

$\mathfrak{c}$ -lineability and $\mathfrak{c}$ -spaceability of some families of real functions

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My talk is devoted to the recent trend in mathematical analysis of the search for algebraic structures, such as linear spaces or algebras with large dimension, in "small" sets of functions. Given a certain property we say that the subset M of a vector space X which satisfies this property is κ -lineable if $M \cup \{0\}$ contains a vector space of dimension κ , where κ is a cardinal number. If X is a topological vector space then we say that $M \subset X$ is κ -spaceable if $M \cup \{0\}$ contains a **closed** vector space of dimension κ . Obviously, if M is κ -spaceable then it is κ -lineable. If M contains an infinite-dimensional (closed) vector space, it shall be simply called lineable (respectively, spaceable). These notions of lineability and spaceability were created by Gurariy and first introduced in [3, 30]. Recently many examples of vector spaces of functions enjoying certain special properties have been constructed. We refer the interested reader to [1, 2, 4, 5, 6, 7, 8, 10, 11, 14, 15, 27, 28] for a wider range of results in lineability and spaceability of some families of functions.

I am to talk about $\mathfrak{c}$ -lineability and $\mathfrak{c}$ -spaceability of some families $\mathcal{F}$ of functions defined on an interval I with values in $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, i.e. $\mathcal{F} \subset \mathbb{K}^I$. We always consider supremum metric $\varrho(f, g) = \min\{1, \sup\{|f(x) - g(x)| : x \in I\}\}$ in $\mathcal{F}$ (convergence in this metric is uniform convergence). If $\mathcal{F}$ is a linear space of some functions defined on an interval I and $A \subset \mathcal{F}$ then $\text{span}(A)$ is a linear space generated by A and $\text{cl}(A)$ is a closure of A in a supremum metric.

We formulate general conditions under which non-empty family $\mathcal{F} \subset \mathbb{K}^I$ of functions is $\mathfrak{c}$ -spaceable or $\mathfrak{c}$ -lineable. We consider the families of function of the form $\mathcal{F} = \mathcal{F}_1 \setminus \mathcal{F}_2$. In most cases, families of functions for which lineability and spaceability are studied have such a form. The most interesting results are obtained in the case where family $\mathcal{F}_2$ is seemingly "very close" to $\mathcal{F}_1$ or if it consists of "almost all" functions. The presented results are a generalization of the ideas presented in [4, 6].

Fichtenholz-Kantorovich Theorem

Theorem (Fichtenholz-Kantorovich [13])

For any set X there exists a family $\mathcal{B} \subset \mathcal{P}(X)$ of cardinality $2^{\overline{\overline{X}}}$ such that for any finite sequences $B_1, \dots, B_n \in \mathcal{B}$ and $\varepsilon_1, \dots, \varepsilon_n \in \{0, 1\}$,

$$B_1^{\varepsilon_1} \cap \dots \cap B_n^{\varepsilon_n} \neq \emptyset$$

(where $B^1 = B$ and $B^0 = X \setminus B$). Moreover, all the sets in $\mathcal{B}$ can be chosen with the cardinality $\overline{\overline{X}}$.

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Remark

In the sequel, we always apply Theorem 1 in the case where $X = \mathbb{N}$. By mentioned theorem there exists $\mathcal{B} \subset \mathcal{P}(\mathbb{N})$ of cardinality $\mathfrak{c}$ such that $B_0^{\varepsilon_0} \cap B_1^{\varepsilon_1} \cap \dots \cap B_n^{\varepsilon_n} \neq \emptyset$ for any finite sequences $B_0, B_1, \dots, B_n \in \mathcal{B}$ and $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n \in \{0, 1\}$. In particular, $B_0 \setminus (B_1 \cup B_2 \cup \dots \cup B_n) \neq \emptyset$. By $\mathcal{B}$ we always denotes such a family.

Definition

Let $I \in \mathbf{P}$, $J = [a, b] \in \mathbf{P}^c$ and $f: J \rightarrow \mathbb{K}$ be non-zero, bounded and $f(a) = f(b)$. Take any $B \subset \mathbb{N}$ and a sequence $([a_n, b_n])_{n \geq 1}$ such that $\inf I < a_1 < \dots < b_n < a_{n+1} < b_{n+1} < \dots < \sup I$ and $\lim_{n \rightarrow \infty} b_n = \sup I$. We define $\psi_{B,f}^I: I \rightarrow \mathbb{K}$ by

$$\psi_{B,f}^I(x) = \begin{cases} \frac{1}{n^2} [(f \circ \varphi_n)(x) - f(a)] & \text{for } x \in [a_n, b_n], n \in B, \\ 0 & \text{for } x \in I \setminus \bigcup_{n \in B} [a_n, b_n], \end{cases}$$

where $\varphi_n: [a_n, b_n] \rightarrow J$ is an increasing homeomorphism for $n \geq 1$.

