

On some generalized topologies satisfying all separation axioms

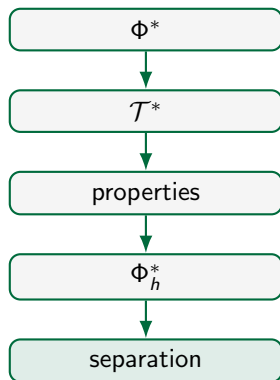
Jacek Hejduk Anna Loranty

Faculty of Mathematics and Computer Science,
Łódź University, POLAND

40th International Summer Conference on Real Functions Theory
Stará Lesná 2026

Presentation outline

1. Generalized topologies and density-type operators
2. Construction of the family $\mathcal{T}^*$
3. Properties: interior, closure and boundary
4. Passage to the operator Φ_h^* through measurable hulls
5. Separation axioms and examples on $\mathbb{R}$



J. Hejduk, A. Loranty *On some generalized topologies satisfying all separation axioms*, Results Math, 2024.

A generalized topology

Definition

Let $X \neq \emptyset$. A family $\Gamma \subseteq 2^X$ is called a **generalized topology** if

$$\emptyset \in \Gamma \quad \text{and} \quad \bigcup_{t \in T} G_t \in \Gamma \quad \text{whenever } G_t \in \Gamma.$$

If $X \in \Gamma$, then (X, Γ) is called a **strong generalized topological space**.

Á. Császár *Generalized open sets*, Acta Mathematica Hungarica 75, 1997, p. 65–87.

Á. Császár *Generalized topologies, generalized continuity* Acta Mathematica Hungarica 96(4), 2002, p. 351–357.

Á. Császár *Separation axioms for generalized topologies*, Acta Mathematica Hungarica 104, 2004, p. 63–69.

Á. Császár *Normal generalized topologies*, Acta Mathematica Hungarica 115, 2007, p. 309–313.

G. E. Xun, G. E. Ying, *μ -Separations in generalized topological spaces*, Applied Mathematic 25(2), 2010, p. 243–252.

An operator Φ^*

Let X be a non-empty set and $\mathcal{J}^* \subset 2^X$ is an admissible σ -ideal of sets. Let's us consider an operator

$$\Phi^* : 2^X \rightarrow 2^X$$

satisfies the following conditions:

- 1* $\Phi^*(\emptyset) = \emptyset, \Phi^*(X) = X,$
- 2* if $A \subseteq B$ then $\Phi^*(A) \subseteq \Phi^*(B)$ for any $A, B \subset X,$
- 3* if $A \triangle B \in \mathcal{J}^*$ then $\Phi^*(A) = \Phi^*(B)$ for any $A, B \subset X,$
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Example 1

It is easy to see that the operator $\Phi_1^* : 2^X \rightarrow 2^X$ defined in the following way: $\Phi_1^*(A) = X$ if $A \notin \mathcal{J}^*$ and $\Phi_1^*(A) = \emptyset$ if $A \in \mathcal{J}^*$ satisfies properties 1*-4*.

Example 2

Let λ^* denote the outer Lebesgue measure on $\mathbb{R}$. A point $x_0 \in \mathbb{R}$ is an outer density point of a set $A \subset \mathbb{R}$ iff

$$\lim_{h \rightarrow 0^+} \frac{\lambda^*(A \cap [x_0 - h, x_0 + h])}{2h} = 1.$$

One can easily show that the operator $\Phi_d^* : 2^{\mathbb{R}} \rightarrow 2^{\mathbb{R}}$ given by the formula

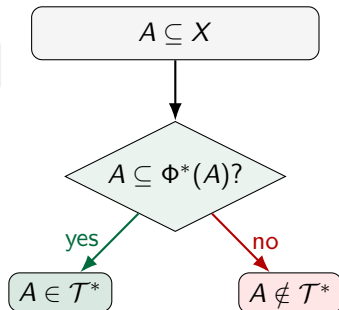
$$\Phi_d^*(A) = \{x \in \mathbb{R} : x \text{ is an outer density point of } A\}$$

for $A \subset \mathbb{R}$ satisfies properties 1*-4*.

Construction of the generalized topology $\mathcal{T}^*$

$$\mathcal{T}^* = \{A \subseteq X : A \subseteq \Phi^*(A)\}.$$

- $\mathcal{T}^*$ is a strong generalized topology.
- In general, $\mathcal{T}^* \neq 2^X$.



