

Cardinals Inspired by Convergence of Sequences of Functions

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Quasinormal convergence of sequences of functions

X – topological normal space
 $f, f_n : X \rightarrow \mathbb{R}$ – continuous functions for $n \in \omega = \mathbb{N}$

Pointwise convergence

$(f_n) \xrightarrow{P} f$ if $\forall \epsilon > 0 \quad \forall x \in X (\{n : |f_n(x) - f(x)| \geq \epsilon\} \in \text{Fin})$

Quasinormal convergence

$(f_n) \xrightarrow{QN} f$ if $\exists (\epsilon_n) \rightarrow 0^+ \forall x \in X (\{n : |f_n(x) - f(x)| \geq \epsilon_n\} \in \text{Fin})$

Relationship between pointwise and quasinormal convergence

- $(f_n) \xrightarrow{QN} f \implies (f_n) \xrightarrow{P} f$ [always “yes”]
- $(f_n) \xrightarrow{QN} f \not\Leftarrow (f_n) \xrightarrow{P} f$ [in general “no”]

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Definition (Bukovský-Reclaw-Repický, 1991)

X is a **QN-space** if X **does not distinguishes** *pointwise* and *quasinormal* convergence i.e. for any sequence $f_n : X \rightarrow \mathbb{R}$ of continuous functions,

$$(f_n) \xrightarrow{P} f \implies (f_n) \xrightarrow{QN} f.$$

Definition (Bukovský-Reclaw-Repický, 1991)

$$\text{non}(\text{QN-space}) = \min \{ |X| : X \text{ in } \text{not a QN-space} \}$$

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Purely combinatorial characterization of non(QN-space)

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Definition (the **bounding number**)

$$\begin{aligned} \mathfrak{b} &= \mathfrak{b}(\omega^\omega, \leq^*) \\ &= \min \left\{ |\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \neg \left(\exists g \in \omega^\omega \forall f \in \mathcal{F} (f \leq^* g) \right) \right\} \end{aligned}$$

where $f \leq^* g \iff \{n : \neg(f(n) \leq g(n))\} \in \text{Fin}$

Theorem

$$\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$$

Theorem (Bukovska ($\geq$), Bukovský-Reclaw-Repický ($\leq$), 1991)

$$\text{non}(\text{QN-space}) = \mathfrak{b}$$

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Definition

A family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an **ideal** on ω if

- 1 $\emptyset \in \mathcal{I}$ and $\omega \notin \mathcal{I}$,
- 2 $A \subseteq B \in \mathcal{I} \implies A \in \mathcal{I}$,
- 3 $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$,
- 4 $\mathcal{I}$ contains all finite subsets of ω .

Examples

- 1 **Fin** = $\{A \subseteq \omega : A \text{ is finite}\}$
- 2 $\mathcal{I}_{1/n} = \left\{ A \subseteq \omega : \sum_{n \in A} \frac{1}{n} < \infty \right\}$ (the summable ideal)
- 3 $\mathcal{Z} = \left\{ A \subseteq \omega : \lim_{n \rightarrow \infty} \frac{|A \cap \{1, \dots, n\}|}{n} = 0 \right\}$ (the density zero ideal)
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Topological properties of ideals on ω

Definition

An **ideal** $\mathcal{I}$ is **Borel** if the set

$$\{\mathbf{1}_A : A \in \mathcal{I}\}$$

is a *Borel* subset of the **Cantor space** $2^\omega = \{0, 1\}^\omega$.

The same for F_σ , $F_{\sigma\delta}$, **analytic**, the **Baire property** and other topological properties.

Examples

- 1 Fin and $\mathcal{I}_{1/n}$ are F_σ ideal
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Ideal convergence of sequences of functions

Let $\mathcal{I}$ be an ideal on ω .