Construction of Ψ_f^I

Definition

Let $I \in \mathbf{P}$, $J = [a, b] \in \mathbf{P}^c$ and $f: J \rightarrow \mathbb{K}$ be non-zero, bounded and $f(a) = f(b)$. Take any $B \subset \mathbb{N}$ and a sequence $([a_n, b_n])_{n \geq 1}$ such that $\inf I < a_1 < \dots < b_n < a_{n+1} < b_{n+1} < \dots < \sup I$ and $\lim_{n \rightarrow \infty} b_n = \sup I$. We define $\psi_{B,f}^I: I \rightarrow \mathbb{K}$ by

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Definition

Let $I \in \mathbf{P}$, $J = [a, b] \in \mathbf{P}^c$ and $f: J \rightarrow \mathbb{K}$ be non-zero, bounded and $f(a) = f(b)$. Take a sequence $([a_n, b_n])_{n \geq 1}$ such that $\inf I < a_1 < \dots < b_n < a_{n+1} < b_{n+1} < \dots < \sup I$ and $\lim_{n \rightarrow \infty} b_n = \sup I$. By $\Psi_f^I \subset \mathbb{K}^I$ we denote a set

$$\Psi_f^I = \left\{ \psi_{B,f}^I : B \in \mathcal{B} \right\}.$$

Remark

Actually, $\psi_{B,f}^I$ depends on the choice of a sequence $([a_n, b_n])_{n \geq 1}$ and Ψ_f^I depends on the choice of $\mathcal{B}$ and $([a_n, b_n])_{n \geq 1}$. But, as we will show, their properties depend neither on the choice of a sequence $([a_n, b_n])_{n \geq 1}$ nor a set $\mathcal{B}$. That's why we write $\psi_{B,f}^I$ and Ψ_f^I instead of $\psi_{B,f,([a_n, b_n])_{n \geq 1}}^I$ and $\Psi_{f,([a_n, b_n])_{n \geq 1}, \mathcal{B}}^I$.

Theorem

Let $I \in \mathbf{P}$, $J = [a, b] \in \mathbf{P}^c$ and $f: J \rightarrow \mathbb{K}$ be non-zero, bounded and $f(a) = f(b)$. Then

- 1 $\overline{\overline{\Psi_f^I}} = \mathfrak{c}$ and Ψ_f^I is linearly independent.
- 2 If f is continuous then $\text{cl}(\text{span}(\Psi_f^I))$ consists of continuous functions.
- 3 If f is Baire one function then $\text{cl}(\text{span}(\Psi_f^I))$ consists of Baire one functions.

Theorem

Let $I \in \mathbf{P}$, $J = [a, b] \in \mathbf{P}^c$ and $f: J \rightarrow \mathbb{K}$ be non-zero, bounded and $f(a) = f(b)$. Then

- ① If $h \in \text{span}(\Psi_f^I) \setminus \{0\}$ then there exists a sequence $(c_n)_{n \geq 1} \subset \mathbb{K}$ such that $\sup\{|c_n|: n \geq 1\} < \infty$ and

$$h(x) = \begin{cases} \frac{c_n}{n^2} [(f \circ \varphi_n)(x) - f(a)] & \text{for } x \in [a_n, b_n], n \geq 1, \\ 0 & \text{for } x \in I \setminus \bigcup_{n \geq 1} [a_n, b_n]. \end{cases}$$

In particular, $\lim_{x \rightarrow \sup I} h(x) = 0$.

