

Ultrafilters and Ramsey theory

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The following celebrated result gave rise to **Ramsey theory**, which, in an informal sense, claims that complete chaos is impossible.

Theorem (Ramsey)

For every finite coloring of edges of the countably infinite complete graph, there exists an infinite complete subgraph all of whose edges have the exact same color.

Ramsey theory has the following general template:

- 1** The Large Object: You start with an infinite structure $\mathcal{S}$ (e.g., a graph, the natural numbers, a group, or a topological space).
- 2** The Partition (Coloring): You randomly divide the structure $\mathcal{S}$ into a finite number of pieces, or, in other words, consider a finite coloring of $\mathcal{S}$.
- 3** The Special Piece: You find a (monochromatic) piece which contains a specific pattern.

In this talk we shall present an interplay between ultrafilters and the so-called “algebraic” Ramsey theory, which deals with algebraic structures and patterns.

Throughout this talk it will be convenient to assume that $\mathbb{N} = \{1, 2, 3, \dots\}$.

Theorem (van der Waerden)

For every finite coloring of $\mathbb{N}$ there exists a monochromatic set A containing arbitrarily long arithmetic progressions, i.e. for each $n \in \mathbb{N}$ there exist $a, d \in \mathbb{N}$ such that $\{a, a + d, a + 2d, \dots, a + (n - 1)d\} \subseteq A$.

A subset $A \subseteq \mathbb{N}$ is said to have **density zero** if

$$\lim_{n \rightarrow \infty} \frac{|A \cap \{1, \dots, n\}|}{n} = 0.$$

We say that a subset $A \subseteq \mathbb{N}$ has **positive upper density** if A is not of density zero.

Theorem (Szemerédi)

Let $A \subseteq \mathbb{N}$ be of positive upper density. Then for every finite coloring of A there exists a monochromatic set which contains arbitrarily long arithmetic progressions.

Theorem (Green, Tao)

For every finite coloring of the set of prime numbers there exists a monochromatic set containing arbitrarily long arithmetic progressions.

The longest arithmetic progression of prime numbers known so far has length 27:

$$224584605939537911 + 81292139 \cdot (23\#) \cdot n, \quad n \in \{0, 1, \dots, 26\},$$

where $23\#$ denotes the product of all primes ≤ 23 .

Theorem (Schur)

For every finite coloring of $\mathbb{N}$ there exist $x, y \in \mathbb{N}$ such that $x, y, x + y$ are monochromatic.

For each $n \in \mathbb{N}$ the **Schur number** $S(n)$ is the minimal positive integer such that for each n -coloring of $\{1, \dots, S(n)\}$ there exist $x, y \in \mathbb{N}$ such that $x, y, x + y$ are monochromatic. It is known that

- $S(1) = 2$;
- $S(2) = 5$;
- $S(3) = 14$;
- $S(4) = 45$;
- $S(5) = 161$;
- $537 \leq S(6) \leq 1928$.

The equality $S(5) = 161$ was proven in 2017 by Marijn Heule, using computers. Moreover, their computation used 2 petabytes (2,000,000 GBs) of space.

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Definition

For a set X a family $\mathbf{u} \subseteq \mathcal{P}(X)$ is called an **ultrafilter** if it satisfies the following conditions:

- 1 $\emptyset \notin \mathbf{u}$ and $X \in \mathbf{u}$;
- 2 if $A, B \in \mathbf{u}$, then $A \cap B \in \mathbf{u}$;
- 3 if $A \in \mathbf{u}$ and $A \subseteq B$, then $B \in \mathbf{u}$;
- 4 for each finite partition $X = A_1 \sqcup \dots \sqcup A_n$ there exists $i \leq n$ such that $A_i \in \mathbf{u}$.

A set endowed with a binary associative operation is called a **semigroup**.

Definition (Protasov)

An ultrafilter $\mathbf{u}$ on an infinite semigroup is called **Schur** if each $A \in \mathbf{u}$ contains points a, b such that $ab \in A$.

Observation

The existence of a Schur ultrafilter on $(\mathbb{N}, +)$ implies Schur's theorem.

The presented above observation can be developed into a general framework which connects Ramsey theory and ultrafilters.

General framework

Let X be an infinite structure (item 1 in Ramsey theory template). Choose a pattern (or property) p you are interested in. Construct an ultrafilter $\mathbf{u}$ on X such that **every** element of $\mathbf{u}$ has this pattern/property p . This will give us a Ramsey-type theorem. Namely, for each finite coloring of X there exists a monochromatic piece which has pattern/property p .

