

Sequence spaces of Baire-one functions and their porosity

Marek Balcerzak^{*} && Gertruda Ivanova^{**}

^{*}Institute of Mathematics,
Łódź University of Technology, Poland

^{**}Faculty of Mathematics and Computer Science
University of Łódź, Poland

40th International Summer Conference on Real Functions Theory

(X, ϱ) – compact metric space

(X, ϱ) – compact metric space

$B_1(X, X)$ – family of Baire class-one self-maps on X

(X, ϱ) – compact metric space

$B_1(X, X)$ – family of Baire class-one self-maps on X

A non-autonomous B_1 -dynamical system, or B_1 -NDS, - sequence of $B_1(X, X)$.

A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

(X, ϱ) – compact metric space

$B_1(X, X)$ – family of Baire class-one self-maps on X

A non-autonomous B_1 -dynamical system, or B_1 -NDS, - sequence of $B_1(X, X)$.

A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

$\mathcal{F}$ – a family of all such systems

(X, ϱ) – compact metric space

$B_1(X, X)$ – family of Baire class-one self-maps on X

A non-autonomous B_1 -dynamical system, or B_1 -NDS, - sequence of $B_1(X, X)$.

A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

$\mathcal{F}$ – a family of all such systems

$F = (f_n)_{n \in \mathbb{N}} \in \mathcal{F}$ and $G = (g_n)_{n \in \mathbb{N}} \in \mathcal{F}$:

$$\varrho_{\mathcal{F}}(F, G) = \sup_{n \in \mathbb{N}} \varrho_u(f_n, g_n),$$

where ϱ_u is the uniform metric on $B_1(X, X)$

(that is, $\varrho_u(f, g) := \sup_{x \in X} \varrho(f(x), g(x))$ for $f, g: X \rightarrow X$)

$\mathcal{F}(X)$ – the family of sequences in $\mathcal{F}$ that are pointwise convergent

$\mathcal{F}(X)$ – the family of sequences in $\mathcal{F}$ that are pointwise convergent

Proposition

The space $(\mathcal{F}, \varrho_{\mathcal{F}})$ is complete and its subspace $\mathcal{F}(X)$ is closed.

$X = I$ – the unit interval $[0, 1]$

$X = I$ – the unit interval $[0, 1]$

A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

- $\mathcal{F}_u(I)$ – the family of all uniformly convergent sequences;

$X = I$ – the unit interval $[0, 1]$

A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

- $\mathcal{F}_u(I)$ – the family of all uniformly convergent sequences;
- $\mathcal{F}_{eq}(I)$ – the family of pointwise convergent equi- B_1 sequences;

$X = I$ – the unit interval $[0, 1]$

A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

- $\mathcal{F}_u(I)$ – the family of all uniformly convergent sequences;
- $\mathcal{F}_{eq}(I)$ – the family of pointwise convergent equi- B_1 sequences;
i.e., systems $(f_n)_{n \in \mathbb{N}}$ where for every $\varepsilon > 0$, there exists a mapping
(a gauge) $\delta: I \rightarrow \mathbb{R}^+$

$X = I$ – the unit interval $[0, 1]$

A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

- $\mathcal{F}_u(I)$ – the family of all uniformly convergent sequences;
- $\mathcal{F}_{eq}(I)$ – the family of pointwise convergent equi- B_1 sequences;
i.e., systems $(f_n)_{n \in \mathbb{N}}$ where for every $\varepsilon > 0$, there exists a mapping (a gauge) $\delta: I \rightarrow \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, the condition $\varrho(x, y) < \min\{\delta(x), \delta(y)\}$ implies $\varrho(f_n(x), f_n(y)) < \varepsilon$ for any $x, y \in I$;

$X = I$ – the unit interval $[0, 1]$

A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

- $\mathcal{F}_u(I)$ – the family of all uniformly convergent sequences;
- $\mathcal{F}_{eq}(I)$ – the family of pointwise convergent equi- B_1 sequences;
i.e., systems $(f_n)_{n \in \mathbb{N}}$ where for every $\varepsilon > 0$, there exists a mapping (a gauge) $\delta: I \rightarrow \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, the condition $\varrho(x, y) < \min\{\delta(x), \delta(y)\}$ implies $\varrho(f_n(x), f_n(y)) < \varepsilon$ for any $x, y \in I$;
- $\mathcal{F}_{p.w.}(I)$ – the family of all Baire-one sequences that converge pointwise to a function $f \in B_1(I, I)$;

$X = I$ – the unit interval $[0, 1]$

A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

- $\mathcal{F}_u(I)$ – the family of all uniformly convergent sequences;
- $\mathcal{F}_{eq}(I)$ – the family of pointwise convergent equi- B_1 sequences; i.e., systems $(f_n)_{n \in \mathbb{N}}$ where for every $\varepsilon > 0$, there exists a mapping (a gauge) $\delta: I \rightarrow \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, the condition $\varrho(x, y) < \min\{\delta(x), \delta(y)\}$ implies $\varrho(f_n(x), f_n(y)) < \varepsilon$ for any $x, y \in I$;
- $\mathcal{F}_{p.w.}(I)$ – the family of all Baire-one sequences that converge pointwise to a function $f \in B_1(I, I)$;
- $\mathcal{F}(I)$ – the family of all Baire-one sequences that converge pointwise.

$X = I$ – the unit interval $[0, 1]$

A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

- $\mathcal{F}_u(I)$ – the family of all uniformly convergent sequences;
- $\mathcal{F}_{eq}(I)$ – the family of pointwise convergent equi- B_1 sequences; i.e., systems $(f_n)_{n \in \mathbb{N}}$ where for every $\varepsilon > 0$, there exists a mapping (a gauge) $\delta: I \rightarrow \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, the condition $\varrho(x, y) < \min\{\delta(x), \delta(y)\}$ implies $\varrho(f_n(x), f_n(y)) < \varepsilon$ for any $x, y \in I$;
- $\mathcal{F}_{p.w.}(I)$ – the family of all Baire-one sequences that converge pointwise to a function $f \in B_1(I, I)$;
- $\mathcal{F}(I)$ – the family of all Baire-one sequences that converge pointwise.

$$\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I) \subsetneq \mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)$$

A. Alikhani-Koopaei, *On dynamics of families of equi-Baire one functions on metric spaces*, Topology Appl. **322** (2022), 108267.

$$\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I) \subsetneq \mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)$$

