

J. Bąk, G.H., E. Jabłońska, M. Terepeta

On K -measurable set-valued maps

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C.J. Himmelberg, *Measurable relations*, Fund. Math. 87 (1975)

For $Y \neq \emptyset$ denote $n(Y) := 2^Y \setminus \{\emptyset\}$.

For a given s.v. map $F: X \rightarrow n(Y)$ and $S \subset Y$, the set

$$F^-(S) := \{x \in X: F(x) \cap S \neq \emptyset\}$$

is called the *inverse of S by F* ,

$$F^+(S) := \{x \in X: F(x) \subset S\}$$

is called the *core of S by F* .

Definition

Let $(X, \mathcal{A})$ be a measurable space and Y be a topological space. An s.v. map $F: X \rightarrow n(Y)$ is called:

- (i) *weakly $\mathcal{A}$ -measurable*, if $F^-(U) \in \mathcal{A}$ for every open set $U \subset Y$,
- (ii) *$\mathcal{A}$ -measurable*, if $F^-(C) \in \mathcal{A}$ for every closed set $C \subset Y$ or, equivalently, if $F^+(U) \in \mathcal{A}$ for every open set $U \subset Y$.

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G.H., E. Jabłońska, W. Jabłoński, M. Terepeta, *On multiadditive set-valued maps "modulo K "*. Aequat. Math. 99 (2025)

Y - real vector space

$A + B := \{a + b : a \in A, b \in B\}$ for $A, B \in n(Y)$.

$K \subset Y$ is called a *convex cone*, if $K + K \subset K$, $tK \subset K$ for all $t \geq 0$, and $K \cap (-K) = \{0\}$.

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- If $K = \{0\}$, then [weak] $\{0\}$ - $\mathcal{A}$ -measurability means the classical notion of [weak] $\mathcal{A}$ -measurability.
If, moreover, Y is a metric space and F has compact values, then weak $\{0\}$ - $\mathcal{A}$ -measurability and $\{0\}$ - $\mathcal{A}$ -measurability coincide.
- Let $K = [0, \infty)$, $Y = \mathbb{R}$, and $F: \mathbb{R} \rightarrow n(\mathbb{R})$ be an s.v. map such that $m(x) := \inf F(x) > -\infty$ for $x \in \mathbb{R}$. Then both notions of weak K - $\mathcal{A}$ -measurability and K - $\mathcal{A}$ -measurability are equivalent to the classical $\mathcal{A}$ -measurability of the real function m .

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Each [weakly] $\mathcal{A}$ -measurable s.v. map is [weakly] K - $\mathcal{A}$ -measurable for every convex cone $K \subset Y$.

However, [weak] K - $\mathcal{A}$ -measurability with a cone $K \subset Y$ does not imply [weak] $\mathcal{A}$ -measurability (even if F has compact convex values).

Example

Let $K = [0, \infty)$ and $F: \mathbb{R} \rightarrow n(\mathbb{R})$ be given by

$$F(x) := \begin{cases} [0, 2], & x \in A, \\ [0, 4], & x \in \mathbb{R} \setminus A, \end{cases}$$

with some set $A \subset \mathbb{R}$ which is not $\mathcal{L}$ -measurable (i.e. Lebesgue measurable).

F is neither $\mathcal{L}$ -measurable nor weakly $\mathcal{L}$ -measurable, but F is weakly K - $\mathcal{L}$ -measurable and K - $\mathcal{L}$ -measurable.

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$\mathcal{A}$ -measurability implies weak $\mathcal{A}$ -measurability provided Y is perfectly normal.

For $Y = \mathbb{R}^n$:

Proposition

*Let $(X, \mathcal{A})$ be a measurable space and $K \subset \mathbb{R}^n$ be a **closed** convex cone. If an s.v. map $F: X \rightarrow n(\mathbb{R}^n)$ is K - $\mathcal{A}$ -measurable, then it is weakly K - $\mathcal{A}$ -measurable.*

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K -continuity in the Kuratowski sense

Let X be a topological space, Y be real topological vector spaces, K be a convex cone in Y , and $x_0 \in X$. An s.v. map $F: X \rightarrow n(Y)$ is called:

- K -upper semicontinuous at x_0 , if for every nonempty open set $V \subset Y$ such that $x_0 \in F^+(V + K)$ there is a neighbourhood U_{x_0} of x_0 satisfying $U_{x_0} \subset F^+(V + K)$,
- K -lower semicontinuous at x_0 , if for every nonempty open set $V \subset Y$ such that $x_0 \in F^-(V - K)$ there is a neighbourhood U_{x_0} of x_0 satisfying $U_{x_0} \subset F^-(V - K)$.