This motivates a search for “special” (in some sense) ultrafilters. In this talk, we will concentrate on Schur, infinitary Schur, and idempotent ultrafilters.

Definition

An ultrafilter $\mathbf{u}$ on a semigroup S is called

- **Schur** if for any $U \in \mathbf{u}$ there exist $x, y \in U$ such that $xy \in U$;
- **infinitary Schur** if for any $U \in \mathbf{u}$ there exist $x \in U$ and an infinite subset $A \subseteq U$ such that $xA \subseteq U$;
- **idempotent** if for each $U \in \mathbf{u}$ there exist $x \in U$ and $V_x \in \mathbf{u}$ such that $V_x \subseteq U$ and $xV_x \subseteq U$.

An ultrafilter $\mathbf{u}$ is called **free** if $\bigcap \mathbf{u} = \emptyset$.

It is clear that for a free ultrafilter $\mathbf{u}$ on a semigroup S the following implications hold:

$$\mathbf{u} \text{ is idempotent} \Rightarrow \mathbf{u} \text{ is infinitary Schur} \Rightarrow \mathbf{u} \text{ is Schur.}$$

It is known that on each infinite semigroup S there exist at least 2^c free idempotent ultrafilters.

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Definition

A free ultrafilter $\mathbf{u}$ on $\mathbb{N}$ is called

- **weak P-point** if for any sequence of free ultrafilters $(\mathbf{v}_n)_{n \in \mathbb{N}}$ on $\mathbb{N}$ all distinct from $\mathbf{u}$, there exists a set $A \in \mathbf{u}$ such that $A \notin \mathbf{v}_n$ for every $n \in \mathbb{N}$;
- **P-point** if for every partition $\{C_n : n \in \omega\}$ of $\mathbb{N}$ such that $C_n \notin \mathbf{u}$ for all $n \in \omega$, there exists $A \in \mathbf{u}$ such that the set $A \cap C_n$ is finite for every $n \in \omega$;
- **selective** if for every partition $\{C_n : n \in \omega\}$ of $\mathbb{N}$ such that $C_n \notin \mathbf{u}$ for all $n \in \omega$, there exists $A \in \mathbf{u}$ such that $|A \cap C_n| \leq 1$ for all $n \in \omega$.

It can be checked that

$$\text{selective} \Rightarrow \text{P-point} \Rightarrow \text{weak P-point}.$$

Kunen showed that the existence of weak P-points on $\mathbb{N}$ is provable in ZFC. On the other hand, Shelah showed that the existence of P-points is independent of ZFC. Same holds for selective ultrafilters.

Theorem (Folklore)

If $\mathbf{u}$ is a free idempotent ultrafilter on $\mathbb{N}$, then $\mathbf{u}$ is not a weak P-point.

Theorem (B., Zlatoš)

Let $\mathbf{u}$ be a free ultrafilter on $\mathbb{N}$. Then the following holds:

- 1 if $\mathbf{u}$ is infinitary Schur, then $\mathbf{u}$ is not a P-point;
- 2 if $\mathbf{u}$ is Schur, then $\mathbf{u}$ is not selective.

Also, we constructed an infinitary Schur ultrafilter on $\mathbb{N}$ which is not idempotent.

Question 1

Does there exist a free Schur ultrafilter on $\mathbb{N}$ which is not infinitary Schur?

Item 1 of the theorem above suggests the following natural question:

Question 2

Does there exist a free Schur ultrafilter on $\mathbb{N}$ which is a P-point?

By $\beta(\mathbb{N})$ we denote the set of all ultrafilters on $\mathbb{N}$ equipped with the Stone topology generated by the base $\{\langle A \rangle : A \subseteq X\}$, where $\langle A \rangle = \{\mathbf{u} \in \beta(X) : A \in \mathbf{u}\}$. Let $\text{Sch}(\mathbb{N})$ denote the subspace of $\beta(\mathbb{N})$ consisting of all free Schur ultrafilters.

Theorem (B.)