A. Alikhani-Koopaei, *On dynamics of families of equi-Baire one functions on metric spaces*, Topology Appl. **322** (2022), 108267.

$$\mathcal{F}_u(I) \subsetneq \underbrace{\mathcal{F}_{eq}(I)}_{\text{closed}} \subsetneq \mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)$$

A. Alikhani-Koopaei, *On dynamics of families of equi-Baire one functions on metric spaces*, Topology Appl. **322** (2022), 108267.

$$\underbrace{\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I)}_{\text{closed}} \subsetneq \underbrace{\mathcal{F}_{p.w.}(I)}_{\text{closed}} \subsetneq \mathcal{F}(I)$$

A. Alikhani-Koopaei, *On dynamics of families of equi-Baire one functions on metric spaces*, Topology Appl. **322** (2022), 108267.

$$\underbrace{\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I)}_{\text{closed}} \subsetneq \underbrace{\mathcal{F}_{p.w.}(I)}_{\text{closed}} \subsetneq \underbrace{\mathcal{F}(I)}_{\text{closed}}$$

A. Alikhani-Koopaei, *On dynamics of families of equi-Baire one functions on metric spaces*, Topology Appl. **322** (2022), 108267.

$$\underbrace{\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I)}_{\text{closed}} \subsetneq \underbrace{\mathcal{F}_{p.w.}(I)}_{\text{closed}} \subsetneq \underbrace{\mathcal{F}(I)}_{\text{closed}}$$

A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

$$\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I) \subsetneq \mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)$$

A. Alikhani-Koopaei, *On dynamics of families of equi-Baire one functions on metric spaces*, Topology Appl. **322** (2022), 108267.

$$\underbrace{\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I)}_{\text{closed}} \subsetneq \underbrace{\mathcal{F}_{p.w.}(I)}_{\text{closed}} \subsetneq \underbrace{\mathcal{F}(I)}_{\text{closed}}$$

A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

$$\underbrace{\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I)}_{\text{nd}} \subsetneq \mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)$$

A. Alikhani-Koopaei, *On dynamics of families of equi-Baire one functions on metric spaces*, Topology Appl. **322** (2022), 108267.

$$\underbrace{\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I)}_{\text{closed}} \subsetneq \underbrace{\mathcal{F}_{p.w.}(I)}_{\text{closed}} \subsetneq \underbrace{\mathcal{F}(I)}_{\text{closed}}$$

A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

$$\underbrace{\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I)}_{\text{nd}} \subsetneq \underbrace{\mathcal{F}_{p.w.}(I)}_{\text{nd}} \subsetneq \mathcal{F}(I)$$

(X, ϱ) – arbitrary metric space

$B(x, r)$ – open ball centred at x with a radius $r > 0$

$B(x, 0) := \emptyset$

(X, ϱ) – arbitrary metric space

$B(x, r)$ – open ball centred at x with a radius $r > 0$

$B(x, 0) := \emptyset$

Fix $M \subset X$, $x \in X$ and $r > 0$:

$$\gamma(x, r, M) := \sup\{t \geq 0 : \exists_{z \in X} B(z, t) \subset B(x, r) \setminus M\},$$

(X, ϱ) – arbitrary metric space

$B(x, r)$ – open ball centred at x with a radius $r > 0$

$B(x, 0) := \emptyset$

Fix $M \subset X$, $x \in X$ and $r > 0$:

$$\gamma(x, r, M) := \sup\{t \geq 0 : \exists z \in X \ B(z, t) \subset B(x, r) \setminus M\},$$

$$\underline{p}(M, x) := 2 \liminf_{r \rightarrow 0^+} \frac{\gamma(x, r, M)}{r},$$

(X, ϱ) – arbitrary metric space

$B(x, r)$ – open ball centred at x with a radius $r > 0$

$B(x, 0) := \emptyset$

Fix $M \subset X$, $x \in X$ and $r > 0$:

$$\gamma(x, r, M) := \sup\{t \geq 0 : \exists_{z \in X} B(z, t) \subset B(x, r) \setminus M\},$$

$$\underline{p}(M, x) := 2 \liminf_{r \rightarrow 0^+} \frac{\gamma(x, r, M)}{r},$$

$$\underline{p}(M) := \inf\{\underline{p}(M, x) : x \in M\}.$$

Definition

Fix $q \in (0, 1)$. We say that M is **lower porous** whenever $\underline{p}(M, x) > 0$ for each $x \in M$.

(X, ϱ) – arbitrary metric space

$B(x, r)$ – open ball centred at x with a radius $r > 0$

$B(x, 0) := \emptyset$

Fix $M \subset X$, $x \in X$ and $r > 0$:

$$\gamma(x, r, M) := \sup\{t \geq 0 : \exists z \in X \ B(z, t) \subset B(x, r) \setminus M\},$$

$$\underline{p}(M, x) := 2 \liminf_{r \rightarrow 0^+} \frac{\gamma(x, r, M)}{r},$$

$$\underline{p}(M) := \inf\{\underline{p}(M, x) : x \in M\}.$$

Definition

Fix $q \in (0, 1)$. We say that M is **lower porous** whenever $\underline{p}(M, x) > 0$ for each $x \in M$. The set M is called **lower q -porous** if $\underline{p}(M) = q$.

(X, ϱ) – arbitrary metric space

$B(x, r)$ – open ball centred at x with a radius $r > 0$

$B(x, 0) := \emptyset$

Fix $M \subset X$, $x \in X$ and $r > 0$:

$$\gamma(x, r, M) := \sup\{t \geq 0 : \exists z \in X \ B(z, t) \subset B(x, r) \setminus M\},$$

$$\underline{p}(M, x) := 2 \liminf_{r \rightarrow 0^+} \frac{\gamma(x, r, M)}{r},$$

$$\underline{p}(M) := \inf\{\underline{p}(M, x) : x \in M\}.$$

Definition

Fix $q \in (0, 1)$. We say that M is **lower porous** whenever $\underline{p}(M, x) > 0$ for each $x \in M$. The set M is called **lower q -porous** if $\underline{p}(M) = q$. The set M is **at least lower q -porous** if $\underline{p}(M) \geq q$.