Theorem

Let $(X, \mathcal{A})$ be a measurable Borel space (i.e. $\mathcal{A}$ contains all open sets). Let Y be a real topological vector space, K be a convex cone and $F: X \rightarrow n(Y)$.

- If F is K -lower semicontinuous on X in the Kuratowski sense, then F is weakly K - $\mathcal{A}$ -measurable.
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K -continuity in the Nikodem sense

Let X be a topological space, Y be a real topological vector space, K be a convex cone in Y , and $x_0 \in X$. An s.v. map $F: X \rightarrow n(Y)$ is called:

- K -upper semicontinuous at x_0 , if for every neighbourhood $W_0 \subset Y$ of 0 there is a neighbourhood U_{x_0} of x_0 such that

$$F(x) \subset F(x_0) + W_0 + K \quad \text{for } x \in U_{x_0},$$

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Corollary

Let $(X, \mathcal{A})$ be a measurable Borel space, Y be a real topological vector space, K be a convex cone and $F: X \rightarrow n(Y)$.

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K -continuity vs. [weak] K - $\mathcal{A}$ measurability

Clearly, there are s.v. maps which are [weakly] K - $\mathcal{A}$ -measurable, but not K -continuous.

It is enough to take $X = Y = \mathbb{R}$, $K = [0, \infty)$, $\mathcal{A} = \mathcal{L}$, and $F(x) = [m(x), \infty)$ for $x \in \mathbb{R}$, where $m: \mathbb{R} \rightarrow \mathbb{R}$ is a Lebesgue measurable discontinuous function.

a composition of a continuous function with a measurable function

For nonempty sets X, Y, Z define a composition $G \circ F: Z \rightarrow n(Y)$ of s.v. maps $F: Z \rightarrow n(X)$ and $G: X \rightarrow n(Y)$ in the following way:

$$(G \circ F)(z) = \bigcup_{x \in F(z)} G(x), \quad z \in Z.$$

Proposition

Let $(Z, \mathcal{A})$ be a measurable space, X, Y be real topological vector spaces, and K be a convex cone in Y . Assume that $F: Z \rightarrow n(X)$ and $G: X \rightarrow n(Y)$.

- (i) If F is weakly $\mathcal{A}$ -measurable and G is K -lower semicontinuous on X in the Kuratowski sense, then $G \circ F: Z \rightarrow n(Y)$ is weakly K - $\mathcal{A}$ -measurable.*
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Properties of [weakly] K - $\mathcal{A}$ -measurable s.v. maps

Let $(X, \mathcal{A})$ be a measurable space, Y be a real topological vector space and $K \subset Y$ be a convex cone. If $F: X \rightarrow n(Y)$ is [weakly] K - $\mathcal{A}$ -measurable, then

- (provided that X is a real topological vector space) for any nonempty open set $D \subset X$, $x_0 \in D$, and $y_0 \in Y$

$$F_{x_0, y_0}(x) := F(x + x_0) - y_0, \quad x \in D_0 := D - x_0,$$

is also [weakly] K - $\mathcal{A}$ -measurable.

- for any $t \geq 0$

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Properties of [weakly] K - $\mathcal{A}$ -measurable s.v. maps

Let $(X, \mathcal{A})$ be a measurable space, Y be a real topological vector space and $K \subset Y$ be a convex cone. If $F: X \rightarrow n(Y)$ is [weakly] K - $\mathcal{A}$ -measurable, then

- (provided that X is a real topological vector space) for any nonempty open set $D \subset X$, $x_0 \in D$, and $y_0 \in Y$

$$F_{x_0, y_0}(x) := F(x + x_0) - y_0, \quad x \in D_0 := D - x_0,$$

is also [weakly] K - $\mathcal{A}$ -measurable.

- for any $t \geq 0$

$$(tF)(x) := tF(x), \quad x \in X,$$

is also [weakly] K - $\mathcal{A}$ -measurable.



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Properties of [weakly] K - $\mathcal{A}$ -measurable s.v. maps

Let $(X, \mathcal{A})$ be a measurable space, Y be a real topological vector space and $K \subset Y$ be a convex cone.

