

Cardinal characteristics of selected ideals on countable sets

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Ministerstwo Nauki
i Szkolnictwa Wyższego

Definition

A family $\mathcal{I} \subseteq \mathcal{P}(X)$ is an **ideal** on X if

- ① $X \notin \mathcal{I}$,
- ② $A \subseteq B \in \mathcal{I} \implies A \in \mathcal{I}$,
- ③ $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$,
- ④ $\mathcal{I}$ contains all finite subsets of X .

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An ideal $\mathcal{I}$ is **tall** if for every infinite $A \subseteq X$ there is an infinite $B \in \mathcal{I}$ such that $B \subseteq A$.

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Examples

① $\text{Fin}(X) = \{A \subseteq X : A \text{ is finite}\}$

② $\text{Fin} = \text{Fin}(\omega)$

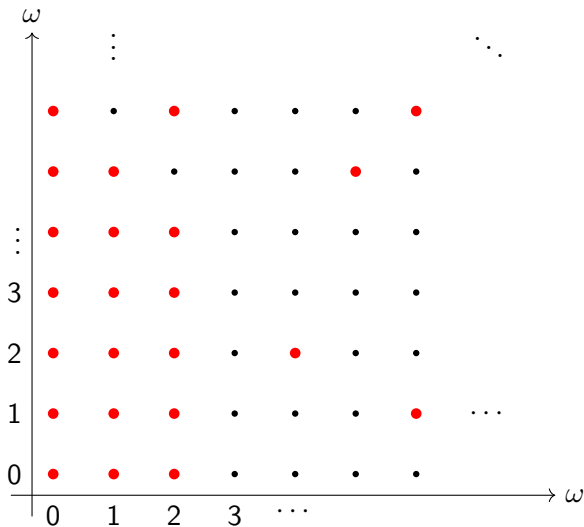
③ $\mathcal{I}_{1/n} = \left\{ A \subseteq \omega : \sum_{n \in A} \frac{1}{n} < \infty \right\}$

④ $\mathcal{I}_d = \left\{ A \subseteq \omega : \lim_{n \rightarrow \infty} \frac{|A \cap \{1, \dots, n\}|}{n} = 0 \right\}$

⑤ $\text{Fin} \otimes \text{Fin}$ is the ideal on $\omega \times \omega$ defined by

$$A \in \text{Fin} \otimes \text{Fin} \iff \exists i_0 \in \omega \forall i \geq i_0 (|\{j \in \omega : (i, j) \in A\}| < \aleph_0).$$

$$\mathcal{A} \in \text{Fin} \otimes \text{Fin} \iff \exists i_0 \in \omega \forall i \geq i_0 (|\{j \in \omega : (i, j) \in \mathcal{A}\}| < \aleph_0)$$



Definition (R. Filipów, MK, A. Kwela, 2026)

Let X be a sequentially compact space. Let $D \subseteq X$ be a countable infinite subset of X . We define

$$\text{conv}(D) = \{A \subseteq D : A^d \text{ is finite}\}.$$

where $A^d = \{p \in X : p \in \text{cl}(A \setminus \{p\})\}$

Definition

- ① If $X = [0, 1]$ with the Euclidean topology, we define the ideal

$$\text{conv} = \text{conv}(\mathbb{Q} \cap [0, 1]).$$

- ② For a countable ordinal $\alpha \geq 2$ and the space $X = \omega^\alpha + 1$ with the order topology, we define the ideal

$$\text{conv}_\alpha = \text{conv}(\omega^\alpha + 1).$$

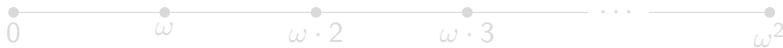
Combinatorial characterization

Proposition (R. Filipów, MK, A. Kwela, 2026)

Let $A \subseteq \omega^\alpha + 1$. T.F.A.E.

- 1 $A \in \text{conv}_\alpha$.
- 2 For every increasing sequence $(\lambda_n)_{n < \omega}$ in ω^α , the set $A \cap [\lambda_n, \lambda_{n+1}]$ is finite for all but finitely many n .

$$\alpha = 2, \lambda_n = \omega \cdot n$$



Corollary (R. Filipów, MK, A. Kwela, 2026)

The ideals conv_2 and $\text{Fin} \otimes \text{Fin}$ are isomorphic.

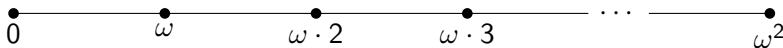
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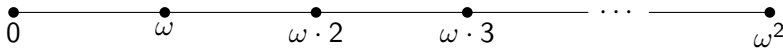
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Katětov order (Katětov, 1968)

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on X and Y respectively.