- ② If $h_0 \in \text{cl}(\text{span}(\Psi_f^I)) \setminus \{0\}$ then there exists a sequence $(d_n)_{n \geq 1} \subset \mathbb{K}$ such that $\lim_{n \rightarrow \infty} d_n = 0$ and

$$h_0(x) = \begin{cases} d_n [(f \circ \varphi_n)(x) - f(a)] & \text{for } x \in [a_n, b_n], n \geq 1, \\ 0 & \text{for } x \in I \setminus \bigcup_{n \geq 1} [a_n, b_n]. \end{cases}$$

Definition

We say that a family $\{\mathcal{F}(I): \mathcal{F}(I) \subset \mathbb{R}^I \text{ for } I \in \mathbf{P}\}$ is compatible if for every $I_1, I_2 \in \mathbf{P}$, for every increasing linear homeomorphism $\varphi: I_2 \rightarrow I_1$, for every $a, b \in \mathbb{R}$ and for every $f: I_1 \rightarrow \mathbb{R}$ if $f \in \mathcal{F}(I_1)$ then $a(f \circ \varphi) + b \in \mathcal{F}(I_2)$.

Remark

All families of functions considered by the authors are compatible.

Compatibility and localizability

Definition

We say that a family $\{\mathcal{F}(I): \mathcal{F}(I) \subset \mathbb{R}^I \text{ for } I \in \mathbf{P}\}$ is compatible if for every $I_1, I_2 \in \mathbf{P}$, for every increasing linear homeomorphism $\varphi: I_2 \rightarrow I_1$, for every $a, b \in \mathbb{R}$ and for every $f: I_1 \rightarrow \mathbb{R}$ if $f \in \mathcal{F}(I_1)$ then $a(f \circ \varphi) + b \in \mathcal{F}(I_2)$.

Remark

All families of functions considered by the authors are compatible.

Definition

We say that $\{\mathcal{F}(I): I \in \mathbf{P}\}$ is localizable if for every $I \in \mathbf{P}$ and for every $f: I \rightarrow \mathbb{R}$ if there exists $J \in \mathbf{P}$, $J \subset \text{int } I$ such that $f|_J \notin \mathcal{F}(J)$ then $f \notin \mathcal{F}(I)$.

Theorem

Let a family $\{\mathcal{F}(I) : I \in \mathbf{P}\}$ be compatible and localizable. Let there exist $J \in \mathbf{P}^c$ with endpoints a, b and non-zero bounded $f \notin \mathcal{F}(J)$ such that $f(a) = f(b)$. Then

$$\mathcal{F}(I) \cap \left(\text{cl}(\text{span}(\Psi_f^I)) \setminus \{0\} \right) = \emptyset.$$

Now, we show three conditions under which if $f \in \mathcal{F}(I)$ then $\text{cl}(\text{span}(\Phi_f^I)) \subset \mathcal{F}(I)$ or $\text{span}(\Phi_f^I) \subset \mathcal{F}(I)$.

First condition

Definition

Let I be any interval. We say that $\{\mathcal{F}(I): I \in \mathbf{P}\}$ is defined locally continuously if for every $x \in \mathbb{K}$ there is property Φ_x such that:

- 1 $\mathcal{F}(I) = \{f: I \rightarrow \mathbb{K}: f \text{ satisfy } \Phi_x \text{ for } x \in I\}$ for every $I \in \mathbf{P}$;
- 2 for every $I \in \mathbf{P}$, $x_0 \in I$, $f, g: I \rightarrow \mathbb{K}$ and for every $\varepsilon > 0$ if $f|_{(x_0-\varepsilon, x_0+\varepsilon) \cap I} = g|_{(x_0-\varepsilon, x_0+\varepsilon) \cap I}$ then f satisfies Φ_{x_0} if and only if g satisfies Φ_{x_0} ;
- 3 for every $I \in \mathbf{P}$ if $f: I \rightarrow \mathbb{R}$ is continuous at some $x_0 \in I$ then f satisfies Φ_{x_0} .

Theorem

Let a family $\{\mathcal{F}(I): I \in \mathbf{P}\}$ be compatible. Moreover, let there exist $J \in \mathbf{P}^c$ with endpoints a, b and non-zero bounded $f \in \mathcal{F}(J)$ such that $f(a) = f(b)$ and f is continuous at a and b . If $\{\mathcal{F}(I): I \in \mathbf{P}\}$ is defined locally continuously then $\text{cl}(\text{span}(\Psi_f^I)) \subset \mathcal{F}(I)$ for every $I \in \mathbf{P}$.