Ideal convergence of a sequence of reals (x_n)

$(x_n) \rightarrow x$ if $\forall \varepsilon > 0 (\{n : |x_n - x| \geq \varepsilon\} \in \mathbf{Fin})$

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Ideal pointwise convergence of a sequence of functions

$(f_n) \xrightarrow{P} f$ if $\forall x \in X ((f_n(x)) \rightarrow f(x))$

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$\mathcal{I}$ -QN-spaces and non($\mathcal{I}$ -QN space)

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Definition – the first guess ...

$$\mathfrak{b}(\omega^\omega, \leq^{\mathcal{I}}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \neg(\exists g \in \omega^\omega \forall f \in \mathcal{F} (f \leq^{\mathcal{I}} g))\}$$

where $f \leq^{\mathcal{I}} g \iff \{n : \neg(f(n) \leq g(n))\} \in \mathcal{I}$

Theorem – ... is wrong

$$\text{non}(\mathcal{I}\text{-QN-space}) \neq \mathfrak{b}(\omega^\omega, \leq^{\mathcal{I}})$$

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Definition

$$\text{non}(\mathcal{I}\text{-QN-space}) = \min\{|X| : X \text{ in } \underline{\text{not}} \text{ a } \mathcal{I}\text{-QN-space}\}$$

Definition – the first guess ...

$$\mathfrak{b}(\omega^\omega, \leq^{\mathcal{I}}) = \min \{ |\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \neg(\exists g \in \omega^\omega \forall f \in \mathcal{F} (f \leq^{\mathcal{I}} g)) \}$$

where $f \leq^{\mathcal{I}} g \iff \{n : \neg(f(n) \leq g(n))\} \in \mathcal{I}$

Theorem – ... is wrong

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It does not work, since every family $\mathcal{F}$ is $\geq^{\mathcal{I}}$ -bounded by the constant zero function.

Definition (...and restrict the field of the relation $\geq^{\mathcal{I}}$)

$$\mathcal{D}_{\mathcal{I}} = \{ f \in \omega^\omega : f^{-1}(n) \in \mathcal{I} \text{ for every } n \}$$

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$$\aleph_1 \leq \mathfrak{b}_{QN}(\mathcal{I}) \leq \mathfrak{c}$$

Theorem (Canjar, 1989)

Consistently, there are ideals $\mathcal{I}$ and $\mathcal{J}$ with

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Rothberger gaps

Definition

(κ, λ) -gap in a poset $(P, \leq)$ is a pair $(\mathcal{A}, \mathcal{B})$ of subsets of P such that

- $|\mathcal{A}| = \kappa$ and $|\mathcal{B}| = \lambda$,
- $a \leq b$ for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$,
- there is no element $c \in P$ with $a < c < b$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Theorem (Rothberger, 1941)

$$\mathfrak{b} = \min \{ \lambda : \exists (\omega, \lambda)\text{-gap in } (\mathcal{P}(\omega), \subseteq^*) \}$$

i.e.

- there exists $(\omega, \mathfrak{b})$ -gap in $(\mathcal{P}(\omega), \subseteq^*)$,
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Definition (Brendle-Mejía, 2014)

$$\mathfrak{b}_R(\mathcal{I}) = \min\{\lambda : \exists (\omega, \lambda)\text{-gap in } (\mathcal{P}(\omega), \subseteq^{\mathcal{I}})\},$$

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$$A \subseteq^{\mathcal{I}} B \iff A \setminus B \in \mathcal{I}.$$

- (Rothberger) $\mathfrak{b}_R(\mathbf{Fin}) = \mathfrak{b}$
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Two $\mathfrak{b}$ or not two $\mathfrak{b}$?

Theorem (F.-Kwela, 2026)

$$\mathfrak{b}_{QN}(\mathcal{I}) \leq \mathfrak{b}_R(\mathcal{I})$$

Question: Two $\mathfrak{b}$ or not two $\mathfrak{b}$?

$$\mathfrak{b}_{QN}(\mathcal{I}) \stackrel{?}{=} \mathfrak{b}_R(\mathcal{I})$$

Silly answer

In general, “two $\mathfrak{b}$ ”, because if $\mathcal{I}$ is a maximal ideal,

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Theorem (F.-Kwela, 2026)

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Precisely, $\mathcal{I} = \mathcal{J} \oplus (\mathcal{K} \otimes \{\emptyset\})$ with $\mathfrak{b}_{QN}(\mathcal{J}) = \mathfrak{b} < \mathfrak{b}_{QN}(\mathcal{K})$, where $\mathcal{J}$ and $\mathcal{K}$ exists by Canjar's theorem.