- If $F_n: X \rightarrow n(Y)$ are [weakly] K - $\mathcal{A}$ -measurable for every $n \in \mathbb{N}$, then

$$\left(\bigcup_{n \in \mathbb{N}} F_n \right)(x) := \bigcup_{n \in \mathbb{N}} F_n(x), \quad x \in X,$$

is also [weakly] K - $\mathcal{A}$ -measurable.

- If $F_n: X \rightarrow n(Y)$ are **weakly** K - $\mathcal{A}$ -measurable for every $n \in \mathbb{N}$, then, under assumption that Y is second countable,

$$\left(\prod_{n \in \mathbb{N}} F_n \right)(x) := \prod_{n \in \mathbb{N}} F_n(x), \quad x \in X,$$

is **weakly** $(\prod_{n \in \mathbb{N}} K)$ - $\mathcal{A}$ -measurable.

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Contrary to the case when $K = \{0\}$, the intersection

$$(F \cap G)(x) := F(x) \cap G(x), \quad x \in X,$$

of two [weakly] K - $\mathcal{A}$ -measurable s.v. maps defined on X needn't be [weakly] K - $\mathcal{A}$ -measurable even if F, G have compact values.

Example

Let $X = Y = \mathbb{R}$ and $K = [0, \infty)$. For a set $A \subset \mathbb{R}$ such that $A \cap (-4, -2] \notin \mathcal{L}$ define $F, G: \mathbb{R} \rightarrow n(\mathbb{R})$ as follows:

$$F(x) = \begin{cases} [x, x+1] \cup \{x+4\}, & x \in A, \\ [x, x+3], & x \in \mathbb{R} \setminus A, \end{cases}$$

$$G(x) = \begin{cases} [x+2, x+5], & x \in A, \\ [x+2, x+4], & x \in \mathbb{R} \setminus A. \end{cases}$$

Then, F, G are [weakly] K - $\mathcal{L}$ -measurable. But

$$(F \cap G)(x) = \begin{cases} \{x+4\}, & x \in A, \\ [x+2, x+3], & x \in \mathbb{R} \setminus A, \end{cases}$$

is neither weakly K - $\mathcal{L}$ -measurable nor K - $\mathcal{L}$ -measurable.

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- Let $(X, \mathcal{A})$ be a measurable space, Y be a real topological vector space, and $K \subset Y$ be a convex cone. An s.v. map $F: X \rightarrow n(Y)$ is **weakly** K - $\mathcal{A}$ -measurable if and only if

$$(\text{cl } F)(x) := \text{cl } F(x), \quad x \in X,$$

is also **weakly** K - $\mathcal{A}$ -measurable.

- The interior

$$(\text{int } F)(x) := \text{int } F(x), \quad x \in X,$$

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$$F(x) = \begin{cases} \{x+1\} \cup [x+4, x+5], & x \in A, \\ \{x+1\} \cup [x+3, x+4], & x \in \mathbb{R} \setminus A. \end{cases}$$

Then, F is [weakly] K - $\mathcal{L}$ -measurable. But,

$$(\text{int } F)(x) = \begin{cases} (x+4, x+5), & x \in A, \\ (x+3, x+4), & x \in \mathbb{R} \setminus A \end{cases}$$

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$$F(x) = \begin{cases} \{x+1\} \cup [x+4, x+5], & x \in A, \\ \{x+1\} \cup [x+3, x+4], & x \in \mathbb{R} \setminus A. \end{cases}$$

Then, F is [weakly] K - $\mathcal{L}$ -measurable. But,

$$(\text{int } F)(x) = \begin{cases} (x+4, x+5), & x \in A, \\ (x+3, x+4), & x \in \mathbb{R} \setminus A \end{cases}$$

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Contrary to the case when $K = \{0\}$, K - $\mathcal{A}$ -measurability of F does not imply [weakly] K - $\mathcal{A}$ -measurability either of

$$(Y \setminus F)(x) := Y \setminus F(x) \quad \text{or} \quad (\text{bd } F)(x) := \text{bd } F(x), \quad x \in X.$$

Example

Let $X = Y = \mathbb{R}$ and $K = [0, \infty)$. For a set $A \subset \mathbb{R}$ such that $A \cap (-1, 0] \notin \mathcal{L}$, define $F: \mathbb{R} \rightarrow n(\mathbb{R})$ as follows:

$$F(x) = \begin{cases} (-\infty, x+1], & x \in A, \\ (-\infty, x], & x \in \mathbb{R} \setminus A. \end{cases}$$

Then F is [weakly] K - $\mathcal{L}$ -measurable. But,

$$(\mathbb{R} \setminus F)(x) = \begin{cases} (x+1, \infty), & x \in A, \\ (x, \infty), & x \in \mathbb{R} \setminus A, \end{cases}$$

and

$$(\text{bd } F)(x) = \begin{cases} \{x+1\}, & x \in A, \\ \{x\}, & x \in \mathbb{R} \setminus A, \end{cases}$$

are neither weakly K - $\mathcal{L}$ -measurable nor K - $\mathcal{L}$ -measurable.