- 1 There exists a free Schur ultrafilter on $\mathbb{N}$ which is not infinitary Schur. Moreover, the set of all free Schur ultrafilters which are not infinitary Schur is an open dense subset of $\text{Sch}(\mathbb{N})$.
- 2 Assuming CH, there exists a free Schur ultrafilter on $\mathbb{N}$ which is a P-point. Moreover, under CH, the set of all Schur P-points is dense in $\text{Sch}(\mathbb{N})$.

A natural question is whether CH can be weakened to a milder set-theoretic assumption. Obviously, the weakest possible assumption is the existence of a P-point. The following proposition implies that this assumption is not sufficient.

Proposition (B.)

It is consistent with ZFC that there exists a P-point ultrafilter on $\mathbb{N}$, but no Schur P-points on $\mathbb{N}$.

The key role in the aforementioned solution plays the following combinatorial description of elements of free Schur ultrafilters.

Definition

A subset $A = \{x_m : m \in \mathbb{N}\} \subseteq \mathbb{N}$ is called **Schur** if for every $n, k \in \mathbb{N}$ there exists $S_A(n, k) \in \mathbb{N}$ such that for every n -coloring of $Y = \{x_1, x_2, \dots, x_{S_A(n, k)}\}$ there exist $a \in Y$ and $B = \{b_1, \dots, b_k\} \subseteq Y$ such that $\{a\} \cup B \cup (a + B)$ is monochromatic.

Theorem (B.)

A set $A \subseteq \mathbb{N}$ belongs to some free Schur ultrafilter on $\mathbb{N}$ if and only if A is Schur.

If $A = \mathbb{N}$, then we agree to write $S(n, k)$ instead of $S_{\mathbb{N}}(n, k)$. The numbers $S(n, k)$, $n, k \in \mathbb{N}$ can be viewed as two dimensional Schur numbers, as for each $n \in \mathbb{N}$ the classical Schur number $S(n)$ equals to $S(n, 1)$.

General question

What can be said about the two dimensional Schur numbers $S(n, k)$? Can we at least compare the numbers $S(7, 1)$ and $S(4, 4)$?

Definition

Let G be a group. Then

- (i) for each $g \in G$ the right shift $\rho_g : G \rightarrow G$ is defined by $\rho_g(x) = xg$;
- (ii) for each $g \in G$ the left shift $\lambda_g : G \rightarrow G$ is defined by $\lambda_g(x) = gx$.

Definition

A compact Hausdorff group G is called a **chart** group if each right shift in G is continuous and densely many left shifts are continuous, i.e., the set $\{x \in G : \text{the left shift } \lambda_x \text{ is continuous}\}$ is dense in G .

Chart groups appear in topological dynamics, as an enveloping semigroup of a dynamical system (X, G) , where X is a compact Hausdorff topological space and G is a group which acts on X as a group of homeomorphisms.

Theorem (Ellis)

A locally compact Hausdorff group G in which all left and all right shifts are continuous is a topological group.

There are examples of chart groups that are not topological groups. Nevertheless, Ellis's theorem motivates the following question.

Question

When a chart group is a topological group?

The following theorem lists in chronological order some partial answers to this question.

Theorem

A chart group is a topological group if it has one of the following properties:

- metrizable (Namioka);
- first-countable (Moors, Namioka);
- Fréchet-Urysohn (Glasner, Mergelishvili);
- countable tightness (Reznichenko).

The following proposition shows that the answer to the above question is completely determined by Schur ultrafilters.

Theorem (B. Zlatoš)

For a chart group G the following conditions are equivalent:

- G is a topological group;
- every infinitary Schur ultrafilter on G converges to 1_G ;
- every Schur ultrafilter on G converges to 1_G .

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For an infinite subset $A = \{a(n) : n \in \mathbb{N}\}$ of a commutative semigroup $(S, +)$, let

$$\text{FS}(A) = A \cup \{a(n_1) + a(n_2) + \cdots + a(n_k) : k \geq 2 \text{ and } n_i \neq n_j \text{ for all } i, j \leq k\}.$$

Thus $\text{FS}(A)$ consists of elements of the set A together with all finite sums of pairwise distinct elements of A .

Theorem (Hindman)

For every finite coloring of $\mathbb{N}$ there exists an infinite subset $A \subseteq \mathbb{N}$ such that $\text{FS}(A)$ is monochromatic.

The following theorem shows that the "finite sums" combinatorial pattern describes elements of idempotent ultrafilters, and thus provides another connection between ultrafilters and Ramsey theory.