Definition

We say that $\{\mathcal{F}(I): I \in \mathbf{P}\}$ is countably glueable if every $\mathcal{F}(I)$ contains constant functions and the following condition holds:

- for every $I \in \mathbf{P}$, for every $f: I \rightarrow \mathbb{K}$, for every countable family of closed intervals $\{[a_n, b_n]: [a_n, b_n] \subset I \text{ and } n \in A\}$ such that $\text{int } I \subset \bigcup_{n \in A} [a_n, b_n]$ and $[a_n, b_n] \cap [a_k, b_k]$ contains at most one point for every $n \neq k$, if $f|_{[a_n, b_n]} \in \mathcal{F}([a_n, b_n])$ for $n \in A$ and f is continuous at $x \in I \setminus \text{int}(I)$ then $f \in \mathcal{F}(I)$.

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- for every $I \in \mathbf{P}$, for every $f: I \rightarrow \mathbb{K}$, for every countable family of closed intervals $\{[a_n, b_n]: [a_n, b_n] \subset I \text{ and } n \in A\}$ such that $\text{int } I \subset \bigcup_{n \in A} [a_n, b_n]$ and $[a_n, b_n] \cap [a_k, b_k]$ contains at most one point for every $n \neq k$, if $f|_{[a_n, b_n]} \in \mathcal{F}([a_n, b_n])$ for $n \in A$ and f is continuous at $x \in I \setminus \text{int}(I)$ then $f \in \mathcal{F}(I)$.

Theorem

Let a family $\{\mathcal{F}(I): I \in \mathbf{P}\}$ be compatible. Let there exist $J \in \mathbf{P}^c$ with endpoints a, b and non-zero bounded $f \in \mathcal{F}(J)$ such that $f(a) = f(b)$. If $\{\mathcal{F}(I): I \in \mathbf{P}\}$ is countably glueable then $\text{cl}(\text{span}(\Psi_f^I)) \subset \mathcal{F}(I)$ for every $I \in \mathbf{P}$.

Theorem

Let families $\{\mathcal{F}_1(I): I \in \mathbf{P}\}$ and $\{\mathcal{F}_2(I): I \in \mathbf{P}\}$ be compatible, $\{\mathcal{F}_2(I): I \in \mathbf{P}\}$ be localizable. Moreover, assume that there exist $J \in \mathbf{P}^c$ with endpoints a, b and non-zero bounded $f \in \mathcal{F}_1(J) \setminus \mathcal{F}_2(J)$ such that $f(a) = f(b)$ and f is continuous at a and b . If $\{\mathcal{F}_1(I): I \in \mathbf{P}\}$ is defined locally continuously then $\mathcal{F}_1(I) \setminus \mathcal{F}_2(I)$ is $\mathfrak{c}$ -spaceable for every $I \in \mathbf{P}$.

Main theorems

Theorem

Let families $\{\mathcal{F}_1(I): I \in \mathbf{P}\}$ and $\{\mathcal{F}_2(I): I \in \mathbf{P}\}$ be compatible, $\{\mathcal{F}_2(I): I \in \mathbf{P}\}$ be localizable. Moreover, assume that there exist $J \in \mathbf{P}^c$ with endpoints a, b and non-zero bounded $f \in \mathcal{F}_1(J) \setminus \mathcal{F}_2(J)$ such that $f(a) = f(b)$ and f is continuous at a and b . If $\{\mathcal{F}_1(I): I \in \mathbf{P}\}$ is defined locally continuously then $\mathcal{F}_1(I) \setminus \mathcal{F}_2(I)$ is $\mathfrak{c}$ -spaceable for every $I \in \mathbf{P}$.

Theorem

Let families $\{\mathcal{F}_1(I): I \in \mathbf{P}\}$ and $\{\mathcal{F}_2(I): I \in \mathbf{P}\}$ be compatible, $\{\mathcal{F}_2(I): I \in \mathbf{P}\}$ be localizable. Moreover, assume that there exist $J \in \mathbf{P}^c$ with endpoints a, b and non-zero bounded $f \in \mathcal{F}_1(J) \setminus \mathcal{F}_2(J)$ such that $f(a) = f(b)$. If $\{\mathcal{F}_1(I): I \in \mathbf{P}\}$ is countably glueable then $\mathcal{F}_1(I) \setminus \mathcal{F}_2(I)$ is $\mathfrak{c}$ -spaceable for every $I \in \mathbf{P}$.