A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026),

109849.

$$\mathcal{F}_u(I) \subsetneq \underbrace{\mathcal{F}_{eq}(I)}_{\text{nd}} \subsetneq \underbrace{\mathcal{F}_{p.w.}(I)}_{\text{nd}} \subsetneq \mathcal{F}(I)$$

$$\mathcal{F}_u(I) \subsetneq \underbrace{\mathcal{F}_{eq}(I)}_{\text{porous?}} \subsetneq \mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)$$

$$\mathcal{F}_u(I) \subsetneq \underbrace{\mathcal{F}_{eq}(I)}_{\text{porous?}} \subsetneq \mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)$$

Compare [Example 3.2] in: A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

Example

For each $\alpha \in (0, 1/2)$ there exists a sequence $F^\alpha = (f_n^\alpha)_{n \in \mathbb{N}}$ in $\mathcal{F}_{eq}(I) \setminus \mathcal{F}_u(I)$ such that for each $n \in \mathbb{N}$ we have $f_n^\alpha|_{I \setminus (a, 1]} \equiv 0$.

Compare [Example 3.2] in: A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*,
Topology Appl. **305** (2026), 109849.

Example

For each $\alpha \in (0, 1/2)$ there exists a sequence $F^\alpha = (f_n^\alpha)_{n \in \mathbb{N}}$ in $\mathcal{F}_{eq}(I) \setminus \mathcal{F}_u(I)$ such that for each $n \in \mathbb{N}$ we have $f_n^\alpha|_{I \setminus (a, 1]} \equiv 0$.

$$\alpha \in (0, 1/2), \quad a_i := a + (1 - a)/n, \quad n \in \mathbb{N}$$

Example

For each $\alpha \in (0, 1/2)$ there exists a sequence $F^\alpha = (f_n^\alpha)_{n \in \mathbb{N}}$ in $\mathcal{F}_{eq}(I) \setminus \mathcal{F}_u(I)$ such that for each $n \in \mathbb{N}$ we have $f_n^\alpha|_{I \setminus (a, 1]} \equiv 0$.

$\alpha \in (0, 1/2)$, $a_i := a + (1 - a)/n$, $n \in \mathbb{N}$

$$f_n^\alpha(x) = \begin{cases} \alpha, & \text{if } x = a_i, \quad i = 1, 2, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

Example

For each $\alpha \in (0, 1/2)$ there exists a sequence $F^\alpha = (f_n^\alpha)_{n \in \mathbb{N}}$ in $\mathcal{F}_{eq}(I) \setminus \mathcal{F}_u(I)$ such that for each $n \in \mathbb{N}$ we have $f_n^\alpha|_{I \setminus (a, 1]} \equiv 0$.

$\alpha \in (0, 1/2)$, $a_i := a + (1 - a)/n$, $n \in \mathbb{N}$

$$f_n^\alpha(x) = \begin{cases} \alpha, & \text{if } x = a_i, \quad i = 1, 2, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

$$\delta(x) = \begin{cases} \frac{1}{2}(a_i - a_{i+1}), & \text{if } x = a_i, \quad i \in \mathbb{N}, \\ \frac{1}{2} \min \{x - a_{i+1}, a_i - x\}, & \text{if } x \in (a_{i+1}, a_i), \quad i \in \mathbb{N}, \\ 1, & \text{otherwise.} \end{cases}$$

Example

For each $\alpha \in (0, 1/2)$ there exists a sequence $F^\alpha = (f_n^\alpha)_{n \in \mathbb{N}}$ in $\mathcal{F}_{eq}(I) \setminus \mathcal{F}_u(I)$ such that for each $n \in \mathbb{N}$ we have $f_n^\alpha|_{I \setminus (a, 1]} \equiv 0$.

$\alpha \in (0, 1/2)$, $a_i := a + (1 - a)/n$, $n \in \mathbb{N}$

$$f_n^\alpha(x) = \begin{cases} \alpha, & \text{if } x = a_i, \quad i = 1, 2, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

$$\delta(x) = \begin{cases} \frac{1}{2}(a_i - a_{i+1}), & \text{if } x = a_i, \quad i \in \mathbb{N}, \\ \frac{1}{2} \min \{x - a_{i+1}, a_i - x\}, & \text{if } x \in (a_{i+1}, a_i), \quad i \in \mathbb{N}, \\ 1, & \text{otherwise.} \end{cases}$$

$$\lim_{n \rightarrow \infty} f_n^\alpha(x) = f^\alpha(x) \quad \text{for each } x \in I$$

$$\sup_{x \in I} |f_n^\alpha(x) - f^\alpha(x)| = \alpha > 0$$

Lemma

Let $q \in (2/3, 1]$ and $r \in (0, 1/2)$.

There exists $F \in \mathcal{F}_u(I)$ such that for each $G \in \mathcal{F}_{eq}(I)$ satisfying $B(G, rq/2) \subset B(F, r)$, we have $\mathcal{F}_u(I) \cap B(G, rq/2) \neq \emptyset$.

Lemma

Let $q \in (2/3, 1]$ and $r \in (0, 1/2)$.

There exists $F \in \mathcal{F}_u(I)$ such that for each $G \in \mathcal{F}_{eq}(I)$ satisfying $B(G, rq/2) \subset B(F, r)$, we have $\mathcal{F}_u(I) \cap B(G, rq/2) \neq \emptyset$.

Let $F = (f_n)_{n \in \mathbb{N}} \in \mathcal{F}_u(I)$ be the sequence defined by $f_n \equiv 0$ for all $n \in \mathbb{N}$.

Lemma

Let $q \in (2/3, 1]$ and $r \in (0, 1/2)$.

There exists $F \in \mathcal{F}_u(I)$ such that for each $G \in \mathcal{F}_{eq}(I)$ satisfying $B(G, rq/2) \subset B(F, r)$, we have $\mathcal{F}_u(I) \cap B(G, rq/2) \neq \emptyset$.

Let $F = (f_n)_{n \in \mathbb{N}} \in \mathcal{F}_u(I)$ be the sequence defined by $f_n \equiv 0$ for all $n \in \mathbb{N}$.

Fix $G = (g_n)_{n \in \mathbb{N}} \in \mathcal{F}_{eq}(I)$ such that $B(G, rq/2) \subset B(F, r)$,

Lemma

Let $q \in (2/3, 1]$ and $r \in (0, 1/2)$.

There exists $F \in \mathcal{F}_u(I)$ such that for each $G \in \mathcal{F}_{eq}(I)$ satisfying $B(G, rq/2) \subset B(F, r)$, we have $\mathcal{F}_u(I) \cap B(G, rq/2) \neq \emptyset$.

Let $F = (f_n)_{n \in \mathbb{N}} \in \mathcal{F}_u(I)$ be the sequence defined by $f_n \equiv 0$ for all $n \in \mathbb{N}$.

