

Kuratowski convergence of Clarke subdifferentials

Daniel Fatuła (Jagiellonian University)

Definition

For a given metric space (X, d) and a sequence of sets $\{A_n\}_{n=1}^{+\infty} \subset X$ we define the Kuratowski lower and upper limits

$$\liminf_{n \rightarrow +\infty} A_n := \{x \in X \mid \forall U \text{ neighbourhood of } x : A_n \cap U = \emptyset \text{ for finitely many } n\} =$$

$$= \{x \in X \mid \exists x_n \in A_n : x_n \rightarrow x\}$$

$$\limsup_{n \rightarrow +\infty} A_n := \{x \in X \mid \forall U \text{ neighbourhood of } x : A_n \cap U \neq \emptyset \text{ for infinitely many } n\} =$$

$$= \{x \in X \mid \exists n_1 < n_2 < \dots, \exists x_{n_k} \in A_{n_k} : x_{n_k} \rightarrow x\}.$$

We say that A_n converges to a set A in the Kuratowski sense, if $A = \liminf_{n \rightarrow +\infty} A_n = \limsup_{n \rightarrow +\infty} A_n$. We then write $A_n \xrightarrow{K} A$.

Observation

Let (X, d) be a locally compact metric space. Then a sequence $\{A_n\} \subset P(X)$ tends in the Kuratowski sense to a set A if and only if $A = \overline{A}$ and the following conditions hold:

- For every open set $U \subset X$, if $U \cap A \neq \emptyset$ then $U \cap A_n \neq \emptyset$ for n large enough.*
- For every compact set $K \subset X$, if $K \cap A = \emptyset$ then $K \cap A_n = \emptyset$ for n large enough.*

We note that $\liminf_{n \rightarrow +\infty} A_n$ and $\limsup_{n \rightarrow +\infty} A_n$ are always closed sets. Also, $\liminf_{n \rightarrow +\infty} A_n = \liminf_{n \rightarrow +\infty} \overline{A_n}$ and $\limsup_{n \rightarrow +\infty} A_n = \limsup_{n \rightarrow +\infty} \overline{A_n}$.

Definition

We define a distance function to a nonempty set A in a metric space (X, d) as $\text{dist}(x, A) = \text{dist}_A(x) = \inf\{d(x, a) : a \in A\}$.

Observation

Let $\{A_n\}_{n=1}^{+\infty}$ be a sequence of nonempty sets in a Heine-Borel space X . We take a closed nonempty set $A \subset X$. Then the following conditions are equivalent:

- ① $A_n \xrightarrow{K} A$,
- ② $\text{dist}_{A_n} \rightarrow \text{dist}_A$ pointwise,
- ③ $\text{dist}_{A_n} \rightarrow \text{dist}_A$ locally uniformly.

Theorem (Zarankiewicz)

Let X be a second countable metric space. Consider a sequence of sets $\{A_n\}_{n=1}^{+\infty} \subset P(X)$. Then there exist a subsequence A_{k_n} convergent in the Kuratowski sense to some $A \subset X$.

Definition

We define a multifunction $F : X \rightrightarrows Y$ as any mapping $F : X \rightarrow P(Y)$.

We define a graph of the multifunction as $\Gamma_F := \{(x, y) \in X \times Y \mid y \in F(x)\}$.

Definition

Let X, Y be metric spaces. We take a sequence of multifunctions $F_n : X \rightrightarrows Y$ and $F : X \rightrightarrows Y$. We say that

- F_n tends to F pointwise if for every $x \in X$, $F_n(x) \xrightarrow{K} F(x)$.
- F_n tends to F graphically if $\Gamma_{F_n} \xrightarrow{K} \Gamma_F$.

We shall be using the graphical convergence as it is the natural counterpart of local uniform convergence of continuous functions.

Observation

Let $\Omega \subset \mathbb{R}^p$ be an open set, $f, f_n : \Omega \rightarrow \mathbb{R}^q$ continuous functions. Then the following conditions are equivalent:

- 1 $\Gamma_{f_n} \xrightarrow{K} \Gamma_f$,
- 2 $f_n \rightarrow f$ locally uniformly.