Definition

We say that $\{\mathcal{F}(I): I \in \mathbf{P}^c\}$ is $\mathbf{P}^c$ -summable if there exists $\{\Gamma_I: \mathbb{R}^I \rightarrow [0, \infty]: I \in \mathbf{P}^c\}$ such that:

- ❶ for every $I \in \mathbf{P}^c$, for every $f: I \rightarrow \mathbb{R}$, $f \in \mathcal{F}(I)$ if and only if $\Gamma_I(f) < \infty$;
- ❷ $\Gamma_I(\mathbb{O}_I) = 0$, where $\mathbb{O}_I(x) = 0$ for every $x \in I$;
- ❸ for every $I = [a, b] \in \mathbf{P}^c$, for every $f: I \rightarrow \mathbb{R}$, for every countable family of closed intervals $\{[a_n, b_n]: [a_n, b_n] \subset I \text{ and } n \in A\}$ such that $\text{int } I \subset \bigcup_{n \in A} [a_n, b_n]$, if f is continuous at a and b then $\Gamma_I(f) \leq \sum_{n \in A} \Gamma_{[a_n, b_n]}(f|_{[a_n, b_n]})$;
- ❹ for every $I_1, I_2 \in \mathbf{P}^c$, for every increasing homeomorphism $\varphi: I_2 \rightarrow I_1$, $\varphi(x) = \alpha x + \beta$ and for every $a \geq 0, b \in \mathbb{R}$ an inequality $\Gamma_{I_2}(a(f \circ \varphi) + b) \leq |a| \max\{\frac{1}{\alpha}, 1\} \Gamma_{I_1}(f)$ is fulfilled for every $f: I_1 \rightarrow \mathbb{R}$.

Theorem

Let a family $\{\mathcal{F}(I): I \in \mathbf{P}\}$ be compatible. Let there exist $J \in \mathbf{P}^c$ with endpoints a, b and non-zero bounded $f \in \mathcal{F}(J)$ such that $f(a) = f(b)$. If $\{\mathcal{F}(I): I \in \mathbf{P}\}$ is $\mathbf{P}^c$ -summable then $\text{span}(\Psi_f^I) \subset \mathcal{F}(I)$ for every $I \in \mathbf{P}^c$.

Theorem

Let a family $\{\mathcal{F}(I): I \in \mathbf{P}\}$ be compatible. Let there exist $J \in \mathbf{P}^c$ with endpoints a, b and non-zero bounded $f \in \mathcal{F}(J)$ such that $f(a) = f(b)$. If $\{\mathcal{F}(I): I \in \mathbf{P}\}$ is $\mathbf{P}^c$ -summable then $\text{span}(\Psi_f^I) \subset \mathcal{F}(I)$ for every $I \in \mathbf{P}^c$.

Theorem

Let families $\{\mathcal{F}_1(I): I \in \mathbf{P}\}$ and $\{\mathcal{F}_2(I): I \in \mathbf{P}\}$ be compatible, $\{\mathcal{F}_2(I): I \in \mathbf{P}\}$ be localizable. Assume that there exist $J \in \mathbf{P}^c$ with endpoints a, b and non-zero bounded $f \in \mathcal{F}_1(J) \setminus \mathcal{F}_2(J)$ such that $f(a) = f(b)$. If $\{\mathcal{F}_1(I): I \in \mathbf{P}\}$ is $\mathbf{P}^c$ -summable then $\mathcal{F}_1(I) \setminus \mathcal{F}_2(I)$ is $\mathfrak{c}$ -lineable for every $I \in \mathbf{P}^c$.

Classical Banach-Zaretsky Theorem states that a function $f: [a, b] \rightarrow \mathbb{R}$ is absolutely continuous iff it is continuous, has the Luzin property (N) and is of bounded variation. For every $I \in \mathbf{P}^c$, by $\mathcal{C}(I)$, $\mathcal{AC}(I)$ and $\mathcal{L}(I)$ we denote all functions $f: I \rightarrow \mathbb{R}$ which are continuous, absolutely continuous and has the Luzin property (N), respectively. It is well-known that $\mathcal{BV}(I) \cap \mathcal{C}(I) \setminus \mathcal{AC}(I) = \mathcal{BV}(I) \cap \mathcal{C}(I) \setminus \mathcal{L}(I) \neq \emptyset$, $\mathcal{BV}(I) \cap \mathcal{L}(I) \setminus \mathcal{AC}(I) = \mathcal{BV}(I) \cap \mathcal{L}(I) \setminus \mathcal{C}(I) \neq \emptyset$ and $\mathcal{L}(I) \cap \mathcal{C}(I) \setminus \mathcal{AC}(I) = \mathcal{L}(I) \cap \mathcal{C}(I) \setminus \mathcal{BV} \neq \emptyset$. Applying obtained results, we will easily prove that the first and the second sets are $\mathfrak{c}$ -lineable and the last is $\mathfrak{c}$ -spaceable.