Denote by $\mathcal{B}(Y)$ the family of all nonempty bounded sets in a real topological vector space Y .

- Let $(X, \mathcal{A})$ be a measurable space, Y be a real topological vector space and $K \subset Y$ be a non-trivial convex cone (i.e. $K \neq \{0\}$).
If $F: X \rightarrow \mathcal{B}(Y)$, then $Y \setminus F$ is [weakly] K - $\mathcal{A}$ -measurable.

Applications of K -measurability

- inspired by K. Nikodem, *K -convex and K -concave set-valued functions* (1989).

Assume that X, Y are real topological vector spaces, $D \neq \emptyset$ is an open convex subset of X and K is a convex cone in Y . An s.v. map $F: X \rightarrow n(Y)$ is called:

- K -midconvex on D , if

$$\frac{F(x) + F(y)}{2} \subset F\left(\frac{x+y}{2}\right) + K \quad \text{for } x, y \in D,$$

- K -midconcave on D , if

$$F\left(\frac{x+y}{2}\right) \subset \frac{F(x) + F(y)}{2} + K \quad \text{for } x, y \in D.$$

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Applications of K -measurability - inspired by K. Nikodem

Theorem

Settings:

Let a real topological vector space X with the family $\mathcal{A}$ be a measurable space, Y be a real topological vector space, K be a convex cone in Y , $D \neq \emptyset$ be an open convex subset of X , and $A \subset D$. Let one of the following conditions holds:

- (a) X is a real Polish vector space, $\text{int } A \neq \emptyset$, and $\mathcal{A} = \mathcal{C}$ is the family of all Christensen measurable sets,
- (b) $X = \mathbb{R}^n$, A is a set of positive Lebesgue measure, and $\mathcal{A} = \mathcal{L}$ is the family of all Lebesgue measurable sets,
- (c) X is a real topological vector space, A is of the second category having the Baire property, and $\mathcal{A} = \mathcal{B}_a$ is the family of all sets with the Baire property.

Applications of K -measurability - inspired by K. Nikodem

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Applications of K -measurability - inspired by K. Nikodem

Theorem

The main result:

If $F, G: D \rightarrow \mathcal{B}(Y)$ satisfy one of the following two conditions:

- (i) F is K -midconvex, G is weakly K - $\mathcal{A}$ -measurable, and $G(x) \subset F(x) + K$ for all $x \in A$,*
- (ii) F is convex-valued and K -midconcave, G is K - $\mathcal{A}$ -measurable, and $F(x) \subset G(x) + K$ for all $x \in A$,*

then F is K -continuous in the Nikodem sense on D .

Corollary

Let X be a real topological vector space with the family $\mathcal{A}$ be a measurable space, Y be a real topological vector space, K be a convex cone in Y , and $D \neq \emptyset$ be an open convex subset of X . Let one of the following conditions holds:

- (a) X is a real Polish vector space and $\mathcal{A} = \mathcal{C}$,*
- (b) $X = \mathbb{R}^n$ and $\mathcal{A} = \mathcal{L}$,*
- (c) X is a real topological vector space and $\mathcal{A} = \mathcal{B}_a$.*

If $F: D \rightarrow \mathcal{B}(Y)$ satisfies one of the following two conditions:

- (i) F is K -midconvex and weakly K - $\mathcal{A}$ -measurable,*
- (ii) F is convex-valued K -midconcave and K - $\mathcal{A}$ -measurable,*

then F is K -continuous in the Nikodem sense on D .

Corollary

Let X be a real topological vector space with the family $\mathcal{A}$ be a measurable space, Y be a real topological vector space, K be a convex cone in Y , and $D \neq \emptyset$ be an open convex subset of X . Let one of the following conditions holds:

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If $F: D \rightarrow \mathcal{B}(Y)$ satisfies one of the following two conditions:

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