



Separable non-negativity and equations for pairs of functions on big sets

Zoltán Boros

University of Debrecen

joint work with Rayene Menzer (University of Miskolc)

40th International Summer Conference on Real Functions Theory
Stará Lesná, Slovakia
September 6–11, 2026

Dedicated to the memory of Lev Bukovský and Ján Borsík.

Outline

The alternative equation

Related research projects

Separable non-negativity

Mixed constraints

A cancellative equation

The alternative equation with measure constraint

Theorem 1 [BM26]

For each $j \in \{1, 2\}$, let $(X_j, \mathcal{A}_j, \mu_j)$ be a σ -finite measure space. Suppose that $f_j : X_j \rightarrow \mathbb{C}$ ($j = 1, 2$) fulfill

$$f_1(x)f_2(y) = 0 \tag{1}$$

for all $(x, y) \in D$, where $D \subseteq X_1 \times X_2$ is a $\mu_1 \otimes \mu_2$ measurable subset with positive measure (here $\mu_1 \otimes \mu_2$ denotes the Lebesgue completion of the product measure $\mu_1 \times \mu_2$). Then there exist an index $j \in \{1, 2\}$ and $A_j \in \mathcal{A}_j$ such that $\mu_j(A_j) > 0$ and $f_j(x) = 0$ for every $x \in A_j$.

The alternative equation with category constraint

Theorem 2 [BM26]

Let $k, m \in \mathbb{N}$. Suppose that $f_1 : \mathbb{R}^k \rightarrow \mathbb{C}$ and $f_2 : \mathbb{R}^m \rightarrow \mathbb{C}$ such that

$$f_1(x)f_2(y) = 0 \tag{2}$$

holds for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^{k+m}$ is a second category Baire set. Then there exists a second category Baire set $A_1 \subseteq \mathbb{R}^k$ such that $f_1(x) = 0$ for every $x \in A_1$, or there exists a second category Baire set $A_2 \subseteq \mathbb{R}^m$ such that $f_2(y) = 0$ for every $y \in A_2$.

Generalized polynomials

Definition.

If $f : \mathbb{R}^k \rightarrow \mathbb{C}$ is the diagonalization of an n -additive function $F : \mathbb{R}^{nk} \rightarrow \mathbb{C}$, we say that f is a *generalized monomial of degree n* . It is convenient to assume that generalized monomials of degree zero are precisely constant mappings.

If f is a finite sum of generalized monomials, then f is called a *generalized polynomial*.

Generalized polynomials

Definition.

If $f : \mathbb{R}^k \rightarrow \mathbb{C}$ is the diagonalization of an n -additive function $F : \mathbb{R}^{nk} \rightarrow \mathbb{C}$, we say that f is a *generalized monomial of degree n* . It is convenient to assume that generalized monomials of degree zero are precisely constant mappings.

If f is a finite sum of generalized monomials, then f is called a *generalized polynomial*.

Lemma 1 [Sze85]

If $f : \mathbb{R}^k \rightarrow \mathbb{C}$ is a generalized polynomial, $E \subseteq \mathbb{R}^k$ is a set of positive (k dimensional) Lebesgue measure or a second category Baire set and $f(x) = 0$ for every $x \in E$, then $f(x) = 0$ for all $x \in \mathbb{R}^k$.

Alternative equation for generalized polynomials

Corollary [BM26]

Let $k, n \in \mathbb{N}$ and $f : \mathbb{R}^k \rightarrow \mathbb{C}$, $g : \mathbb{R}^n \rightarrow \mathbb{C}$ be generalized polynomials fulfilling

$$f(x)g(y) = 0 \tag{3}$$

for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^k \times \mathbb{R}^n$ is a measurable set with positive $k + n$ dimensional Lebesgue measure or D is a second category Baire set. Then f or g vanishes identically.

Exponential polynomials

Definition.

A continuous function $f : \mathbb{R}^k \rightarrow \mathbb{C}$ is called an *exponential polynomial* if it can be written as

$$f = \sum_{i=1}^n m_i p_i,$$

where each $m_i : \mathbb{R}^k \rightarrow \mathbb{C}$ is a nonzero exponential and each $p_i : \mathbb{R}^k \rightarrow \mathbb{C}$ is a (generalized) polynomial.

Exponential polynomials

Definition.

A continuous function $f : \mathbb{R}^k \rightarrow \mathbb{C}$ is called an *exponential polynomial* if it can be written as

$$f = \sum_{i=1}^n m_i p_i,$$

where each $m_i : \mathbb{R}^k \rightarrow \mathbb{C}$ is a nonzero exponential and each $p_i : \mathbb{R}^k \rightarrow \mathbb{C}$ is a (generalized) polynomial.