Theorem (Galvin-Glazer)

Let S be a commutative semigroup. A subset $X \subseteq S$ belongs to some idempotent ultrafilter on S if and only if there exists an infinite set A such that $\text{FS}(A) \subseteq X$.

Inspired by the Szemerédi's theorem, which generalizes van der Waerden's theorem over subsets of $\mathbb{N}$ with positive upper density, Erdős asked whether there exists an analogue of Hindman's theorem for subsets of positive upper density. Note that we need to search really for an **analogue** of Hindman's theorem, as the direct generalization is impossible due to the following simple observation.

Observation

The odd numbers $2\mathbb{N} + 1$ have density $1/2$. But, since the sum of two odd numbers is even, there is no infinite subset $A \subseteq 2\mathbb{N} + 1$ such that $\text{FS}(A) \subseteq 2\mathbb{N} + 1$.

The following recent result is an essential step towards answering Erdős' question.

Theorem (Kra, Moreira, Richter, Robertson)

Let $A \subseteq \mathbb{N}$ have positive upper density. Then for every $k \in \mathbb{N}$ there exist $t \in \mathbb{N}$ and an infinite set $B \subset \mathbb{N}$ such that

$$\left\{ \sum_{n \in F} n : F \subset B, 0 < |F| \leq k \right\} \subseteq A - t.$$

Definition

Let NC be the family of all those subsets of $\mathbb{R}^n$ that can be covered by a union of a compact subset of $\mathbb{R}^n$ and a nowhere dense subset of $\mathbb{R}^n$. That is NC is a set-theoretic ideal on $\mathbb{R}^n$ generated by all compact subsets and all nowhere dense subsets of $\mathbb{R}^n$.

Theorem (B.)

For every $C \in \text{NC}$ there exists an open set $P \supseteq C$ such that for every finite coloring of $\mathbb{Q}^n \setminus P$ there exists a family $\mathcal{A}$ of infinite subsets of $\mathbb{Q}^n \setminus P$ which satisfies the following conditions:

- (i) the set $\bigcup_{A \in \mathcal{A}} \text{FS}(A)$ is monochromatic;
- (ii) the set $\bigcup_{A \in \mathcal{A}} \text{FS}(A)$ is dense in an open unbounded subset of $\mathbb{R}^n$;
- (iii) $\text{FS}(A)$ is a closed discrete subset of $\mathbb{R}^n$ for all $A \in \mathcal{A}$;
- (iv) $\text{FS}(A) \cap \text{FS}(B) = \emptyset$ for all distinct $A, B \in \mathcal{A}$.

The primary technical tools in our proof are special idempotent ultrafilters.

This result is a corollary of a much more general approach which can be vaguely described as follows.

For a certain group G we are able to turn a triple (τ, S, J) to a Hindman-type theorem similar to the one above, where

- 1 τ is a certain topology on G compatible with the group structure on G (usually locally compact non-compact and Polish);
- 2 S is a certain highly non-compact subsemigroup of (G, τ) ;
- 3 J is a certain (set-theoretic) ideal on G compatible with algebraic and topological structures of G and S .

In particular, in the above theorem we have $G = \mathbb{R}^n$, τ is the usual topology on $\mathbb{R}^n$, $S = \mathbb{Q}^n$ and $J = \text{NC}$.

Proposition (B.)

Let A be an infinite subset of a compact topological group G . Then $1_G \in \overline{\text{FS}(A)}$.

It follows that in the aforementioned extension of Hindman's theorem we cannot substitute $\mathbb{R}$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\mathbb{Q}$ with $\mathbb{Q}/\mathbb{Z}$.

Definition

Let $\mathcal{MC}$ be the ideal on $\mathbb{R}^n$ that is generated by all compact and all meager subsets of $\mathbb{R}^n$.

Proposition (B.)

For every $M \in \mathcal{MC}$ there exists an open set $P \supseteq M$ such that for each finite coloring of $\mathbb{R}^n \setminus P$ there exists an infinite subset $A \subseteq \mathbb{R}^n \setminus P$ such that $\text{FS}(A)$ is monochromatic.

On the other hand, in general, we cannot find a family $\mathcal{A}$ of infinite subsets of $\mathbb{R}^n \setminus P$ so that $\bigcup_{A \in \mathcal{A}} \text{FS}(A)$ is monochromatic and dense in an open subset of $\mathbb{R}^n$.

Thank You for attention!

Ďakujem za pozornosť!