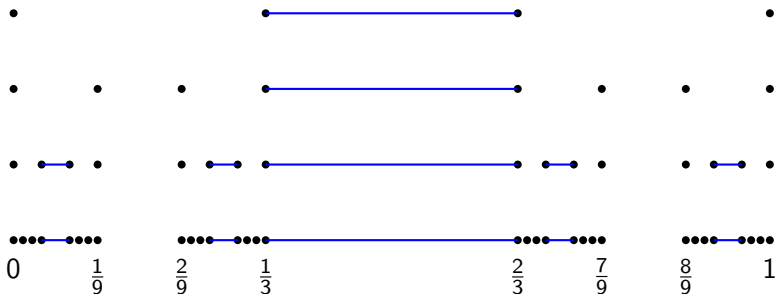
P. Mendes, F. Oliveira. *On the topological structure of the arithmetic sum of two Cantor sets*. Nonlinearity 7 no. 2 (1994), pp. 329–343.

M-Cantorval example

Approximate construction of an M-Cantorval

$$M = C \cup \bigcup_{k \in \mathbb{N}} U_{2k-1},$$

where C is the classical ternary Cantor set and U_n is the union of all open intervals removed at the n -th step of its construction.



Multigeometric sequences and self-similar sets

A *multigeometric sequence* is the sequence of the form

$$(p_1, \dots, p_n; q) := (p_1 q, \dots, p_n q, p_1 q^2, \dots, p_n q^2, \dots),$$

where $p_1, \dots, p_n$ are real numbers and $q \in (0, 1)$.

If $\Sigma \subseteq \mathbb{R}$ is a finite set and $q \in (0, 1)$, then we consider a set

$$E(\Sigma, q) := \left\{ \sum_{i=1}^{\infty} \sigma_i q^i : (\sigma_i) \in \Sigma^{\mathbb{N}} \right\}.$$

Fact

For any $p_1, \dots, p_n \in \mathbb{R}$ and $q \in (0, 1)$, we have

$$A(p_1, \dots, p_n; q) = E(\Sigma, q),$$

where $\Sigma = \{ \sum_{i=1}^n \varepsilon_i p_i : (\varepsilon_i) \in \{0, 1\}^n \}$.

Guthrie–Nymann sequence as an example

If $(x_n) = (p_1, p_2; q) = (3, 2; \frac{1}{4})$, i.e.

$$(x_n) = \left(\frac{3}{4}, \frac{1}{2}, \frac{3}{16}, \frac{1}{8}, \dots, \frac{3}{4^n}, \frac{2}{4^n}, \dots \right),$$

then $\Sigma = \{0, 2, 3, 5\}$ and

$$A\left(3, 2; \frac{1}{4}\right) = E\left(\{0, 2, 3, 5\}, \frac{1}{4}\right)$$

is a Cantorval.

J. A. Guthrie, J. E. Nymann. *The topological structure of the set of subsums of an infinite series*. Colloq. Math. 55 no. 2 (1988), pp. 323–327.

Partial results on $E(\Sigma, q)$

Adopting the notation

$$\Delta(A) := \max\{|a - b| : a, b \in A, (a, b) \cap A = \emptyset\},$$

$$I(A) := \frac{\Delta(A)}{\Delta(A) + \text{diam } A},$$

$$i(A) := \inf\{I(B) : B \subseteq A, |B| \geq 2\},$$

we have the following:

Theorem

- ① $E(\Sigma, q)$ is a Cantor set, if $q < \frac{1}{|\Sigma|}$.
- ② $E(\Sigma, q)$ is an interval if and only if $q \geq I(\Sigma)$.
- ③ $E(\Sigma, q)$ contains an interval, if $q \geq i(\Sigma)$.

T. Banach, A. Bartoszewicz, M. Filipczak, E. Szymonik. *Topological and measure properties of some self-similar sets*. Topol. Methods Nonlinear Anal. 46 no. 2 (2015), pp. 1013–1028.

Case $q = \frac{1}{|\Sigma|}$

Theorem (Kenyon)

Let $u \in \mathbb{R}$ and $E = E(\{0, 1, u\}, \frac{1}{3})$. Then E contains an interval if and only if $u \in \mathbb{Q}$, where $u = \frac{p}{q}$ is lowest terms and $3 \mid (p + q)$.