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Two $\mathfrak{b}$ or not two $\mathfrak{b}$ in Borel realm?

Question

$\mathfrak{b}_R(\mathcal{I}) \leq \mathfrak{c}$ for **Borel** ideals?

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$\mathfrak{b}_{QN}(\mathcal{I}) \stackrel{?}{=} \mathfrak{b}_R(\mathcal{I})$ for **Borel** ideals with $\mathfrak{b}_R(\mathcal{I}) \leq \mathfrak{c}$?

Guessing

All examples suggest: **not** two $\mathfrak{b}$.

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Bounds for some classes of Borel ideals

Theorem (Todorčević, 1998)

If $\mathcal{I}$ is F_σ ideal or $F_{\sigma\delta}$ P-ideal,

$$\mathfrak{b}_{QN}(\mathcal{I}) \leq \mathfrak{b}_R(\mathcal{I}) \leq \mathfrak{b}.$$

Theorem (F.-Kwela, 2022)

If $\mathcal{I}$ is ω -diagonalizable by $\mathcal{I}^*$ -universal sets,

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In particular, the above holds for

- Borel weak P-ideals,
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Does the above theorem hold for $\mathfrak{b}_R(\mathcal{I})$?

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- Borel weak P-ideals,
- Π_4^0 ideals (for instance, F_σ , $F_{\sigma\delta}$).

Question

Does the above theorem hold for $\mathfrak{b}_R(\mathcal{I})$?

The value $\mathfrak{b}$ for some Borel ideals

Theorem

- $\mathfrak{b}_R(\mathcal{I}) = \mathfrak{b}$ for **analytic P-ideals** (Brendle-Mejía)
- $\mathfrak{b}_{QN}(\mathcal{I}) = \mathfrak{b}_R(\mathcal{I}) = \mathfrak{b}$ for **meager P-ideals** (F.-Kwela)
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Corollary

$$\mathfrak{b}_{QN}(\{\emptyset\} \otimes \text{Fin}) = \mathfrak{b}_R(\{\emptyset\} \otimes \text{Fin}) = \mathfrak{b}$$

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$$\mathfrak{b}_{QN}(\text{Fin}^\alpha) \stackrel{?}{=} \mathfrak{b}_R(\text{Fin}^\alpha) \stackrel{?}{=} \mathfrak{b} \quad \text{for } \alpha \geq \omega$$



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The value $\aleph_1$ for some Borel ideals

Consistently, $\aleph_1 < \mathfrak{b} = \aleph_2 = \mathfrak{c}$.

Theorem (Brendle-Mejía, 2014)

If $\mathcal{I}$ is an F_σ ideal which is fragmented but not gradually fragmented (e.g. $\mathcal{I} = \mathcal{ED}_{\text{Fin}}$),

$$\mathfrak{b}_{QN}(\mathcal{I}) = \mathfrak{b}_R(\mathcal{I}) = \aleph_1.$$

Theorem (F.-Kwela, 2022)

If $\mathcal{A} \subseteq [\omega]^\omega$ is such that $|\mathcal{A}| \geq \aleph_1$ and $|A \setminus (A_1 \cup \dots \cup A_n)| = \aleph_0$ for each pairwise distinct $A, A_1, \dots, A_n \in \mathcal{A}$,

$$\mathfrak{b}_{QN}(\mathcal{I}(\mathcal{A})) = \aleph_1.$$

For instance, the above holds for

- ideals generated by **uncountable A.D. families** (hence, for **infinite M.A.D.**),
- $\mathcal{ED}$, $\mathcal{I}_b$ (the branching ideal), $\mathcal{S}$ (Solecki ideal).