$\mathcal{J} \leq_K \mathcal{I} \iff$ there exists a function $f : X \rightarrow Y$ such that

$$A \in \mathcal{J} \implies f^{-1}[A] \in \mathcal{I}.$$

$\mathcal{J} \approx \mathcal{I} \iff$ there exists a **bijection** $f : X \rightarrow Y$ such that

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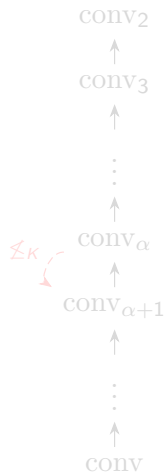
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The $\leq_K$ -hierarchy

$$\text{conv} = \{A \subseteq \mathbb{Q} \cap [0, 1] : A^d \text{ is finite}\}$$
$$\text{conv}_\alpha = \{A \subseteq \omega^\alpha + 1 : A^d \text{ is finite}\}$$

Proposition (R. Filipów, MK, A. Kwela, 2026)

- 1 If $\alpha < \beta$, then $\text{conv}_\beta \leq_K \text{conv}_\alpha$ and $\text{conv}_\alpha \not\leq_K \text{conv}_\beta$.
- 2 $\text{conv} \leq_K \text{conv}_\alpha$ and $\text{conv}_\alpha \not\leq_K \text{conv}$ for every α .

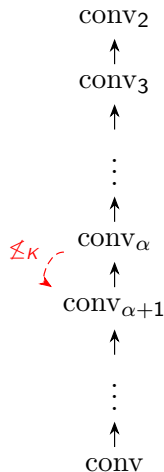


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Proposition

For limit ordinal α , the ideal conv_α is not the greatest lower bound of $\{\text{conv}_\beta : \beta < \alpha\}$.

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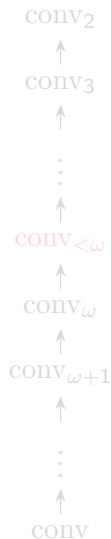
Definition

For every limit ordinal α define:

$$\text{conv}_{<\alpha} = \{A \subseteq \omega^\alpha + 1 : \forall \beta < \alpha (A \cap \omega^\beta \in \text{conv}_\beta)\}$$

Proposition

- 1 The ideal $\text{conv}_{<\alpha}$ is the greatest lower bound of $\{\text{conv}_\beta : \beta < \alpha\}$.
- 2 $\text{conv}_\alpha \leq_K \text{conv}_{<\alpha}$ and $\text{conv}_{<\alpha} \not\leq_K \text{conv}_\alpha$.



The $\leq_K$ -hierarchy

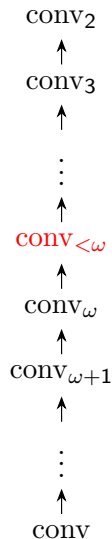
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Question

Is there a greatest lower bound for the ideals conv_α for all α ?

Definition (F. Hernández, M. Hrušák, 2007)

Let $\mathcal{I}$ be a tall ideal.

$$\text{add}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \neg \exists B \in \mathcal{I} \forall A \in \mathcal{A} (A \subseteq^* B)\}$$

$$\text{non}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \wedge \forall B \in \mathcal{I} \exists A \in \mathcal{A} (|A \cap B| < \aleph_0)\}$$

$$\text{cof}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall B \in \mathcal{I} \exists A \in \mathcal{A} (B \subseteq^* A)\}$$

$$\text{cov}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall B \in [\omega]^\omega \exists A \in \mathcal{A} (|A \cap B| = \aleph_0)\}$$

where $A \subseteq^* B$ means that $A \setminus B$ is finite

Cardinal characteristics

Definition (J. Brendle, S. Shelah, 1999)

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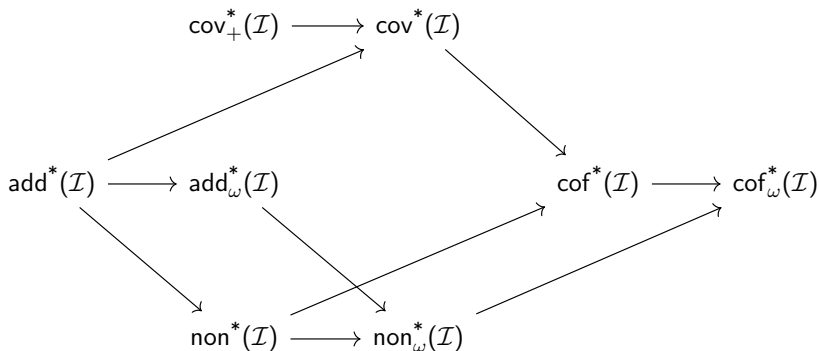
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Cardinal characteristics



Relationships between defined cardinal characteristics ($\kappa \rightarrow \lambda$ means $\kappa \leq \lambda$ in this diagram).