Theorem (Nitecki)

Let $\Sigma \subseteq \mathbb{R}$ be a finite set and $q = \frac{1}{|\Sigma|}$. If $\Sigma \subseteq \mathbb{Z}$ and elements of Σ give all possible remainders modulo $|\Sigma|$, then $E(\Sigma, q)$ contains an interval.

R. Kenyon. *Projecting the one-dimensional Sierpiński gasket*. Israel J. Math. 97 (1997), pp. 221–238.

Z. Nitecki. *Cantorvals and subsum sets of null sequences*. Amer. Math. Monthly 122 no. 9 (2015), pp. 862–870.

Theorem (Głab, Kula)

Let $\Sigma \subseteq \mathbb{R}$ a finite set. Then $E\left(\Sigma, \frac{1}{|\Sigma|}\right)$ contains an interval if and only if there exists $\delta > 0$ such that

$$(1) \quad \forall n \in \mathbb{N} \exists k \in \mathbb{N} \sum_{\sigma \in \Sigma} \exp\left(\frac{2\pi i n \sigma}{\delta |\Sigma|^k}\right) = 0.$$

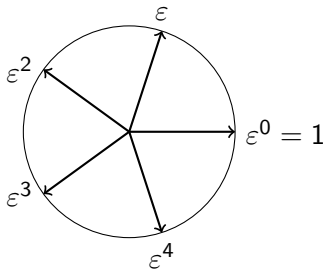
Moreover, if $E(\Sigma, q)$ contains an interval, then elements of $\Sigma - \Sigma$ are commensurable, and any common divisor satisfies condition (1) as δ .

S. Głab, M. Kula, *Kenyon Theorem Revisited*. arXiv:2508.19454 [math.GN]

Example

If $\Sigma = \{0, 1, 2, 3, 4\}$, then condition (1) is satisfied with $\delta = 1$. For any $n \in \mathbb{N}$ we take $k = 1$, and

$$\sum_{\sigma \in \Sigma} \exp\left(\frac{2\pi i n \sigma}{\delta |\Sigma|^k}\right) = \sum_{s=0}^4 \exp\left(\frac{2\pi i n s}{5}\right) = 0.$$



$$\varepsilon = \exp\left(\frac{2\pi i}{5}\right)$$

Nitecki theorem as a corollary I

Fact

If $\Sigma \subseteq \mathbb{Z}$ is a finite set and elements of Σ give all possible remainders modulo $|\Sigma|$, then condition (1) is satisfied with $\delta = 1$.

Nitecki theorem as a corollary II

Proof.

For any $n \in \mathbb{N}$ take $k = 1$. Denote $p = |\Sigma|$, $\varepsilon = \exp\left(\frac{2\pi i}{p}\right)$ and define functions $h: \mathbb{Z}_p \rightarrow \mathbb{Z}_p$, $g: \mathbb{Z}_p \rightarrow \mathbb{C}$ with $h(x) := nx \bmod p$, $g(s) := \varepsilon^s$. Then h is group homomorphism, $h[\mathbb{Z}_p] = \{y \in \mathbb{Z}_p: d|y\}$ a non-trivial subgroup for some divisor d of p and

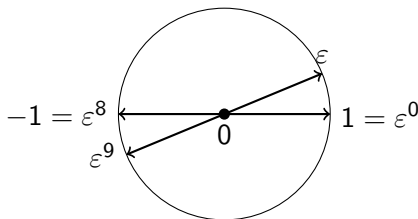
$$\begin{aligned} \sum_{\sigma \in \Sigma} \exp\left(\frac{2\pi i n \sigma}{|\Sigma|}\right) &= \sum_{x \in \mathbb{Z}_p} g(h(x)) = \sum_{s \in h[\mathbb{Z}_p]} g(s) |h^{-1}(s)| = \\ &= |\ker h| \sum_{j=0}^{\frac{p}{d}-1} \varepsilon^{dj} = 0, \end{aligned}$$

because $(\varepsilon^d)^{\frac{p}{d}} = \varepsilon^p = 1$.