Fix $G = (g_n)_{n \in \mathbb{N}} \in \mathcal{F}_{eq}(I)$ such that $B(G, rq/2) \subset B(F, r)$, define $H := (h_n)_{n \in \mathbb{N}}$ with $h_n \equiv r/3$ for all $n \in \mathbb{N}$,

Lemma

Let $q \in (2/3, 1]$ and $r \in (0, 1/2)$.

There exists $F \in \mathcal{F}_u(I)$ such that for each $G \in \mathcal{F}_{eq}(I)$ satisfying $B(G, rq/2) \subset B(F, r)$, we have $\mathcal{F}_u(I) \cap B(G, rq/2) \neq \emptyset$.

Let $F = (f_n)_{n \in \mathbb{N}} \in \mathcal{F}_u(I)$ be the sequence defined by $f_n \equiv 0$ for all $n \in \mathbb{N}$.

Fix $G = (g_n)_{n \in \mathbb{N}} \in \mathcal{F}_{eq}(I)$ such that $B(G, rq/2) \subset B(F, r)$, define $H := (h_n)_{n \in \mathbb{N}}$ with $h_n \equiv r/3$ for all $n \in \mathbb{N}$, then

$$0 \leq g_n(x) < \frac{2r}{3} \text{ for all } n \in \mathbb{N} \text{ and } x \in I.$$

Lemma

Let $q \in (2/3, 1]$ and $r \in (0, 1/2)$.

There exists $F \in \mathcal{F}_u(I)$ such that for each $G \in \mathcal{F}_{eq}(I)$ satisfying $B(G, rq/2) \subset B(F, r)$, we have $\mathcal{F}_u(I) \cap B(G, rq/2) \neq \emptyset$.

Let $F = (f_n)_{n \in \mathbb{N}} \in \mathcal{F}_u(I)$ be the sequence defined by $f_n \equiv 0$ for all $n \in \mathbb{N}$.

Fix $G = (g_n)_{n \in \mathbb{N}} \in \mathcal{F}_{eq}(I)$ such that $B(G, rq/2) \subset B(F, r)$, define $H := (h_n)_{n \in \mathbb{N}}$ with $h_n \equiv r/3$ for all $n \in \mathbb{N}$, then

$$0 \leq g_n(x) < \frac{2r}{3} \text{ for all } n \in \mathbb{N} \text{ and } x \in I.$$

Consequently,

$$\sup_{x \in I} |g_n(x) - h_n(x)| \leq \frac{r}{3} < \frac{rq}{2} \text{ for each } n \in \mathbb{N},$$

$$H \in B(G, rq/2) \cap \mathcal{F}_u(I).$$

Lemma

Let $q \in (2/3, 1]$ and $r \in (0, 1/2)$.

There exists $F \in \mathcal{F}_u(I)$ such that for each $G \in \mathcal{F}_{eq}(I)$ satisfying $B(G, rq/2) \subset B(F, r)$, we have $\mathcal{F}_u(I) \cap B(G, rq/2) \neq \emptyset$.

Theorem

The family $\mathcal{F}_u(I)$ is lower $2/3$ -porous in $\mathcal{F}_{eq}(I)$.

$$F \in \mathcal{F}_u(I)$$

$$\gamma(F, r, \mathcal{F}_u(I)) := \sup\{t \geq 0 : \exists_{G \in \mathcal{F}_{eq}(I)} B(G, t) \subset B(F, r) \setminus \mathcal{F}_u(I)\} \geq \frac{r}{3},$$

$$\underline{p}(\mathcal{F}_u(I), F) := 2 \liminf_{r \rightarrow 0^+} \frac{\gamma(F, r, \mathcal{F}_u(I))}{r} \geq \frac{2}{3},$$

$$\underline{p}(\mathcal{F}_u(I)) := \inf\{\underline{p}(\mathcal{F}_u(I), F) : F \in \mathcal{F}_u(I)\} \geq \frac{2}{3}.$$

$$\mathcal{F}_u(I) \subsetneq \underbrace{\mathcal{F}_{eq}(I)}_{2/3\text{-l.p.}} \subsetneq \mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)$$

$$\mathcal{F}_u(I) \subsetneq \underbrace{\mathcal{F}_{eq}(I)}_{2/3\text{-l.p.}} \subsetneq \underbrace{\mathcal{F}_{p.w.}(I)}_{?} \subsetneq \mathcal{F}(I)$$

Definition

Let F be a family of real valued functions defined on a metric space (X, ϱ) . The functions $\text{env}^+ F$, $\text{env}^- F$

$$\text{env}^+ F(x) = \sup_{f \in F} f(x) \quad \text{and} \quad \text{env}^- F(x) = \inf_{f \in F} f(x) \quad \text{for } x \in X,$$

are called the **upper envelope** and the **lower envelope** of the family F , respectively.

Theorem (Theorem 3.14)

Suppose that the family $F \subset B_1(X, \mathbb{R})$ is an equi-Baire-one family. If $\text{env}^+ F(x) < \infty$ (resp., $\text{env}^- F(x) > -\infty$) for all $x \in X$, then $\text{env}^+ F$ (resp., $\text{env}^- F$) is a Baire-one function.

$$\mathcal{F}_u(I) \subsetneq \underbrace{\mathcal{F}_{eq}(I)}_{2/3\text{-l.p.}} \subsetneq \underbrace{\mathcal{F}_{p.w.}(I)}_{?} \subsetneq \mathcal{F}(I)$$

Compare [Example 3.5] in A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

Example

For each $\alpha \in (0, 1/2)$ and $(a, b) \subset I$ there exists a sequence $F^\alpha = (f_n^\alpha)_{n \in \mathbb{N}}$ in $\mathcal{F}_{p.w.}(I) \setminus \mathcal{F}_{eq}(I)$ such that for each $n \in \mathbb{N}$ we have $f_n^\alpha|_{I \setminus (a,b)} \equiv 0$.

$$\mathcal{F}_u(I) \subsetneq \underbrace{\mathcal{F}_{eq}(I)}_{2/3\text{-l.p.}} \subsetneq \underbrace{\mathcal{F}_{p.w.}(I)}_{?} \subsetneq \mathcal{F}(I)$$

Compare [Example 3.5] in A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

Example

For each $\alpha \in (0, 1/2)$ and $(a, b) \subset I$ there exists a sequence $F^\alpha = (f_n^\alpha)_{n \in \mathbb{N}}$ in $\mathcal{F}_{p.w.}(I) \setminus \mathcal{F}_{eq}(I)$ such that for each $n \in \mathbb{N}$ we have $f_n^\alpha|_{I \setminus (a,b)} \equiv 0$.