Linearly sensitive functions

In [4] A. Bartoszewicz, M. Filipczak and M. Terepeta considered algebraic properties of linearly sensitive functions.

Definition

[4] Let (P) be a property of functions in $\mathbb{R}^{[0,1]}$. We say that $f: [0,1] \rightarrow \mathbb{R}$ is linearly sensitive with respect to (P) if f has the property (P) and for any real $a \neq 0$ the function $f + a \cdot \text{id}$ does not have the property (P) .

In the mentioned paper the authors considered

- continuous functions linearly sensitive with respect to the Luzin property (N),
- functions linearly sensitive with respect to the Świątkowski condition,
- functions linearly sensitive with respect to the strong Świątkowski condition.

They proved that all these classes are $\mathfrak{c}$ -lineable. Applying obtained results, we strengthen first two results.

Linearly sensitive functions

Theorem

For every $I \in \mathbf{P}$ the family of all continuous functions $g: I \rightarrow \mathbb{R}$ linearly sensitive to the Luzin property (N) is $\mathfrak{c}$ -spaceable.

Theorem

The family of all Baire one functions linearly sensitive to strong Świątkowski condition is $\mathfrak{c}$ -spaceable.

Definition

[24] We say that f satisfies the strong Świątkowski condition if for all $x_1 < x_2$ with $f(x_1) \neq f(x_2)$ and each y between $f(x_1)$ and $f(x_2)$, there is a point $x \in (x_1, x_2)$ such that $f(x) = y$ and f is continuous at x .

In [9] J. Borsík and J. Holos defined families of porouscontinuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$. Applying their ideas it is easy to transfer this concept for real functions defined on any interval I . There are three kinds of porouscontinuous functions: $\mathcal{N}_r$ for $r \in (0, 1]$, $\mathcal{S}_r$ for $r \in [0, 1)$ and $\mathcal{P}_r$ for $r \in [0, 1)$. The main result of [9] summarizes all inclusions between families of porouscontinuous functions in the following formula:

$$\mathcal{C}(I) \subsetneq \mathcal{M}_1(I) \subsetneq \mathcal{P}_{r_2}(I) \subsetneq \mathcal{S}_{r_2}(I) \subsetneq \mathcal{M}_{r_2}(I) \subsetneq \mathcal{P}_{r_1}(I) \subsetneq \mathcal{S}_{r_1}(I) \subsetneq \mathcal{M}_{r_1}(I) \subsetneq \mathcal{P}_0(I) \subsetneq \mathcal{S}_0(I) \subsetneq \mathcal{Q}(I),$$

for every $0 < r_1 < r_2 < 1$.

Theorem

$\mathcal{M}_1(I) \setminus \mathcal{C}(I)$ is $\mathfrak{c}$ -spaceable for every $I \in \mathbf{P}$.

Theorem

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Theorem

$\mathcal{P}_{r_1}(I) \setminus \mathcal{M}_{r_2}(I)$ is $\mathfrak{c}$ -spaceable for $0 \leq r_1 < r_2 \leq 1$ and $I \in \mathbf{P}$.

Theorem

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$\mathcal{P}_{r_1}(I) \setminus \mathcal{M}_{r_2}(I)$ is $\mathfrak{c}$ -spaceable for $0 \leq r_1 < r_2 \leq 1$ and $I \in \mathbf{P}$.

Theorem

$\mathcal{S}_r(I) \setminus \mathcal{P}_r(I)$ is $\mathfrak{c}$ -spaceable for every $r \in [0, 1)$ and $I \in \mathbf{P}$.

Theorem

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$\mathcal{P}_{r_1}(I) \setminus \mathcal{M}_{r_2}(I)$ is $\mathfrak{c}$ -spaceable for $0 \leq r_1 < r_2 \leq 1$ and $I \in \mathbf{P}$.

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Theorem

$\mathcal{Q}(I) \setminus \mathcal{S}_0(I)$ is $\mathfrak{c}$ -spaceable for every $I \in \mathbf{P}$.

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