Example – not all remainders

If $\Sigma = \{0, 1, 8, 9\}$, then condition (1) is satisfied with $\delta = 1$. For example, for $n = 1$ we take $k = 2$.



$$\varepsilon = \exp\left(\frac{2\pi i}{16}\right)$$

Let $E \subseteq \mathbb{R}$ be a bounded set. The family $\mathcal{T}$ is an *E-tiling* of the set $A \subseteq \mathbb{R}$, if $A \subseteq \bigcup \mathcal{T}$ and elements of $\mathcal{T}$ are measure disjoint translations of E .

Since E is bounded, then *E-tiling* $\mathcal{T}$ uniquely determines the set $B \subseteq \mathbb{R}$ such that

$$\mathcal{T} = \{\beta + E : \beta \in B\}.$$

We call the set B the *basis* for the tiling $\mathcal{T}$.

An *E-tiling* $\mathcal{T}$ with basis B is *periodic*, if there exists $t \neq 0$ such that $B + t = B$. The number $t \in \mathbb{R}$ with this property is called a *period* of the tiling $\mathcal{T}$.

If $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measurable spaces and $\mu: \mathcal{A} \rightarrow \mathbb{R}$ is a measure, and $f: X \rightarrow Y$ is measurable, then the formula $\nu(B) = \mu(f^{-1}[B])$ defines the measure $\nu: \mathcal{B} \rightarrow \mathbb{R}$. We call it a *push-forward measure* and denote by $f\#\mu$.

λ denotes one-dimensional Lebesgue measure.

If $\Sigma \subseteq \mathbb{R}$ is a finite set and $q \in (0, 1)$, then for any $n \in \mathbb{N}$ we supplement the definition of $E(\Sigma, q)$ with

$$F_n(\Sigma, q) := \left\{ \sum_{i=1}^n \sigma_i q^i : (\sigma_i) \in \Sigma^n \right\}.$$

Theorem (Głab, Kula)

Let $\Sigma \subseteq \mathbb{R}$ be a finite set and $q = \frac{1}{|\Sigma|}$. Denote $E = E(\Sigma, q)$, $F_n = F_n(\Sigma, q)$ and let μ_∞ be the Borel measure associated with $E(\Sigma, q)$. Then the following are equivalent:

- (I) E has positive Lebesgue measure.
- (II) E contains an interval.
- (III) There exists a countable E -tiling $\mathcal{T}$ of $\mathbb{R}$ and $n \in \mathbb{N}$ such that dq^{-n} is a period of $\mathcal{T}$ for all $d \in \Sigma - \Sigma$.
- (IV) Elements of $\Sigma - \Sigma$ are commensurable and $|F_n| = |\Sigma|^n$ for all $n \in \mathbb{N}$.
- (V) μ_∞ is absolutely continuous with respect to Lebesgue measure.

More equivalent conditions (cont.)

(VI) *There exists $\delta > 0$ such that $f_\delta \# \mu_\infty$ is Lebesgue measure on $\mathcal{B}[0, 1)$, where $f_\delta(x) = \frac{x}{\delta} - \lfloor \frac{x}{\delta} \rfloor$ for $x \in \mathbb{R}$.*

(VII) *There exists $\delta > 0$ such that for all $n \in \mathbb{N}$ we have*

$$\int_{\mathbb{R}} \exp\left(\frac{2\pi i n s}{\delta}\right) d\mu_\infty(s) = 0.$$

(VIII) *There exists $\delta > 0$ such that*

$$(1) \quad \forall n \in \mathbb{N} \exists k \in \mathbb{N} \sum_{\sigma \in \Sigma} \exp\left(\frac{2\pi i n \sigma}{\delta |\Sigma|^k}\right) = 0.$$

Moreover, if the above conditions are fulfilled, then

$$(2) \quad \mu_\infty(B) = \frac{\lambda(B \cap E)}{\lambda(E)} \text{ for all } B \in \mathcal{B}(\mathbb{R}).$$

(IV) $\Rightarrow$ (V) Let $\delta > 0$ be a common divisor of $\Sigma - \Sigma$. Since $|F_n| = |\Sigma|^n$, we have

$$\mu_n(B) = q^n |B \cap F_n|$$

for each $n \in \mathbb{N}$ and Borel set $B \in \mathcal{B}(\mathbb{R})$. Furthermore, $F_n - F_n \subseteq (\delta q^n)\mathbb{Z}$. Hence for any interval $I \subseteq \mathbb{R}$ we have

$$\mu_n(I) = q^n |I \cap F_n| \leq q^n k < \frac{\lambda(I)}{\delta} + q^n,$$

where $k \in \mathbb{N}$ is chosen in such a way that $(k-1)\delta q^n < \lambda(I) \leq k\delta q^n$.