$\alpha \in (0, 1/2)$, $(a, b) \subset I$, $(r_n)_{n \in \mathbb{N}}$ – an enumeration of the set $(a, b) \cap \mathbb{Q}$

$$\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I) \subsetneq \mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)$$

$\underbrace{\hspace{1.5cm}}_{2/3\text{-l.p.}} \quad \underbrace{\hspace{1.5cm}}_{?}$

Compare [Example 3.5] in A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

Example

For each $\alpha \in (0, 1/2)$ and $(a, b) \subset I$ there exists a sequence $F^\alpha = (f_n^\alpha)_{n \in \mathbb{N}}$ in $\mathcal{F}_{p.w.}(I) \setminus \mathcal{F}_{eq}(I)$ such that for each $n \in \mathbb{N}$ we have $f_n^\alpha|_{I \setminus (a,b)} \equiv 0$.

$\alpha \in (0, 1/2)$, $(a, b) \subset I$, $(r_n)_{n \in \mathbb{N}}$ – an enumeration of the set $(a, b) \cap \mathbb{Q}$

$$f_n^\alpha(x) = \begin{cases} \alpha, & \text{if } x = r_n, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I) \subsetneq \mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)$$

$\underbrace{\hspace{10em}}_{2/3\text{-l.p.}}$
 $\underbrace{\hspace{10em}}_{?}$

Compare [Example 3.5] in A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

Example

For each $\alpha \in (0, 1/2)$ and $(a, b) \subset I$ there exists a sequence $F^\alpha = (f_n^\alpha)_{n \in \mathbb{N}}$ in $\mathcal{F}_{p.w.}(I) \setminus \mathcal{F}_{eq}(I)$ such that for each $n \in \mathbb{N}$ we have $f_n^\alpha|_{I \setminus (a,b)} \equiv 0$.

$\alpha \in (0, 1/2)$, $(a, b) \subset I$, $(r_n)_{n \in \mathbb{N}}$ – an enumeration of the set $(a, b) \cap \mathbb{Q}$

$$f_n^\alpha(x) = \begin{cases} \alpha, & \text{if } x = r_n, \\ 0, & \text{otherwise.} \end{cases}$$

$$\lim_{n \rightarrow \infty} f_n^\alpha(x) = 0 \quad \text{for each } x \in I,$$

$$\underbrace{\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I)}_{2/3\text{-l.p.}} \subsetneq \underbrace{\mathcal{F}_{p.w.}(I)}_{?} \subsetneq \mathcal{F}(I)$$

Compare [Example 3.5] in A. Alikhani-Koopaei, *On dynamics of convergent sequences of functions on metric spaces*, Topology Appl. **305** (2026), 109849.

Example

For each $\alpha \in (0, 1/2)$ and $(a, b) \subset I$ there exists a sequence $F^\alpha = (f_n^\alpha)_{n \in \mathbb{N}}$ in $\mathcal{F}_{p.w.}(I) \setminus \mathcal{F}_{eq}(I)$ such that for each $n \in \mathbb{N}$ we have $f_n^\alpha|_{I \setminus (a,b)} \equiv 0$.

$\alpha \in (0, 1/2)$, $(a, b) \subset I$, $(r_n)_{n \in \mathbb{N}}$ – an enumeration of the set $(a, b) \cap \mathbb{Q}$

$$f_n^\alpha(x) = \begin{cases} \alpha, & \text{if } x = r_n, \\ 0, & \text{otherwise.} \end{cases}$$

$$\lim_{n \rightarrow \infty} f_n^\alpha(x) = 0 \quad \text{for each } x \in I,$$

$$\text{env}^+ F^\alpha(x) = \sup_{n \in \mathbb{N}} f_n^\alpha(x) = \begin{cases} \alpha, & \text{if } x \in \{r_i : i \in \mathbb{N}\}, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma

Let $q \in (2/3, 1]$ and $r \in (0, 1/2)$.

There exists $F \in \mathcal{F}_u(I)$ such that for each $G \in \mathcal{F}_{eq}(I)$ satisfying $B(G, rq/2) \subset B(F, r)$, we have $\mathcal{F}_u(I) \cap B(G, rq/2) \neq \emptyset$.

$$\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I) \subsetneq \mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)$$

Theorem

The family $\mathcal{F}_{eq}(I)$ is lower $2/3$ -porous in $\mathcal{F}_{p.w.}(I)$.

$$F \in \mathcal{F}_{eq}(I)$$

$$\gamma(F, r, \mathcal{F}_{eq}(I)) := \sup\{t \geq 0 : \exists G \in \mathcal{F}_{p.w.}(I) \ B(G, t) \subset B(F, r) \setminus \mathcal{F}_{eq}(I)\} \geq \frac{r}{3},$$

$$\underline{p}(\mathcal{F}_{eq}(I), F) := 2 \liminf_{r \rightarrow 0^+} \frac{\gamma(F, r, \mathcal{F}_{eq}(I))}{r} \geq \frac{2}{3},$$

$$\underline{p}(\mathcal{F}_{eq}(I)) := \inf\{\underline{p}(\mathcal{F}_{eq}(I), F) : F \in \mathcal{F}_{eq}(I)\} \geq \frac{2}{3}.$$

$$\mathcal{F}_u(I) \subsetneq \underbrace{\mathcal{F}_{eq}(I)}_{2/3\text{-l.p.}} \subsetneq \underbrace{\mathcal{F}_{p.w.}(I)}_{2/3\text{-l.p.}} \subsetneq \mathcal{F}(I)$$

$$\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I) \subsetneq \mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)$$

$\underbrace{\hspace{1.5cm}}_{2/3\text{-l.p.}} \quad \underbrace{\hspace{1.5cm}}_{2/3\text{-l.p.}} \quad \underbrace{\hspace{1.5cm}}_{?}$

$$\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I) \subsetneq \mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)$$

$\underbrace{\hspace{10em}}_{2/3\text{-l.p.}}$
 $\underbrace{\hspace{10em}}_{2/3\text{-l.p.}}$
 $\underbrace{\hspace{10em}}_{?}$

Example

For each $\alpha \in (0, 1/2)$ and $(a, b) \subset I$ there exists a sequence $F^\alpha = (f_n^\alpha)_{n \in \mathbb{N}}$ in $\mathcal{F}(I) \setminus \mathcal{F}_{p.w.}(I)$ such that for each $n \in \mathbb{N}$ we have $f_n^\alpha|_{I \setminus (a,b)} \equiv 0$.