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Values for some topologically defined Borel ideals

Definition

conv = $\{A \subseteq \mathbb{Q} \cap [0, 1] : \text{cl}(A) \text{ has finitely many limit points}\}$

ctbl = $\{A \subseteq \mathbb{Q} \cap [0, 1] : \text{cl}(A) \text{ is countable}\}$

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Corollary

If $CB_\alpha = \{A \subseteq \mathbb{Q} \cap [0, 1] : (\text{cl}(A))^{(\alpha)} = \emptyset\}$,

- **conv** = $CB_2 \subseteq \dots \subseteq CB_\alpha \subseteq \dots \subseteq$ **ctbl** for $2 \leq \alpha < \omega_1$,
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Ideals defined in category and measure manner

Definition

$$\mathbf{nwd} = \{A \subseteq \mathbb{Q} \cap [0, 1] : \text{cl}(A) \text{ is nowhere dense}\}$$

$$\mathbf{null} = \{A \subseteq \mathbb{Q} \cap [0, 1] : \text{cl}(A) \text{ has measure zero}\}$$

Theorem

- $\mathfrak{b}_R(\mathbf{nwd}) = \text{add}(\mathcal{M})$ (Kankaanpää, 2013)
- $\mathfrak{b}_{QN}(\mathbf{nwd}) = \text{add}(\mathcal{M})$ (Kwela, 2026)

The first guess: $\mathfrak{b}_{QN}(\mathbf{null}) = \mathfrak{b}_R(\mathbf{null}) = \text{add}(\mathcal{N})$ is **wrong**, because

Theorem (F.-Brendle-Cieřlak-Kwela, 202?)

$$\mathfrak{b}_{QN}(\mathbf{null}) = \mathfrak{b}_R(\mathbf{null}) = \text{add}(\mathcal{M})$$

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Ideal convergence of sequences of functions

Ideal pointwise convergence

$$(f_n) \xrightarrow{P} f \text{ if } \forall_{\varepsilon>0} \forall_{x \in X} \quad \exists_k \forall_{n>k} \quad (|f_n(x) - f(x)| < \varepsilon)$$

$$(f_n) \xrightarrow{\mathcal{I}-P} f \text{ if } \forall_{\varepsilon>0} \forall_{x \in X} \quad \exists A \in \mathcal{I} \forall_{n \in \omega \setminus A} \quad (|f_n(x) - f(x)| < \varepsilon)$$

Ideal quasi-normal convergence

$$(f_n) \xrightarrow{QN} f \text{ if } \exists_{(\varepsilon_n) \rightarrow 0} \forall_{x \in X} \quad \exists_k \forall_{n>k} \quad (|f_n(x) - f(x)| < \varepsilon_n)$$

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Ideal uniform convergence

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Ideal σ -uniform convergence

$$(f_n) \xrightarrow{\sigma-U} f \text{ if there is a partition } X_1, X_2, \dots \text{ of } X \text{ such that}$$

Ideal convergence of sequences of functions

Ideal pointwise convergence

$$(f_n) \xrightarrow{\mathcal{I}-P} f \text{ if } \forall_{\varepsilon>0} \forall_{x \in X} \exists_{A \in \mathcal{I}} \forall_{n \in \omega \setminus A} (|f_n(x) - f(x)| < \varepsilon)$$

Ideal quasi-normal convergence

$$(f_n) \xrightarrow{QN} f \text{ if } \exists_{(\varepsilon_n) \rightarrow 0} \forall_{x \in X} \exists_k \forall_{n>k} (|f_n(x) - f(x)| < \varepsilon_n)$$
$$(f_n) \xrightarrow{\mathcal{I}-QN} f \text{ if } \exists_{(\varepsilon_n) \xrightarrow{\mathcal{I}} 0} \forall_{x \in X} \exists_{A \in \mathcal{I}} \forall_{n \in \omega \setminus A} (|f_n(x) - f(x)| < \varepsilon_n)$$