Definition

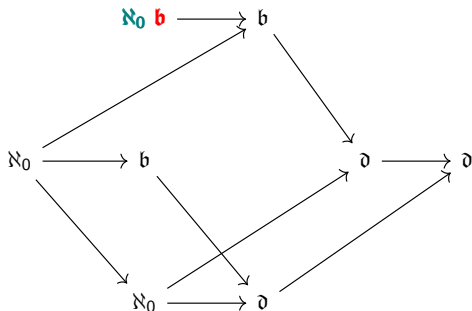
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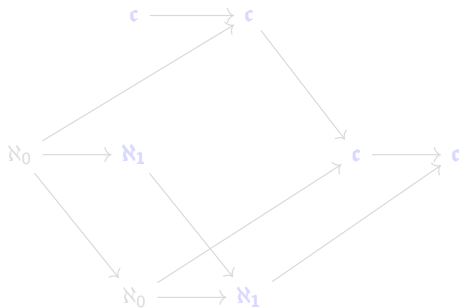
Diagrams

Cieślak and Martínez-Celis, Farkas and Zdomskyy, Filipów and MK and Kwela, Meza-Alcántara



$\text{Fin} \otimes \text{Fin} \approx \text{conv}_2$

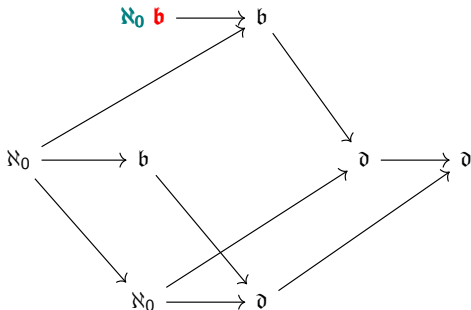
conv_α for $\alpha \geq 3$



conv

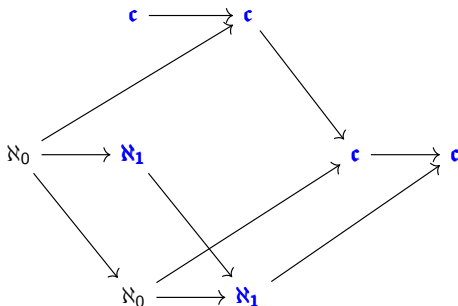
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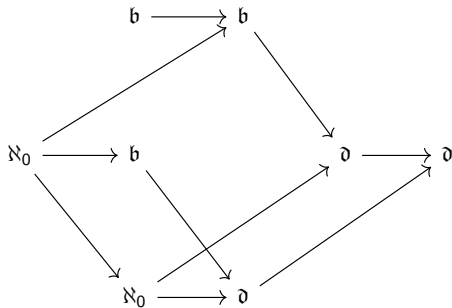
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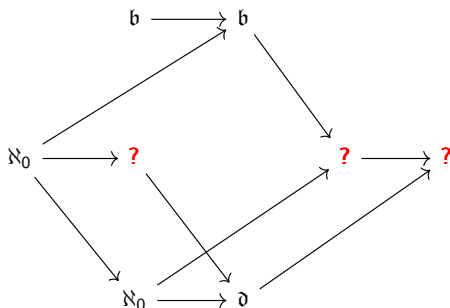


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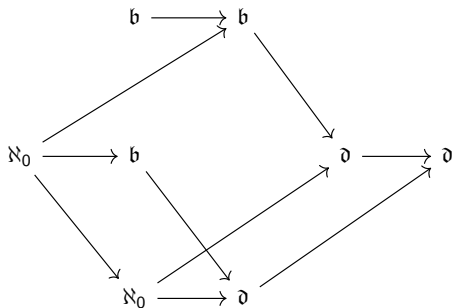


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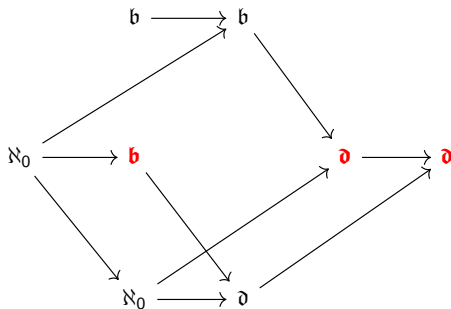
$\text{conv}_{<\alpha}$

Diagrams



conv_α for $\alpha \geq 3$

Conjecture



$\text{conv}_{<\alpha}$

Thank You!

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