



On the optimized certainty equivalent

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$$u(x) \leq x \quad \text{for} \quad x \in \mathbb{R}.$$

Moreover, let

$$\mathcal{U}^< := \{u \in \mathcal{U} : u(x) < x \text{ for } x \neq 0\}.$$

For any $u \in \mathcal{U}$, put

$$S_u(X) = \sup \{t + E[u(X - t)] : t \in \mathbb{R}\} \quad \text{for } X \in \mathcal{X} \quad (1)$$

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that is,

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In Ben-Tal A., Teboulle M. (2007) several properties of (OCE) were investigated. In particular, the authors proved that a utility function can be **recovered** from its (OCE).

More precisely, if $u \in \mathcal{U}^<$, then

$$(P_1^+) \quad u(x) = \lim_{p \rightarrow 0^+} \frac{S_u(x; p)}{p} \quad \text{for } x \in (0, \infty).$$

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However, **the later statement is not true in general.**

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Hence, for all $x \in (0, \infty)$, we have

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Hence, for all $x \in (0, \infty)$, we have

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Therefore, (P_1^+) is valid, but (P_2^+) does not hold.

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Finally, we present some applications of our results.

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$$pu(x) \leq S_u(x; p) \leq px \quad \text{for } x \in \mathbb{R}, p \in [0, 1]. \quad (7)$$

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Assume that $u \in \mathcal{U}$. Then the conditions (P_1^+) and (P_1^-) are satisfied if and only if either $u \in \mathcal{U}^<$ or $u(x) = x$ **for** $x \in \mathbb{R}$.

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$$u(x) = \begin{cases} \gamma^- x & \text{for } x \in (-\infty, 0), \\ \gamma^+ x & \text{for } x \in [0, \infty). \end{cases} \quad (16)$$

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Value at Risk is monotone, translative and positively homogeneous, but it is not subadditive (and so it is not coherent).