Ideal uniform convergence

$$(f_n) \xrightarrow{U} f \text{ if } \forall_{\varepsilon>0} \exists_k \forall_{n>k} \forall_{x \in X} (|f_n(x) - f(x)| < \varepsilon)$$
$$(f_n) \xrightarrow{\mathcal{I}-U} f \text{ if } \forall_{\varepsilon>0} \exists_{A \in \mathcal{I}} \forall_{n \in \omega \setminus A} \forall_{x \in X} (|f_n(x) - f(x)| < \varepsilon)$$

Ideal σ -uniform convergence

$$(f_n) \xrightarrow{\sigma-U} f \text{ if there is a partition } X_1, X_2, \dots \text{ of } X \text{ such that}$$
$$\forall_{k \in \mathbb{N}} (f_n \upharpoonright X_k) \xrightarrow{U} f$$

Ideal convergence of sequences of functions

Ideal pointwise convergence

$$(f_n) \xrightarrow{\mathcal{I}-P} f \text{ if } \forall_{\varepsilon>0} \forall_{x \in X} \exists_{A \in \mathcal{I}} \forall_{n \in \omega \setminus A} (|f_n(x) - f(x)| < \varepsilon)$$

Ideal quasi-normal convergence

$$(f_n) \xrightarrow{\mathcal{I}-QN} f \text{ if } \exists_{(\varepsilon_n) \xrightarrow{\mathcal{I}} 0} \forall_{x \in X} \exists_{A \in \mathcal{I}} \forall_{n \in \omega \setminus A} (|f_n(x) - f(x)| < \varepsilon_n)$$

Ideal uniform convergence

$$(f_n) \xrightarrow{U} f \text{ if } \forall_{\varepsilon>0} \exists_k \forall_{n>k} \forall_{x \in X} (|f_n(x) - f(x)| < \varepsilon)$$
$$(f_n) \xrightarrow{\mathcal{I}-U} f \text{ if } \forall_{\varepsilon>0} \exists_{A \in \mathcal{I}} \forall_{n \in \omega \setminus A} \forall_{x \in X} (|f_n(x) - f(x)| < \varepsilon)$$

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Ideal convergence of sequences of functions

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$$(f_n) \xrightarrow{\mathcal{I}-P} f \text{ if } \forall_{\varepsilon > 0} \forall_{x \in X} \exists_{A \in \mathcal{I}} \forall_{n \in \omega \setminus A} (|f_n(x) - f(x)| < \varepsilon)$$

Ideal quasi-normal convergence

$$(f_n) \xrightarrow{\mathcal{I}-QN} f \text{ if } \exists_{(\varepsilon_n) \xrightarrow{\mathcal{I}} 0} \forall_{x \in X} \exists_{A \in \mathcal{I}} \forall_{n \in \omega \setminus A} (|f_n(x) - f(x)| < \varepsilon_n)$$

Ideal uniform convergence

$$(f_n) \xrightarrow{\mathcal{I}-U} f \text{ if } \forall_{\varepsilon > 0} \exists_{A \in \mathcal{I}} \forall_{n \in \omega \setminus A} \forall_{x \in X} (|f_n(x) - f(x)| < \varepsilon)$$

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$$\forall_{k \in \mathbb{N}} (f_n \upharpoonright X_k) \xrightarrow{\mathcal{I}-U} f$$