Elementary properties of $\mathcal{T}^*$

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- For every $A \subseteq X$, the set $A \cap \Phi^*(A)$ belongs to $\mathcal{T}^*$.
- The smallest σ -algebra $\sigma(\mathcal{T}^* \cup \mathcal{J}^*)$ generated by the family $\mathcal{T}^* \cup \mathcal{J}^*$ is equal to 2^X .

Interior, closure and boundary

Structural formulas

$$\text{int}_{\mathcal{T}^*}(A) = A \cap \Phi^*(A),$$

$$\text{cl}_{\mathcal{T}^*}(A) = A \cup (X \setminus \Phi^*(X \setminus A)).$$

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- $\text{int}_{\mathcal{T}^*}(A) = \emptyset$ iff $A \in \mathcal{J}^*$;

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- $\text{Fr}_{\mathcal{T}^*}(A) \in \mathcal{J}^*$.

Nowhere dense and strongly nowhere dense sets in GT

Let (X, Γ) be a generalized topological space.

Definition

We say that a set $A \subset X$ is Γ -nowhere dense if $\text{int}_\Gamma(\text{cl}_\Gamma(A)) = \emptyset$.

Definition

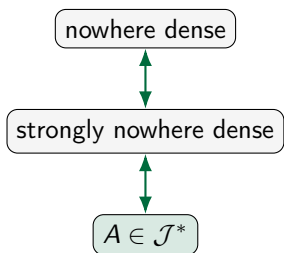
A set $A \subset X$ is Γ -strongly nowhere dense if for $V \in \Gamma \setminus \{\emptyset\}$ there exists $U \in \Gamma \setminus \{\emptyset\}$ such that $U \subset V$ and $A \cap U = \emptyset$.

It is easy to see that if A is Γ -strongly nowhere dense then it is Γ -nowhere dense. The converse theorem is not true in general.

In the space $(X, \mathcal{T}^*)$ the following conditions are equivalent:

1. A is strongly nowhere dense,
2. A is nowhere dense,
3. $A \in \mathcal{J}^*$.

Consequently, the families of nowhere dense sets, strongly nowhere dense sets, meager sets and strongly meager sets all coincide with $\mathcal{J}^*$.



Compactness and countability

Global properties of the space $(X, \mathcal{T}^*)$

- A set A is $\mathcal{T}^*$ -compact if and only if A is finite.

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- The space is not separable.

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Global properties of the space $(X, \mathcal{T}^*)$

- A set A is $\mathcal{T}^*$ -compact if and only if A is finite.
- The space does not satisfy the first countability axiom.
- The space does not satisfy the second countability axiom.
- The space is not separable.
- If $\mathcal{J}$ contains an uncountable set then $\langle X, \mathcal{T}^* \rangle$ is not a Lindelöf space.

From now, we will consider a measurable space $\langle X, \mathcal{S}, \mathcal{J} \rangle$, where $\mathcal{S}$ is a σ -algebra of subsets of X and $\mathcal{J} \subset \mathcal{S}$ is a admissible σ -ideal.

Let $\Phi : \mathcal{S} \rightarrow 2^X$ be any operator satisfying the following properties:

- 1° $\Phi(\emptyset) = \emptyset$ and $\Phi(X) = X$,
- 2° for any $A, B \in \mathcal{S}$ if $A \subset B$ then $\Phi(A) \subset \Phi(B)$,
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Definition

A measurable hull of a set $A \subset X$ is any set $B \in \mathcal{S}$ such that $A \subset B$ and for any $C \subset B \setminus A$ if $C \in \mathcal{S}$ then $C \in \mathcal{J}$. The set B described above is called an $\mathcal{S}$ -measurable hull of a set A .

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Remark

It is easy to see that if B_1 and B_2 are $\mathcal{S}$ -measurable hulls of a set $A \subset X$ then $B_1 \triangle B_2 \in \mathcal{J}$, so $\Phi(B_1) = \Phi(B_2)$.

Measurable hulls and the operator Φ_h^*

Let $(X, \mathcal{S}, \mathcal{I})$ be a measurable space with an admissible σ -ideal.

Let $\Phi : \mathcal{S} \rightarrow 2^X$ satisfies conditions $1^\circ - 4^\circ$.

Let every $A \subseteq X$ has an $\mathcal{S}$ -measurable hull B .