$\alpha \in (0, 1/2)$, $(a, b) \subset I$, $(r_i)_{i \in \mathbb{N}}$ – an enumeration of the rational numbers in (a, b)

$$\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I) \subsetneq \mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)$$

$\underbrace{\hspace{10em}}_{2/3\text{-l.p.}}$
 $\underbrace{\hspace{10em}}_{2/3\text{-l.p.}}$
 $\underbrace{\hspace{10em}}_{?}$

Example

For each $\alpha \in (0, 1/2)$ and $(a, b) \subset I$ there exists a sequence $F^\alpha = (f_n^\alpha)_{n \in \mathbb{N}}$ in $\mathcal{F}(I) \setminus \mathcal{F}_{p.w.}(I)$ such that for each $n \in \mathbb{N}$ we have $f_n^\alpha|_{I \setminus (a,b)} \equiv 0$.

$\alpha \in (0, 1/2)$, $(a, b) \subset I$, $(r_i)_{i \in \mathbb{N}}$ – an enumeration of the rational numbers in (a, b)

$$f_n^\alpha(x) = \begin{cases} \alpha, & \text{if } x = r_i, \quad i = 1, 2, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I) \subsetneq \mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)$$

$\underbrace{\hspace{10em}}_{2/3\text{-l.p.}}$
 $\underbrace{\hspace{10em}}_{2/3\text{-l.p.}}$
 $\underbrace{\hspace{10em}}_{?}$

Example

For each $\alpha \in (0, 1/2)$ and $(a, b) \subset I$ there exists a sequence $F^\alpha = (f_n^\alpha)_{n \in \mathbb{N}}$ in $\mathcal{F}(I) \setminus \mathcal{F}_{p.w.}(I)$ such that for each $n \in \mathbb{N}$ we have $f_n^\alpha|_{I \setminus (a,b)} \equiv 0$.

$\alpha \in (0, 1/2)$, $(a, b) \subset I$, $(r_i)_{i \in \mathbb{N}}$ – an enumeration of the rational numbers in (a, b)

$$f_n^\alpha(x) = \begin{cases} \alpha, & \text{if } x = r_i, \quad i = 1, 2, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

$$f^\alpha(x) = \lim_{n \rightarrow \infty} f_n^\alpha(x) = \begin{cases} \alpha, & \text{if } x = r_i, \quad i \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma

Let $q \in (2/3, 1]$ and $r \in (0, 1/2)$.

There exists $F \in \mathcal{F}_u(I)$ such that for each $G \in \mathcal{F}_{eq}(I)$ satisfying $B(G, rq/2) \subset B(F, r)$, we have $\mathcal{F}_u(I) \cap B(G, rq/2) \neq \emptyset$.

$$\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I) \subsetneq \mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)$$

Theorem

The family $\mathcal{F}_{p.w.}(I)$ is lower $2/3$ -porous in $\mathcal{F}(I)$.

$$\mathcal{F}_u(I) \subsetneq \underbrace{\mathcal{F}_{eq}(I)}_{2/3\text{-l.p.}} \subsetneq \underbrace{\mathcal{F}_{p.w.}(I)}_{2/3\text{-l.p.}} \subsetneq \mathcal{F}(I)$$

$$\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I) \subsetneq \mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)$$

$\underbrace{\hspace{1.5cm}}_{2/3\text{-l.p.}} \quad \underbrace{\hspace{1.5cm}}_{2/3\text{-l.p.}} \quad \underbrace{\hspace{1.5cm}}_{2/3\text{-l.p.}}$

$$\underbrace{\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I)}_{2/3\text{-l.p.}} \subsetneq \underbrace{\mathcal{F}_{p.w.}(I)}_{2/3\text{-l.p.}} \subsetneq \underbrace{\mathcal{F}(I)}_{2/3\text{-l.p.}}$$

A. Alikhani-Koopaei, *On dynamics of families of equi-Baire one functions on metric spaces*, Topology Appl. **322** (2022), 108267.

$$\underbrace{\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I)}_{\text{nd}} \subsetneq \underbrace{\mathcal{F}_{p.w.}(I)}_{\text{nd}} \subsetneq \mathcal{F}(I)$$

$$\underbrace{\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I)}_{2/3\text{-l.p.}} \subsetneq \underbrace{\mathcal{F}_{eq}(I) \subsetneq \mathcal{F}_{p.w.}(I)}_{2/3\text{-l.p.}} \subsetneq \underbrace{\mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)}_{2/3\text{-l.p.}} \subsetneq \underbrace{\mathcal{F}(I) \subsetneq \mathcal{F}}_{2/3\text{-l.p.}}$$

$$f: I \rightarrow \mathbb{R},$$

$f: I \rightarrow \mathbb{R}$,

$B_1(I, \mathbb{R})$ – the family of all Baire class-one functions $f: I \rightarrow \mathbb{R}$

$f: I \rightarrow \mathbb{R}$,

$B_1(I, \mathbb{R})$ – the family of all Baire class-one functions $f: I \rightarrow \mathbb{R}$

$b\mathcal{F}$ – the family of all sequences of uniformly bounded functions in $B_1(I, \mathbb{R})$

$b\mathcal{F}$ is a normed space with the norm

$$\|F\|_{\infty} := \sup_{n \in \mathbb{N}} \sup_{x \in I} |f_n(x)|,$$

where $F = (f_n)_{n \in \mathbb{N}}$, $F \in b\mathcal{F}$.