Lemma 2 [Sze91, Theorem 5.3]

If $f : \mathbb{R}^k \rightarrow \mathbb{C}$ is an exponential polynomial, $E \subseteq \mathbb{R}^k$ is a set of positive (k dimensional) Lebesgue measure and $f(x) = 0$ for every $x \in E$, then $f(x) = 0$ for all $x \in \mathbb{R}^k$.

Alternative equation for exponential polynomials

Corollary [BM26]

Let $k, n \in \mathbb{N}$ and $f : \mathbb{R}^k \rightarrow \mathbb{C}$, $g : \mathbb{R}^n \rightarrow \mathbb{C}$ be exponential polynomials fulfilling

$$f(x)g(y) = 0 \tag{4}$$

for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^k \times \mathbb{R}^n$ is a measurable set with positive $k + n$ dimensional Lebesgue measure. Then f or g vanishes identically.

Further results motivated by alternative equations

These investigations of alternative equations inspired research results into 3 different directions.

- ▶ We (B–Menzer) investigated non-negativity of separable expressions (i.e., products) on big subsets of product spaces.
- ▶ Balcerzak and Popławski considered a real generalized polynomial f fulfilling $f(x)f(y) = 0$ for all $(x, y) \in D$, where $D \subseteq \mathbb{R} \times \mathbb{R}$ is a Borel set which is non-meager in a mixed sense, i.e., it does not belong to the Fubini product of the σ -ideals of Lebesgue zero sets and first category sets, respectively.
- ▶ Motivated by a question of Peter Eliaš, we have investigated the functional equation

$$F(f(x), g(y)) = 0, \tag{5}$$

where F is a given operation.

Separable non-negativity with measure constraint

Theorem 3

For each $j \in \{1, 2\}$, let $(X_j, \mathcal{A}_j, \mu_j)$ be a σ -finite measure space. Suppose that $f_j : X_j \rightarrow \mathbb{R}$ ($j = 1, 2$) fulfill

$$f_1(x)f_2(y) \geq 0 \tag{6}$$

for all $(x, y) \in D$, where $D \subseteq X_1 \times X_2$ is a $\mu_1 \otimes \mu_2$ measurable subset with positive measure. Then we have one of the following options:

- ▶ There exist an index $j \in \{1, 2\}$ and $A_j \in \mathcal{A}_j$ such that $\mu_j(A_j) > 0$ and $f_j(x) = 0$ for every $x \in A_j$.
- ▶ There exist $A_j \in \mathcal{A}_j$ such that $\mu_j(A_j) > 0$ and $f_j(x) \geq 0$ for every $x \in A_j$ ($j = 1, 2$).
- ▶ There exist $A_j \in \mathcal{A}_j$ such that $\mu_j(A_j) > 0$ and $f_j(x) \leq 0$ for every $x \in A_j$ ($j = 1, 2$).

Separable non-negativity with category constraint

Theorem 4

Let $k, m \in \mathbb{N}$. Suppose that $f_1 : \mathbb{R}^k \rightarrow \mathbb{R}$ and $f_2 : \mathbb{R}^m \rightarrow \mathbb{R}$ such that

$$f_1(x)f_2(y) \geq 0 \tag{7}$$

holds for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^{k+m}$ is a second category Baire set. Then we have one of the following options:

- ▶ There exists a second category Baire set $A_1 \subseteq \mathbb{R}^k$ such that $f_1(x) = 0$ for every $x \in A_1$, or there exists a second category Baire set $A_2 \subseteq \mathbb{R}^m$ such that $f_2(y) = 0$ for every $y \in A_2$.
- ▶ There exist second category Baire sets $A_1 \subseteq \mathbb{R}^k$ and $A_2 \subseteq \mathbb{R}^m$ such that $f_1(x) \geq 0$ for every $x \in A_1$ and $f_2(y) \geq 0$ for every $y \in A_2$.
- ▶ There exist second category Baire sets $A_1 \subseteq \mathbb{R}^k$ and $A_2 \subseteq \mathbb{R}^m$ such that $f_1(x) \leq 0$ for every $x \in A_1$ and $f_2(y) \leq 0$ for every $y \in A_2$.

Separable non-negativity for monomials on $\mathbb{R}$

Lemma 3 [BM25]

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a generalized monomial of an even degree, $E \subseteq \mathbb{R}$ is a set of positive Lebesgue measure or a second category Baire set and $f(x) \geq 0$ for every $x \in E$, then $f(x) \geq 0$ for all $x \in \mathbb{R}$.

Separable non-negativity for monomials on $\mathbb{R}$

Lemma 3 [BM25]

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a generalized monomial of an even degree, $E \subseteq \mathbb{R}$ is a set of positive Lebesgue measure or a second category Baire set and $f(x) \geq 0$ for every $x \in E$, then $f(x) \geq 0$ for all $x \in \mathbb{R}$.