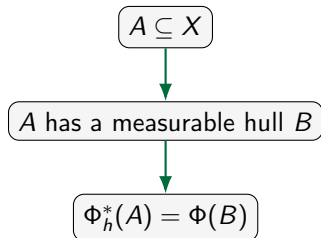
Define

$$\Phi_h^*(A) = \Phi(B),$$

where B is a measurable hull of A .

Property

The operator $\Phi^* : 2^X \rightarrow 2^X$ satisfies conditions $1^* - 4^*$.



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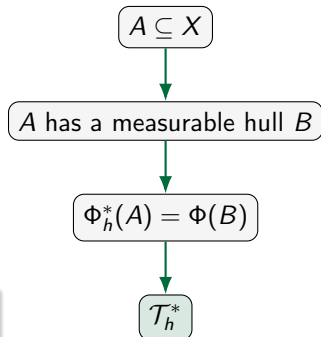
where B is a measurable hull of A .

Property

The operator $\Phi^* : 2^X \rightarrow 2^X$ satisfies conditions $1^* - 4^*$.

Hence

$\mathcal{T}_h^* = \{A \subseteq X : A \subseteq \Phi_h^*(A)\}$ is a strong generalized topology.



The non-separation property

Key assumption

A space $(X, \mathcal{S}, \mathcal{J})$ has the **non-separation property** with respect to $\mathcal{S}$ if for every $E \notin \mathcal{J}$ there exist disjoint subsets of E which cannot be separated by sets from $\mathcal{S}$.

The non-separation property

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Due to Sierpiński and Luzin (1934) we have

The space $(\mathbb{R}, \mathbb{L}, \mathcal{L})$ has the non-separation property.

The space $(\mathbb{R}, Ba, \mathcal{I})$ has the non-separation property.

The non-separation property

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Theorem (Grzegorek, Labuda (2019) [4])

Let $\langle X, \mathcal{S}, \mathcal{J} \rangle$ have a non-separation property with respect to $\mathcal{S}$ and satisfy CCC. Every infinite set $C \subset X$ can be decomposed into an infinite family of sets such that a hull of C is a hull of each set from this family. In particular, any set $C \notin \mathcal{J}$ can be decomposed into subsets C_1 and C_2 such that a hull of C is simultaneously a hull of C_1 and C_2 .

Total disconnectedness

Theorem

If $(X, \mathcal{S}, \mathcal{J})$ has the non-separation property with respect to $\mathcal{S}$ and satisfies the countable chain condition, then the generalized topological space

$$(X, \mathcal{T}_h^*)$$

is totally disconnected.

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The space $(X, \mathcal{T}_h^*)$ is Hausdorff.

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The space $(X, \mathcal{T}_h^*)$ is completely regular.

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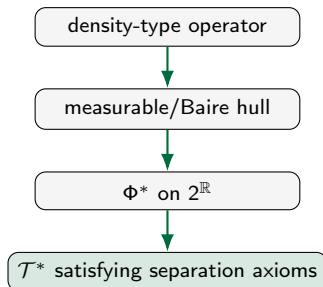
Theorem

The space $(X, \mathcal{T}_h^*)$ is perfectly normal.

Examples

The final part of the paper shows that the abstract construction is realized by natural operators on $\mathbb{R}$:

- outer Lebesgue density operators,
- operators defined by $\lim$, $\limsup$ and $\liminf$ of local measure ratios,
- threshold versions with parameters $\alpha \in [0, 1)$,
- lower density operators related to the Baire property and the ideal of meager sets.



Example: an outer density on $\mathbb{R}$

Let λ be the Lebesgue measure and λ^* be the outer Lebesgue measure.

Lebesgue density type operator

For any Lebesgue measurable set $B \subseteq \mathbb{R}$ define

$$\Phi_0(B) = \left\{ x \in \mathbb{R} : \lim_{h \rightarrow 0^+} \frac{\lambda(B \cap [x - h, x + h])}{2h} = 1 \right\}.$$

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Let $A \subseteq \mathbb{R}$ and B be a Lebesgue measurable hull of A

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For $A \subseteq \mathbb{R}$

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Example: an outer density on $\mathbb{R}$

Let λ be the Lebesgue measure and λ^* be the outer Lebesgue measure.