Diagram of relationships between these convergences

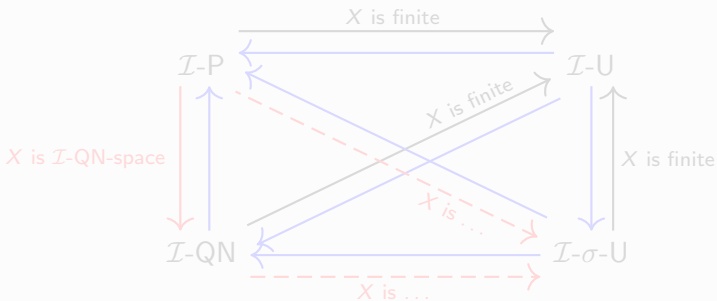
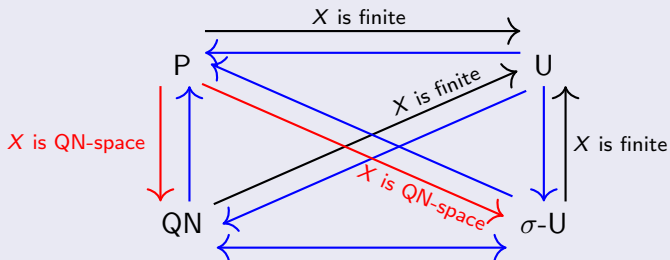
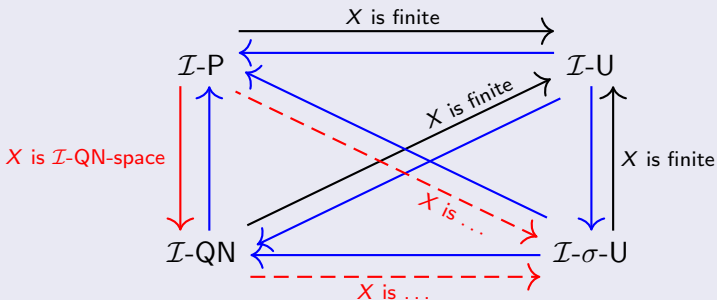
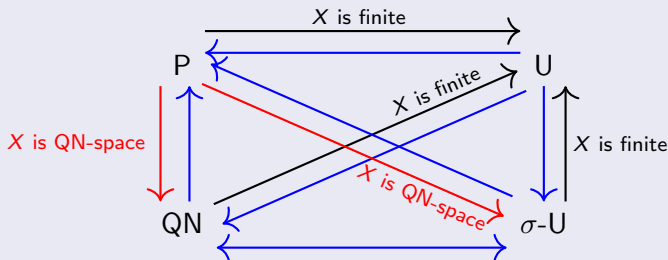
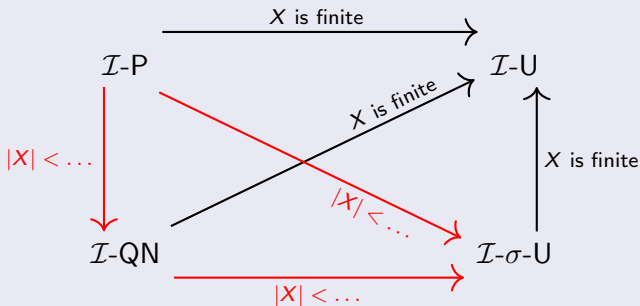


Diagram of relationships between these convergences



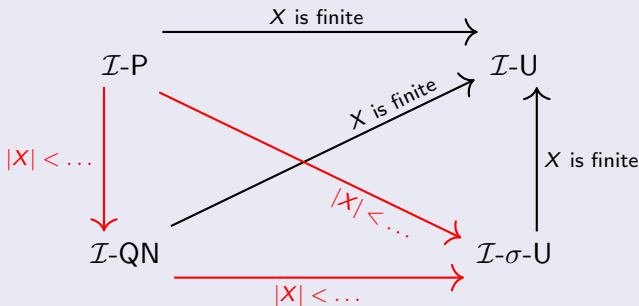
Minimal size of a space which distinguishes convergence



Definition

- $\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-QN}) = \min\{|X| : \exists f_n: X \rightarrow \mathbb{R} (f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \wedge f_n \not\xrightarrow{\mathcal{I}\text{-QN}} 0)\}$
- $\text{non}(\mathcal{I}\text{-QN}, \mathcal{I}\text{-}\sigma\text{-U}) = \min\{|X| : \exists f_n: X \rightarrow \mathbb{R} (f_n \xrightarrow{\mathcal{I}\text{-QN}} 0 \wedge f_n \not\xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0)\}$
- $\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \min\{|X| : \exists f_n: X \rightarrow \mathbb{R} (f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \wedge f_n \not\xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0)\}$