Let Ω be an open set in $\mathbb{R}^p$.

Theorem (Rademacher)

If $f : \Omega \rightarrow \mathbb{R}^q$ is a locally Lipschitz function, then f is differentiable almost everywhere.

Definition

For a locally Lipschitz function $f : \Omega \rightarrow \mathbb{R}^q$ we define its Clarke subdifferential as

$$\partial f(x) := \text{conv}\left\{ \lim_{x_n \rightarrow x} f'(x_n) \mid x_n \in \Omega \setminus Z, x_n \rightarrow x, \{f'(x_n)\}_{n=1}^{+\infty} \text{ is convergent} \right\},$$

where Z is a set of measure 0 containing the non-differentiability points of f (the above definition does not depend on the choice of Z).

Theorem (Lebourg mean value theorem)

Let $\Omega \subset \mathbb{R}^p$ be an open set and $f : \Omega \rightarrow \mathbb{R}$ be a locally Lipschitz function. We take $a, b \in \Omega$ such that $[a, b] = \{ta + (1 - t)b \in \mathbb{R}^p : 0 \leq t \leq 1\} \subset \Omega$. Then there exist $c \in [a, b]$ and $u \in \partial f(c)$ such that $f(b) - f(a) = u(b - a)$.

Corollary

Let $\Omega \subset \mathbb{R}^p$ be a domain. We take $f : \Omega \rightarrow \mathbb{R}^q$ a locally Lipschitz function. We assume that $f'(x) = 0$ for almost all $x \in \Omega$. Then f is constant.

Theorem

Let X be an open and connected subset of $\mathbb{R}^p$. We consider a sequence of L -Lipschitz mappings $f_n : X \rightarrow \mathbb{R}^q$. The sequence ∂f_n tends graphically to a multifunction $D : X \rightrightarrows L(\mathbb{R}^p, \mathbb{R}^q)$ such that $\#D(x) = 1$ for almost all $x \in X$. Finally, assume that there exists a point $\alpha \in X$ such that $f_n(\alpha)$ is convergent. Then the sequence f_n is convergent uniformly on bounded sets to an L -Lipschitz function $f : X \rightarrow \mathbb{R}^q$ and moreover $D(x) = \{f'(x)\}$ for almost all $x \in X$.

Proposition

Let $\{A_n\}_{n=1}^{+\infty} \subset \mathbb{R}^p$ be a sequence of nonempty sets converging to a nonempty set $A \subset \mathbb{R}^p$. Then $\partial \text{dist}_{A_n}^2$ tends graphically to ∂dist_A^2 .

We note that the Kuratowski convergence of A_n does not imply the graphical convergence of $\partial \text{dist}_{A_n}$.

Example: $A_n = (-\infty, -1/n] \cup [1/n, +\infty)$.

Definition

Let $A \subset \mathbb{R}^p$ be a closed set. We define a multifunction $P_A : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ as $P_A(x) = \{a \in A : |x - a| = \text{dist}(x, A)\}$.

Definition

We define medial axis of a nonempty closed set $A \subset \mathbb{R}^p$ as $M_A := \{x \in \mathbb{R}^p : \#P_A(x) > 1\}$.

Proposition

Let A be a nonempty, closed subset of $\mathbb{R}^p$. Then $\partial \text{dist}^2(x, A) = 2(x - \text{conv}P_A(x))$.

Observation

M_A is the set of points where dist_A^2 is not differentiable.

Bibliografia

- [1] L. Birbrair, M. Denkowski, Medial axis and singularities, *Journal of Geometric Analysis* 27 (2017), 2339–2380.
- [2] F. Clarke, Generalized gradients and applications, *Trans. Amer. Math. Soc.* 205 (1975), 247-262.
- [3] G. Lebourg, Generic differentiability of lipschitzian functions. *Transactions of the American Mathematical Society*, Volume 256, (1979), 125-145.
- [4] R. T. Rockafellar, R. Wets, *Variational analysis*. Springer Verlag, 1998.

Thank you for your attention