Similarly as before, we define the following families of subsets of $b\mathcal{F}$:

- $b\mathcal{F}_u(I, \mathbb{R})$ – the family of all uniformly convergent sequences;

Similarly as before, we define the following families of subsets of $b\mathcal{F}$:

- $b\mathcal{F}_u(I, \mathbb{R})$ – the family of all uniformly convergent sequences;
- $b\mathcal{F}_{eq}(I, \mathbb{R})$ – the family of pointwise convergent equi- B_1 sequences;

Similarly as before, we define the following families of subsets of $b\mathcal{F}$:

- $b\mathcal{F}_u(I, \mathbb{R})$ – the family of all uniformly convergent sequences;
- $b\mathcal{F}_{eq}(I, \mathbb{R})$ – the family of pointwise convergent equi- B_1 sequences;
- $b\mathcal{F}_{p.w.}(I, \mathbb{R})$ – the family of all Baire-one sequences that converge pointwise to a function $f \in B_1(I, \mathbb{R})$;

Similarly as before, we define the following families of subsets of $b\mathcal{F}$:

- $b\mathcal{F}_u(I, \mathbb{R})$ – the family of all uniformly convergent sequences;
- $b\mathcal{F}_{eq}(I, \mathbb{R})$ – the family of pointwise convergent equi- B_1 sequences;
- $b\mathcal{F}_{p.w.}(I, \mathbb{R})$ – the family of all Baire-one sequences that converge pointwise to a function $f \in B_1(I, \mathbb{R})$;
- $b\mathcal{F}(I, \mathbb{R})$ – the family of all Baire-one sequences that converge pointwise.

Similarly as before, we define the following families of subsets of $b\mathcal{F}$:

- $b\mathcal{F}_u(I, \mathbb{R})$ – the family of all uniformly convergent sequences;
- $b\mathcal{F}_{eq}(I, \mathbb{R})$ – the family of pointwise convergent equi- B_1 sequences;
- $b\mathcal{F}_{p.w.}(I, \mathbb{R})$ – the family of all Baire-one sequences that converge pointwise to a function $f \in B_1(I, \mathbb{R})$;
- $b\mathcal{F}(I, \mathbb{R})$ – the family of all Baire-one sequences that converge pointwise.

$$b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R}) \subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R}) \subsetneq b\mathcal{F}$$

$$b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R}) \subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R}) \subsetneq b\mathcal{F} \quad (1)$$

If Y is a closed proper linear subspace of a normed space X , then Y is nowhere dense in X .

$$b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R}) \subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R}) \subsetneq b\mathcal{F} \quad (1)$$

If Y is a closed proper linear subspace of a normed space X , then Y is nowhere dense in X .

Proposition

All the subspaces of $b\mathcal{F}$ listed in (1) are closed linear spaces. Consequently, they are Banach spaces.

$$b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R}) \subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R}) \subsetneq b\mathcal{F} \quad (1)$$

If Y is a closed proper linear subspace of a normed space X , then Y is nowhere dense in X .

Proposition

All the subspaces of $b\mathcal{F}$ listed in (1) are closed linear spaces. Consequently, they are Banach spaces.

$$\underbrace{b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R}) \subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R})}_{\text{nd}} \subsetneq b\mathcal{F}(I, \mathbb{R}) \subsetneq b\mathcal{F}$$

$$b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R}) \subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R}) \subsetneq b\mathcal{F} \quad (1)$$

If Y is a closed proper linear subspace of a normed space X , then Y is nowhere dense in X .

Proposition

All the subspaces of $b\mathcal{F}$ listed in (1) are closed linear spaces. Consequently, they are Banach spaces.

$$\underbrace{b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R})}_{\text{nd}} \underbrace{\subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R})}_{\text{nd}} \subsetneq b\mathcal{F}(I, \mathbb{R}) \subsetneq b\mathcal{F}$$

$$b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R}) \subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R}) \subsetneq b\mathcal{F} \quad (1)$$

If Y is a closed proper linear subspace of a normed space X , then Y is nowhere dense in X .

Proposition

All the subspaces of $b\mathcal{F}$ listed in (1) are closed linear spaces. Consequently, they are Banach spaces.

$$\underbrace{b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R})}_{\text{nd}} \underbrace{\subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R})}_{\text{nd}} \underbrace{\subsetneq b\mathcal{F}(I, \mathbb{R})}_{\text{nd}} \subsetneq b\mathcal{F}$$

$$b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R}) \subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R}) \subsetneq b\mathcal{F} \quad (1)$$

If Y is a closed proper linear subspace of a normed space X , then Y is nowhere dense in X .

Proposition

All the subspaces of $b\mathcal{F}$ listed in (1) are closed linear spaces. Consequently, they are Banach spaces.

$$\underbrace{b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R})}_{\text{nd}} \underbrace{\subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R})}_{\text{nd}} \underbrace{\subsetneq b\mathcal{F}(I, \mathbb{R})}_{\text{nd}} \underbrace{\subsetneq b\mathcal{F}}_{\text{nd}}$$

$(X, \|\cdot\|)$ – real normed linear space and $M \subset X$

$(X, \|\cdot\|)$ – real normed linear space and $M \subset X$

X^* – space of all continuous linear functionals on X

$(X, \|\cdot\|)$ – real normed linear space and $M \subset X$

X^* – space of all continuous linear functionals on X

L. Zajíček, *Porosity and σ -porosity*, Real Anal Exchange **13** (1987-88), 314–350.

Definition

Let $R > 0$. We say that $M \subset X$ is **R -ball porous** if for any $x \in M$ and $\alpha \in (0, 1)$, there exists $y \in X$ such that $\|x - y\| = R$ and $B(y, \alpha R) \cap M = \emptyset$.

$(X, \|\cdot\|)$ – real normed linear space and $M \subset X$

X^* – space of all continuous linear functionals on X

L. Zajíček, *Porosity and σ -porosity*, Real Anal Exchange **13** (1987-88), 314–350.

Definition

Let $R > 0$. We say that $M \subset X$ is **R -ball porous** if for any $x \in M$ and $\alpha \in (0, 1)$, there exists $y \in X$ such that $\|x - y\| = R$ and $B(y, \alpha R) \cap M = \emptyset$.

If $M \subset X$ is R -ball porous for every $R > 0$, then M is lower 1-porous.

$(X, \|\cdot\|)$ – real normed linear space and $M \subset X$

X^* – space of all continuous linear functionals on X

L. Zajíček, *Porosity and σ -porosity*, Real Anal Exchange **13** (1987-88), 314–350.

Definition

Let $R > 0$. We say that $M \subset X$ is **R -ball porous** if for any $x \in M$ and $\alpha \in (0, 1)$, there exists $y \in X$ such that $\|x - y\| = R$ and $B(y, \alpha R) \cap M = \emptyset$.

If $M \subset X$ is R -ball porous for every $R > 0$, then M is lower 1-porous.