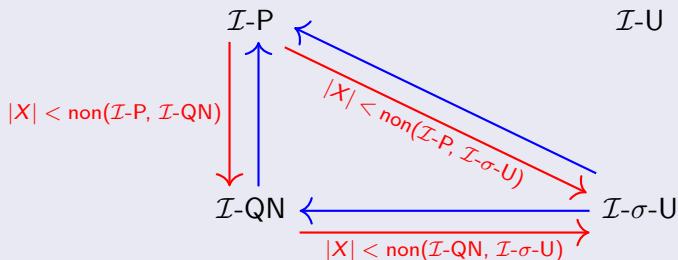
Minimal size of a space which distinguishes convergence



Definition

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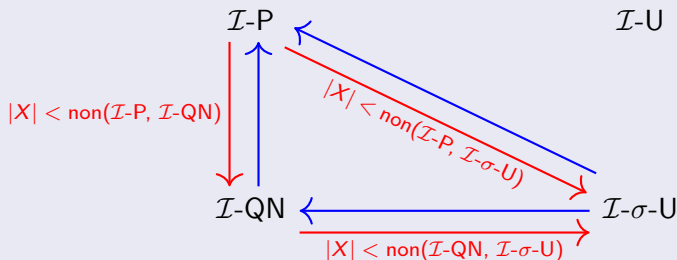
Relationships between minimal size of a space which distinguishes convergence



Theorem

$$\begin{aligned}
 \text{non}(I-P, I-\sigma-U) &= \min \left\{ \text{non}(I-P, I-QN), \quad \text{non}(I-QN, I-\sigma-U) \right\} \\
 &= \min \left\{ \mathfrak{b}_{QN}(I), \quad \text{non}(I-QN, I-\sigma-U) \right\} \\
 &= \min \left\{ \mathfrak{b}(\mathcal{D}_I, \geq^I), \quad \text{non}(I-QN, I-\sigma-U) \right\}
 \end{aligned}$$

Relationships between minimal size of a space which distinguishes convergence



Theorem

$$\begin{aligned}
 \text{non}(I-P, I-\sigma-U) &= \min \left\{ \text{non}(I-P, I-QN), \quad \text{non}(I-QN, I-\sigma-U) \right\} \\
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 &= \min \left\{ \mathfrak{b}(\mathcal{D}_I, \geq^I), \quad \text{non}(I-QN, I-\sigma-U) \right\}
 \end{aligned}$$

Combinatorial characterizations of minimal size of a space which distinguishes convergence

Definition

$$\mathfrak{b}_\sigma(\mathcal{I}) = \min \left\{ |\mathcal{E}| : \mathcal{E} \subseteq \mathcal{M}_{\mathcal{I}} \wedge \forall_{(A_n) \in \mathcal{M}_{\mathcal{I}}} \exists_{(E_n) \in \mathcal{E}} \forall_k \exists_{n > k} (E_n \not\subseteq A_n) \right\}$$

where $\mathcal{M}_{\mathcal{I}}$ is the family of all **increasing** sequences $(A_n)_n$ with $A_n \in \mathcal{I}$

$$\text{add}_\omega(\mathcal{I}) = \min \left\{ |\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall_{(B_n) \in \mathcal{I}^\omega} \exists_{A \in \mathcal{A}} \forall_n (A \not\subseteq B_n) \right\}$$

where $\mathcal{I}^\omega$ is the family of all sequences $(A_n)_n$ with $A_n \in \mathcal{I}$

Theorem (F.-Kwela, 2023)

$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \mathfrak{b}_\sigma(\mathcal{I})$$

$$\text{non}(\mathcal{I}\text{-QN}, \mathcal{I}\text{-}\sigma\text{-U}) = \text{add}_\omega(\mathcal{I})$$

In particular,

$$\mathfrak{b}_\sigma(\mathcal{I}) = \min \left\{ \mathfrak{b}(\mathcal{D}_{\mathcal{I}}, \geq^{\mathcal{I}}), \text{add}_\omega(\mathcal{I}) \right\}$$

