

The Fréchet derivative is in the first Baire class

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Fréchet derivative f' is Baire1

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable. Then $f'(t) = \lim_{n \rightarrow \infty} q_n(t)$ where $q_n(t) = \frac{f(t+1/n) - f(t)}{1/n}$ is the difference quotient.

Hence f' is in the *first Baire class*, that is, a pointwise limit of continuous functions.

Suppose that X is a normed linear space, Y is a Banach space, $G \subset X$ is an open set and $f : G \rightarrow Y$ is an arbitrary function. Let D be the set of points $x \in G$ at which the Fréchet derivative $f'(x)$ exists. Then $f' : D \rightarrow L(X, Y)$ is of the first Baire class.

[Kopecká and Zajíček '26]

- ▶ f doesn't have to be continuous and D can be empty.
- ▶ When $D = X = \mathbb{R}^d$ and $Y = \mathbb{R}$ the result follows from the 1-dim case applied to the partial derivatives.
- ▶ Our result generalizes [Matoušková and Stegall '01] where it is assumed that X is a Banach space and f is Fréchet differentiable everywhere on G .

step functions $\varphi_n \Rightarrow f'$

X is a normed linear space, Y is a Banach space, $G \subset X$ is an open set and $f : G \rightarrow Y$ is a mapping, $D \subset G$ is the set of points at which the Fréchet derivative f' exists.

Instead of converging pointwise by continuous *quotients* $q_n \rightarrow f'$ we converge uniformly by Baire1 *step functions* $\varphi_n \Rightarrow f'$

the *step function* φ_n is a constant on the sets of the following

PARTITION Ω_n

For each $n \in \mathbb{N}$ there exists a family Ω_n of nonempty subsets of D with the following properties:

- (a) The sets in Ω_n are pairwise disjoint.
- (b) The union of all of the sets in Ω_n is D .
- (c) The family Ω_n is σ -discrete.
- (d) Each member of Ω_n is an F_σ set in D .
- (e) For each $E \in \Omega_n$ it holds that $\text{osc}(f', E) \leq 1/n$.

Remark: each $E \in \Omega_n$ is contained in a set of the form

$$\{x \in D : |y - x| < \frac{1}{m} \Rightarrow |f(y) - f(x) - f'(x)(y - x)| \leq \frac{1}{12n} |y - x|\}.$$

Theorem

Suppose that X is a normed linear space, Y is a Banach space, $G \subset X$ is an open set and $f : G \rightarrow Y$ is an arbitrary function. Let D be the set of points $x \in G$ at which the Fréchet derivative $f'(x)$ exists. Then $f' : D \rightarrow L(X, Y)$ is of the first Baire class.

Proof.

Ω_n : σ -discrete partition of D , consisting of F_σ sets,
 $\text{osc}(f', E) \leq 1/n$ on each $E \in \Omega_n$.

For $E \in \Omega_n$ fix $x_E \in E$ and define φ_n on E as the constant $f'(x_E)$.

$\text{osc}(f', E) \leq 1/n \Rightarrow \varphi_n \rightrightarrows f'$

$G \subset L(X, Y)$ open $\Rightarrow \varphi_n^{-1}(G) = \bigcup \{E \in \Omega_n : f'(x_E) \in G\}$

σ -discrete family Ω_n consists of F_σ sets

$$\Rightarrow \left\{ \begin{array}{l} \varphi_n^{-1}(G) \text{ is } F_\sigma \Rightarrow \varphi_n \text{ is Baire1} \\ \varphi_n \text{ has a } \sigma\text{-discrete base} \end{array} \right\} \Rightarrow \varphi_n \text{ is Baire1}$$

φ_n is Baire1 and $\varphi_n \rightrightarrows f' \Rightarrow f'$ is Baire1



partition of D into sets on which f' is almost constant

Suppose that X is a normed linear space, Y is a Banach space, $G \subset X$ is an open set and $f : G \rightarrow Y$ is an arbitrary function. Let D be the set of points $x \in G$ at which the Fréchet derivative $f'(x)$ exists. For $n \in \mathbb{N}$ there exists a σ -discrete partition Ω_n of D , consisting of F_σ sets s.t. $\text{osc}(f', E) \leq 1/n$ on each $E \in \Omega_n$.

Ingredients of the construction:

$$A_m = \{x \in D : |y - x| < \frac{1}{m} \Rightarrow |(f(y) - f(x))/|y - x| - f'(x)| \leq \frac{1}{12n}\}$$

$$D = \bigcup_{m \in \mathbb{N}} A_m$$

$$U \subset X \text{ open with } \text{diam } U < \frac{1}{m} \Rightarrow \text{osc}(f', D \cap U \cap \overline{A_m}) \leq \frac{1}{n}.$$

We can choose for each $m \in \mathbb{N}$ a σ -discrete cover $\mathcal{B}^m$ of X by open sets of diameters less than $\frac{1}{m}$ and write $\mathcal{B}^m = \bigcup_{i=1}^{\infty} \mathcal{B}_i^m$ where each $\mathcal{B}_i^m$ is a discrete family. Set

$$\Omega_i^m = \{B \cap D \cap \overline{A_m} : B \in \mathcal{B}_i^m\} \quad m, i \in \mathbb{N} \text{ and } \Omega^* := \bigcup_{m, i \in \mathbb{N}} \Omega_i^m.$$

Ω^* has all the properties of Ω , except for its sets being pairwise disjoint.

Definition: A family of sets in a topological space is said to be *discrete* if each point of the space has a neighbourhood that meets at most one set of the family.