F. Strobín, *Porosity of convex nowhere dense subsets of normed linear spaces*, Abstr. Appl. Anal. (2009), 243604.

Definition

We say that M is **c -porous** if for any $x \in X$ and every $r > 0$, there are $y \in B(x, r)$ and $\phi \in X^* \setminus \{0\}$ such that

$$\{z \in X : \phi(z) > \phi(y)\} \cap M = \emptyset.$$

Propositions 2.5 and 2.8 in: F. Stobin, *Porosity of convex nowhere dense subsets of normed linear spaces*, Abstr. Appl. Anal. (2009), 243604.

Theorem

A subset M of a normed space X is c -porous iff the convex hull $\text{conv} M$ is nowhere dense. If a set M is c -porous then it is R -ball porous for every $R > 0$.

Propositions 2.5 and 2.8 in: F. Stobin, *Porosity of convex nowhere dense subsets of normed linear spaces*, Abstr. Appl. Anal. (2009), 243604.

Theorem

A subset M of a normed space X is c -porous iff the convex hull $\text{conv} M$ is nowhere dense. If a set M is c -porous then it is R -ball porous for every $R > 0$.

$$b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R}) \subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R}) \subsetneq b\mathcal{F} \quad (2)$$

Propositions 2.5 and 2.8 in: F. Strobín, *Porosity of convex nowhere dense subsets of normed linear spaces*, Abstr. Appl. Anal. (2009), 243604.

Theorem

A subset M of a normed space X is c -porous iff the convex hull $\text{conv} M$ is nowhere dense. If a set M is c -porous then it is R -ball porous for every $R > 0$.

$$b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R}) \subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R}) \subsetneq b\mathcal{F} \quad (2)$$

Corollary

Every space different from $b\mathcal{F}$ in the chain (2) is c -porous in the next space of this chain.

Propositions 2.5 and 2.8 in: F. Strobín, *Porosity of convex nowhere dense subsets of normed linear spaces*, Abstr. Appl. Anal. (2009), 243604.

Theorem

A subset M of a normed space X is c -porous iff the convex hull $\text{conv} M$ is nowhere dense. If a set M is c -porous then it is R -ball porous for every $R > 0$.

$$b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R}) \subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R}) \subsetneq b\mathcal{F} \quad (2)$$

Corollary

Every space different from $b\mathcal{F}$ in the chain (2) is c -porous in the next space of this chain.

Consequently, it is R -ball porous for every $R > 0$

Propositions 2.5 and 2.8 in: F. Stobin, *Porosity of convex nowhere dense subsets of normed linear spaces*, Abstr. Appl. Anal. (2009), 243604.

Theorem

A subset M of a normed space X is c -porous iff the convex hull $\text{conv} M$ is nowhere dense. If a set M is c -porous then it is R -ball porous for every $R > 0$.

$$b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R}) \subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R}) \subsetneq b\mathcal{F} \quad (2)$$

Corollary

Every space different from $b\mathcal{F}$ in the chain (2) is c -porous in the next space of this chain.

Consequently, it is R -ball porous for every $R > 0$ and it is 1-lower porous.

$$b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R}) \subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R}) \subsetneq b\mathcal{F}$$

$$\underbrace{b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R})}_{\text{c-p.}} \subsetneq \underbrace{b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R})}_{\text{c-p.}} \subsetneq \underbrace{b\mathcal{F}}_{\text{c.p.}}$$

$$\underbrace{b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R})}_{\text{c-p.}} \subsetneq \underbrace{b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R})}_{\text{c-p.}} \subsetneq \underbrace{b\mathcal{F}}_{\text{c.p.}}$$

For every $R > 0$

$$\underbrace{b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R})}_{\text{R-b.p.}} \subsetneq \underbrace{b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R})}_{\text{R-b.p.}} \subsetneq \underbrace{b\mathcal{F}}_{\text{R-b.p.}}$$

$$\underbrace{b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R})}_{\text{c-p.}} \subsetneq \underbrace{b\mathcal{F}_{eq}(I, \mathbb{R}) \subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R})}_{\text{c-p.}} \subsetneq \underbrace{b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R})}_{\text{c-p.}} \subsetneq \underbrace{b\mathcal{F}(I, \mathbb{R}) \subsetneq b\mathcal{F}}_{\text{c.p.}}$$

For every $R > 0$

$$\underbrace{b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R})}_{\text{R-b.p.}} \subsetneq \underbrace{b\mathcal{F}_{eq}(I, \mathbb{R}) \subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R})}_{\text{R-b.p.}} \subsetneq \underbrace{b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R})}_{\text{R-b.p.}} \subsetneq \underbrace{b\mathcal{F}(I, \mathbb{R}) \subsetneq b\mathcal{F}}_{\text{R-b.p.}}$$

$$\underbrace{b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R})}_{\text{1-l.p.}} \subsetneq \underbrace{b\mathcal{F}_{eq}(I, \mathbb{R}) \subsetneq b\mathcal{F}_{p.w.}(I, \mathbb{R})}_{\text{1-l.p.}} \subsetneq \underbrace{b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R})}_{\text{1-l.p.}} \subsetneq \underbrace{b\mathcal{F}(I, \mathbb{R}) \subsetneq b\mathcal{F}}_{\text{1-l.p.}}$$

$$\underbrace{b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R})}_{\text{c-p.}} \subsetneq \underbrace{b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R})}_{\text{c-p.}} \subsetneq \underbrace{b\mathcal{F}}_{\text{c.p.}}$$

For every $R > 0$

$$\underbrace{b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R})}_{\text{R-b.p.}} \subsetneq \underbrace{b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R})}_{\text{R-b.p.}} \subsetneq \underbrace{b\mathcal{F}}_{\text{R-b.p.}}$$

$$\underbrace{b\mathcal{F}_u(I, \mathbb{R}) \subsetneq b\mathcal{F}_{eq}(I, \mathbb{R})}_{\text{1-l.p.}} \subsetneq \underbrace{b\mathcal{F}_{p.w.}(I, \mathbb{R}) \subsetneq b\mathcal{F}(I, \mathbb{R})}_{\text{1-l.p.}} \subsetneq \underbrace{b\mathcal{F}}_{\text{1-l.p.}}$$

$$\underbrace{\mathcal{F}_u(I) \subsetneq \mathcal{F}_{eq}(I)}_{\text{2/3-l.p.}} \subsetneq \underbrace{\mathcal{F}_{p.w.}(I) \subsetneq \mathcal{F}(I)}_{\text{2/3-l.p.}} \subsetneq \underbrace{\mathcal{F}}_{\text{2/3-l.p.}}$$