Corollary

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be generalized monomials of even degrees fulfilling

$$f(x)g(y) \geq 0 \tag{8}$$

for all $(x, y) \in D$, where $D \subseteq \mathbb{R} \times \mathbb{R}$ is a measurable set with positive planar Lebesgue measure or D is a second category Baire set. Then inequality (8) holds for all $x, y \in \mathbb{R}$.

Fubini product of σ -ideals

Definition. Let $\mathcal{I}$ and $\mathcal{J}$ be given σ -ideals of subsets of $\mathbb{R}$. For $B \subseteq \mathbb{R}^2$ and $x \in \mathbb{R}$, let

$$B_x := \{y \in \mathbb{R} \mid (x, y) \in B\} \text{ and } \Phi_{\mathcal{J}}(B) := \{x \in \mathbb{R} \mid B_x \notin \mathcal{J}\}.$$

We define $\mathcal{I} \otimes \mathcal{J}$ as the family of all sets $A \subseteq \mathbb{R}^2$ such that there exists a Borel set $B \subseteq \mathbb{R}^2$ with $A \subseteq B$ and $\Phi_{\mathcal{J}}(B) \in \mathcal{I}$.

Fubini product of σ -ideals

Definition. Let $\mathcal{I}$ and $\mathcal{J}$ be given σ -ideals of subsets of $\mathbb{R}$. For $B \subseteq \mathbb{R}^2$ and $x \in \mathbb{R}$, let

$$B_x := \{y \in \mathbb{R} \mid (x, y) \in B\} \text{ and } \Phi_{\mathcal{J}}(B) := \{x \in \mathbb{R} \mid B_x \notin \mathcal{J}\}.$$

We define $\mathcal{I} \otimes \mathcal{J}$ as the family of all sets $A \subseteq \mathbb{R}^2$ such that there exists a Borel set $B \subseteq \mathbb{R}^2$ with $A \subseteq B$ and $\Phi_{\mathcal{J}}(B) \in \mathcal{I}$.

Notation. Let $\mathcal{M}$ and $\mathcal{N}$ denote the σ -ideals of meager sets and Lebesgue null sets in $\mathbb{R}$, respectively.

Results by Balcerzak and Popławski

Theorem [BP26, Theorem 3.5]

Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a generalized polynomial such that $f(x)f(y) = 0$ for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^2$ is a Borel set such that $D \notin \mathcal{M} \otimes \mathcal{N}$ or $D \notin \mathcal{N} \otimes \mathcal{M}$. Then $f(x) = 0$ for each $x \in \mathbb{R}$.

Theorem [BP26, Theorem 3.5]

Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a generalized polynomial such that $f(x)f(y) = 0$ for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^2$ is a Borel set such that $D \notin \mathcal{M} \otimes \mathcal{N}$ or $D \notin \mathcal{N} \otimes \mathcal{M}$. Then $f(x) = 0$ for each $x \in \mathbb{R}$.

Theorem [BP26, Theorem 4.3]

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a generalized monomial of an even degree, fulfilling $f(x)f(y) \geq 0$ for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^2$ is a Borel set such that $D \notin \mathcal{M} \otimes \mathcal{N}$ or $D \notin \mathcal{N} \otimes \mathcal{M}$. Then $f(x)f(y) \geq 0$ holds for all $(x, y) \in \mathbb{R}^2$.

0-cancellative mappings

Let $S \subseteq \mathbb{C} \times \mathbb{C}$ and $F : S \rightarrow \mathbb{C}$. We call F **0-cancellative** if it fulfills each of the following two conditions:

- ▶ for every $z \in \mathbb{C}$, there exists at most one $w \in \mathbb{C}$ such that $(z, w) \in S$ and $F(z, w) = 0$;
- ▶ for every $w \in \mathbb{C}$, there exists at most one $z \in \mathbb{C}$ such that $(z, w) \in S$ and $F(z, w) = 0$.

0-cancellative mappings

Let $S \subseteq \mathbb{C} \times \mathbb{C}$ and $F : S \rightarrow \mathbb{C}$. We call F **0-cancellative** if it fulfills each of the following two conditions:

- ▶ for every $z \in \mathbb{C}$, there exists at most one $w \in \mathbb{C}$ such that $(z, w) \in S$ and $F(z, w) = 0$;
- ▶ for every $w \in \mathbb{C}$, there exists at most one $z \in \mathbb{C}$ such that $(z, w) \in S$ and $F(z, w) = 0$.

Examples.

- ▶ $F(z, w) = z + w \quad (z, w) \in S = \mathbb{C} \times \mathbb{C}$;
- ▶ $F(z, w) = z \cdot w - 1 \quad (z, w) \in S = \mathbb{C} \times \mathbb{C}$;
- ▶ $F(z, w) = \frac{z+1}{w} - 2 \quad (z, w) \in S = \mathbb{C} \times (\mathbb{C} \setminus \{0\})$.