Lebesgue upper density type operator

For any Lebesgue measurable set $B \subseteq \mathbb{R}$ define

$$\Phi_1(B) = \left\{ x \in \mathbb{R} : \limsup_{h \rightarrow 0^+} \frac{\lambda(B \cap [x - h, x + h])}{2h} = 1 \right\}.$$

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Theorem

Let $i \in \{0, 1\}$. The operator Φ_i^* satisfies conditions 1*-4* and the family

$$\mathcal{T}_i^* = \{A \subset \mathbb{R} : A \subset \Phi_i^*(A)\}$$

is a strong generalized topology on $\mathbb{R}$ containing the classical density topology $\mathcal{T}_d$ and the space $\langle \mathbb{R}, \mathcal{T}_i^* \rangle$ satisfies all separation axioms.

Let λ be the Lebesgue measure.

For $\alpha \in [0, 1)$ and any Lebesgue measurable set $B \subseteq \mathbb{R}$ define:

$$\Phi_{1\alpha}(B) = \left\{ x : \lim_{h \rightarrow 0^+} \frac{\lambda(B \cap [x - h, x + h])}{2h} > \alpha \right\},$$

$$\Phi_{2\alpha}(B) = \left\{ x : \liminf_{h \rightarrow 0^+} \frac{\lambda(B \cap [x - h, x + h])}{2h} > \alpha \right\},$$

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Let $A \subseteq \mathbb{R}$ and B be a Lebesgue measurable hull of A

$$\Phi_{i\alpha}^*(A) = \Phi_{i\alpha}(B),$$

for $i = 1, 2, 3$.

Let λ^* be the outer Lebesgue measure.

For $\alpha \in [0, 1)$ and any set $A \subseteq \mathbb{R}$:

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Theorem

For $i \in \{1, 2, 3\}$ the operator $\Phi_{i\alpha}^*$ satisfies conditions 1*-4* and the generalized topological space

$$(\mathbb{R}, \mathcal{T}_{i\alpha}^*)$$

satisfies all separation axioms.

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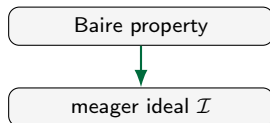
Remark

The analogous conclusion holds for the variants with $\geq \alpha$.

Example: a category-type operators

The final example moves from measure to category.

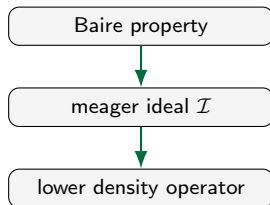
- Let Ba be the σ -algebra of sets with the Baire property.
- Let $\mathcal{I}$ be the σ -ideal of meager sets.



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- Let Ba be the σ -algebra of sets with the Baire property.
- Let $\mathcal{I}$ be the σ -ideal of meager sets.
- Let $\Phi : Ba \rightarrow Ba$ be a lower density operator (the $\mathcal{I}$ -density operator, simple density operator $\Phi_{s\mathcal{I}}$, density operator $\Phi_{\mathcal{I}(J)}$, $\langle s \rangle$ -density operator $\Phi_{\mathcal{I}\langle s \rangle}$.)

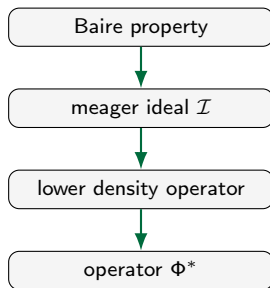


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- If B is a Ba -measurable hull of $A \subset \mathbb{R}$, define

$$\Phi_h^*(A) = \Phi(B).$$

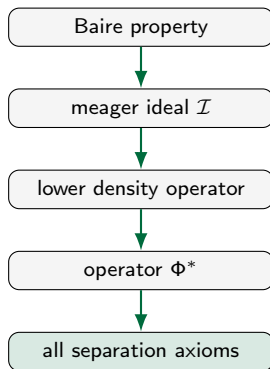


Example: a category-type operators

The final example moves from measure to category.

- Let Ba be the σ -algebra of sets with the Baire property.
- Let $\mathcal{I}$ be the σ -ideal of meager sets.
- Let $\Phi : Ba \rightarrow Ba$ be a lower density operator (the $\mathcal{I}$ -density operator, simple density operator $\Phi_{s\mathcal{I}}$, density operator $\Phi_{\mathcal{I}(J)}$, $\langle s \rangle$ -density operator $\Phi_{\mathcal{I}\langle s \rangle}$.)
- If B is a Ba -measurable hull of $A \subset \mathbb{R}$, define

$$\Phi_h^*(A) = \Phi(B).$$



Theorem

The operator Φ_h^* satisfies conditions 1*-4* and the generalized topological space $(\mathbb{R}, \mathcal{T}_h^*)$ satisfies all separation axioms.

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Thank you for your attention