Combinatorial characterizations of minimal size of a space which distinguishes convergence

Definition

$$\mathfrak{b}_\sigma(\mathcal{I}) = \min \left\{ |\mathcal{E}| : \mathcal{E} \subseteq \mathcal{M}_{\mathcal{I}} \wedge \forall (A_n)_{n \in \mathbb{N}} \in \mathcal{M}_{\mathcal{I}} \exists (E_n)_{n \in \mathbb{N}} \in \mathcal{E} \forall k \exists n > k (E_n \not\subseteq A_n) \right\}$$

where $\mathcal{M}_{\mathcal{I}}$ is the family of all **increasing** sequences $(A_n)_n$ with $A_n \in \mathcal{I}$

$$\text{add}_\omega(\mathcal{I}) = \min \left\{ |\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall (B_n)_{n \in \mathbb{N}} \in \mathcal{I}^\omega \exists A \in \mathcal{A} \forall n (A \not\subseteq B_n) \right\}$$

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Theorem (F.-Kwela, 2023)

$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \mathfrak{b}_\sigma(\mathcal{I})$$

$$\text{non}(\mathcal{I}\text{-QN}, \mathcal{I}\text{-}\sigma\text{-U}) = \text{add}_\omega(\mathcal{I})$$

In particular,

$$\mathfrak{b}_\sigma(\mathcal{I}) = \min \left\{ \mathfrak{b}(\mathcal{D}_{\mathcal{I}}, \geq^{\mathcal{I}}), \text{add}_\omega(\mathcal{I}) \right\}$$

Combinatorial characterizations of minimal size of a space which distinguishes convergence

Definition

$$\mathfrak{b}_\sigma(\mathcal{I}) = \min \left\{ |\mathcal{E}| : \mathcal{E} \subseteq \mathcal{M}_{\mathcal{I}} \wedge \forall (A_n) \in \mathcal{M}_{\mathcal{I}} \exists (E_n) \in \mathcal{E} \forall k \exists n > k (E_n \not\subseteq A_n) \right\}$$

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$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \mathfrak{b}_\sigma(\mathcal{I})$$

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In particular,

$$\mathfrak{b}_\sigma(\mathcal{I}) = \min \left\{ \mathfrak{b}(\mathcal{D}_{\mathcal{I}}, \geq^{\mathcal{I}}), \text{add}_\omega(\mathcal{I}) \right\}$$

References

- J. Brendle, D. A. Mejía, *Rothberger gaps in fragmented ideals*, Fund. Math. 227 (2014), no. 1, 35–68.
- Z. Bukovská, Quasinormal convergence, Math. Slovaca 41 (1991), no. 2, 137–146.
- L. Bukovský, I. Reclaw, M. Repický, *Spaces not distinguishing pointwise and quasinormal convergence of real functions*, Topology Appl. 41 (1991), no. 1-2, 25–40.
- Á. Császár, M. Laczkovich, Some remarks on discrete Baire classes, Acta Math. Acad. Sci. Hungar. 33 (1979), no. 1-2, 51–70.
- B. Farkas, L. Soukup, *More on cardinal invariants of analytic P -ideals*, Comment. Math. Univ. Carolin. 50 (2009), no. 2, 281–295.
- R. Filipów, A. Kwela, *Yet another ideal version of the bounding number*, J. Symb. Log. 87 (2022), no. 3, 1065–1092
- R. Filipów, A. Kwela, *Two $\mathfrak{b}$ or not two $\mathfrak{b}$?*, J. Symb. Log., 2026:1-15, published online
- R. Filipów, M. Staniszewski, *Pointwise versus equal (quasi-normal) convergence via ideals*, J. Math. Anal. Appl. 422 (2015), no. 2, 995–1006.
- A. Kwela, *More on yet another ideal version of the bounding number*, to appear in Journal of Symbolic Logic
- M. Repický, *Spaces not distinguishing ideal convergences of real-valued functions*, Real Anal. Exchange 46 (2021), no. 2, 367–394.