A cancellative equation with measure constraint

Theorem 5

Let $S \subseteq \mathbb{C} \times \mathbb{C}$ and $F : S \rightarrow \mathbb{C}$ be 0-cancellative. For each $j \in \{1, 2\}$, let $(X_j, \mathcal{A}_j, \mu_j)$ be a σ -finite measure space. Suppose that $f_j : X_j \rightarrow \mathbb{C}$ ($j = 1, 2$) fulfill $(f_1(x), f_2(y)) \in S$ for all $(x, y) \in X_1 \times X_2$ and

$$F(f_1(x), f_2(y)) = 0 \tag{9}$$

for all $(x, y) \in D$, where $D \subseteq X_1 \times X_2$ is a $\mu_1 \otimes \mu_2$ measurable subset with positive measure. Then, for each index $j \in \{1, 2\}$, there exist $A_j \in \mathcal{A}_j$ and $c_j \in \mathbb{C}$ such that $(c_1, c_2) \in S$, $F(c_1, c_2) = 0$, $\mu_j(A_j) > 0$ and $f_j(x) = c_j$ for every $x \in A_j$.

Lack of cancellativity

Example.

Let $H \subseteq [0, 1]$ such that $\lambda^*(H) = \lambda^*([0, 1] \setminus H) = 1$. Let us define

$$f(x) = \begin{cases} 1, & \text{if } x \in H, \\ -1, & \text{if } x \in [0, 1] \setminus H, \end{cases}$$

and $F(z, w) = z^2 w^2 - 1$ $((z, w) \in \mathbb{C} \times \mathbb{C})$.

Then $F(f(x), f(y)) = 0$ for all $(x, y) \in [0, 1]^2$, but f is not constant on any set of positive linear Lebesgue measure.

A cancellative equation with category constraint

Theorem 6

Let $S \subseteq \mathbb{C} \times \mathbb{C}$ and $F : S \rightarrow \mathbb{C}$ be 0-cancellative. Let $k, m \in \mathbb{N}$. Suppose that $f_1 : \mathbb{R}^k \rightarrow \mathbb{C}$ and $f_2 : \mathbb{R}^m \rightarrow \mathbb{C}$ such that $(f_1(x), f_2(y)) \in S$ for all $(x, y) \in \mathbb{R}^k \times \mathbb{R}^m \doteq \mathbb{R}^{k+m}$ and

$$F(f_1(x), f_2(y)) = 0 \tag{10}$$

holds for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^{k+m}$ is a second category Baire set. Then there exist second category Baire sets $A_1 \subseteq \mathbb{R}^k$, $A_2 \subseteq \mathbb{R}^m$ and complex numbers $c_1, c_2 \in \mathbb{C}$ such that $(c_1, c_2) \in S$, $F(c_1, c_2) = 0$ and $f_j(u) = c_j$ for every $u \in A_j$ ($j = 1, 2$).

Cancellative equation for polynomials

Corollary

Let $S \subseteq \mathbb{C} \times \mathbb{C}$ and $F : S \rightarrow \mathbb{C}$ be 0-cancellative. Let $k, n \in \mathbb{N}$ and $f : \mathbb{R}^k \rightarrow \mathbb{C}$, $g : \mathbb{R}^n \rightarrow \mathbb{C}$ be generalized polynomials fulfilling

$$F(f(x), g(y)) = 0 \tag{11}$$

for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^k \times \mathbb{R}^n$ is a measurable set with positive $k + n$ dimensional Lebesgue measure or D is a second category Baire set. Then there exist complex numbers $c_1, c_2 \in \mathbb{C}$ such that $(c_1, c_2) \in S$, $F(c_1, c_2) = 0$, $f(x) = c_1$ for every $x \in \mathbb{R}^k$ and $g(y) = c_2$ for every $y \in \mathbb{R}^n$.

References



M. Balcerzak, M. Popławski,

Extending Theorems of Boros and Menzer,
Preprint arXiv:2603.25609, 2026.



Z. Boros, R. Menzer,

An alternative equation for generalized monomials involving
measure,

Publ. Math. Debrecen **104**/1-2 (2024), 171–183.

<https://doi.org/10.5486/PMD.2024.9664>



Z. Boros, R. Menzer,

An alternative equation for generalized polynomials
involving measure and category constraints,

Acta Math. Hungarica **175**/2 (2025), 376–394.

<https://doi.org/10.1007/s10474-024-01498-9>



Z. Boros, R. Menzer,

An alternative equation involving two generalized polynomials
(manuscript).



L. Székelyhidi,

Regularity properties of polynomials on groups,
Acta Math. Hung. **45**/1-2 (1985), 15–19.



L. Székelyhidi,

Convolution Type Functional Equation on Topological Abelian Groups,
World Scientific, Singapore, 1991.

Thank you for your
kind attention!