Since μ_n is weakly convergent to μ_∞ , by Portmanteau Theorem we get

$$\mu_\infty(I) \leq \liminf_{n \rightarrow \infty} \mu_n(I) \leq \frac{\lambda(I)}{\delta}$$

for any open subset $I \subseteq \mathbb{R}$. Hence for any Borel set $B \in \mathcal{B}(\mathbb{R})$ we have

$$\begin{aligned} \lambda(B) &= \inf \left\{ \sum_{k \in \mathbb{N}} \lambda(I_k) : B \subseteq \bigcup_{k \in \mathbb{N}} I_k \right\} \geq \\ &\geq \inf \left\{ \sum_{k \in \mathbb{N}} \delta \mu_\infty(I_k) : B \subseteq \bigcup_{k \in \mathbb{N}} I_k \right\} \geq \delta \mu_\infty(B). \end{aligned}$$

Fact

Let $\Sigma \subseteq \mathbb{R}$ be a finite set and $q \in (0, 1)$. Then for any $x \in \mathbb{R}$, $E(\Sigma + x, q) = E(\Sigma, q) + \frac{x}{1-q}$. Moreover, if μ_∞ is a measure associated with $E(\Sigma, q)$ and μ_∞^x is associated with $E(\Sigma + x, q)$, then for all $B \in \mathcal{B}(\mathbb{R})$ we have

$$\mu_\infty^x(B) = \mu_\infty\left(B - \frac{x}{1-q}\right).$$

Corollary

Assume that for some $x \in \mathbb{R}$ and $\delta > 0$ the set $(\Sigma - x)/\delta$ consists of integers that give all remainders modulo $|\Sigma|$. Then $B = \delta\mathbb{Z}$ is the basis for the E -tiling of $\mathbb{R}$, δ is the greatest common divisor of $\Sigma - \Sigma$ and

$$\lambda \left(E \left(\Sigma, \frac{1}{|\Sigma|} \right) \right) = \delta.$$

When $|\Sigma|$ is prime

Corollary

Let $\Sigma \subseteq \mathbb{R}$ be a finite set such that $|\Sigma| = p$ is prime. Then $E(\Sigma, \frac{1}{p})$ contains an interval if and only if there exists $x \in \mathbb{R}$ and $\delta > 0$ such that elements of $(\Sigma - x)/\delta$ are integers that give all remainder modulo p .

Idea for proof.

$(\Rightarrow)$ Assume that $E(\Sigma, \frac{1}{p})$ contains an interval. Then, by our main theorem, there exists $\delta > 0$ that is the greatest common divisor of $\Sigma - \Sigma$ and

$$\forall n \in \mathbb{N} \exists k \in \mathbb{N} \sum_{\sigma \in \Sigma} \exp\left(\frac{2\pi i n \sigma}{\delta p^k}\right) = 0.$$

Put $\Sigma^* = (\Sigma - \min \Sigma)/\delta \subseteq \mathbb{Z}$ and take $n = 1$.



When $|\Sigma|$ is prime (cont.)

Proof cont.

Then there is $k \in \mathbb{N}$ such that

$$\sum_{s \in \Sigma^*} \exp\left(\frac{2\pi i s}{p^k}\right) = 0.$$

It follows from the properties of cyclotomic polynomials (details if time permits) that members of

$$\left\{ \exp\left(\frac{2\pi i s}{p^k}\right) : s \in \Sigma^* \right\}$$

are vertices of a regular polygon, i.e.

$$\Sigma^* \bmod p = p^{k-1} \{0, 1, \dots, p-1\},$$

and since elements of Σ^* are relatively prime, we get $k = 1$.

($\Leftarrow$) By previous corollary.



-  T. Banach, A. Bartoszewicz, M. Filipczak, E. Szymonik. *Topological and measure properties of some self-similar sets*. Topol. Methods Nonlinear Anal. 46 no. 2 (2015), pp. 1013–1028.
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