38th International Summer Conference on Real Functions Theory

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Preface

38th International Summer Conference on Real Functions Theory has been organized by the Mathematical Institute of the Slovak Academy of Sciences following the tradition in conferences on real functions theory dating back to 1971 when it was founded by professor Tibor Šalát and professor Pavel Kostyrko.

This booklet contains the list of participants, the abstracts and the list of authors. Participants are invited to send their contributions to the journal

Tatra Mountains Mathematical Publications.

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Abstracts

Bohr compactification of topological groups and Schur ultrafilters

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We discuss Schur ultrafilters on groups. Using them, we give a new description of Bohr compactifications of topological groups. This approach allows us to characterize topological chart groups. Namely, a chart group G is a topological group if and only if each Schur ultrafilter on G converges to the unit of G.

Keywords: Schur ultrafilter, Bohr compactification, chart group.

Algebraically Baire regular families of functions in product spaces

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Our investigations are based on the following result.

Theorem 1. Let $k\,,\,m \in \mathbb{N}$. Suppose that $f_1 : \mathbb{R}^k \to \mathbb{C}$ and $f_2 : \mathbb{R}^m \to \mathbb{C}$ such that

$$f_1(x)f_2(y) = 0 \tag{1}$$

holds for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^{k+m}$ is a second category Baire set. Then either there exists a second category Baire set $A_1 \subset \mathbb{R}^k$ such that $f_1(x) = 0$ for every $x \in A_1$, or there exists a second category Baire set $A_2 \subset \mathbb{R}^m$ such that $f_2(y) = 0$ for every $y \in A_2$.

We call a family of functions *algebraically Baire regular* if every member of this family that vanishes on a second category Baire set must be identically equal to zero. Now we can formulate a corollary of the previous theorem to products related to such families of functions.

Theorem 2. For some $k\,,\,m \in \mathbb{N}$, let $\mathcal{F}_1$ and $\mathcal{F}_2$ denote algebraically Baire regular families of functions $f_1 : \mathbb{R}^k \to \mathbb{C}$ and $f_2 : \mathbb{R}^m \to \mathbb{C}$, respectively. Suppose that $f_j \in \mathcal{F}_j$ $(j = 1, 2)$ such that

$$f_1(x)f_2(y) = 0 \tag{2}$$

holds for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^{k+m}$ is a second category Baire set. Then either f_1 or f_2 is identically equal to zero.

We establish a category version of Székelyhidi's theorem [1, Theorem 2], on the zeroes of generalized polynomials and we present the application of Theorem 2 to generalized polynomials as well.

Keywords: product measure, zeros of functions, generalized polynomials.

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On argmin multifunction

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Let X be a Tychonoff topological space, $C(X,\mathbb{R})$ be the space of all continuous real-valued functions defined on X and $K(X)$ be the space of all nonempty compact subsets of X. The multifunction

$$argmin : C(X,\mathbb{R}) \times K(X) \to X$$

is defined as follows:

$$argmin((f,K)) = \{x \in K : f(x) = min\{f(y) : y \in K\}\}.$$

Let τ_U be the topology of uniform convergence on $C(X,\mathbb{R})$ and τ_V the Vietoris topology on $K(X)$. We prove that

$$argmin : (C(X,\mathbb{R}), \tau_U) \times (K(X), \tau_V) \to X$$

is a minimal usco map. If X is a locally Čech-complete Stegall space, the set

$$\{(f,K) \in C(X,\mathbb{R}) \times K(X) : |argmin(f,K)| = 1\}$$

is residual.

On uniformly dense sets of functions

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For a topological space X, let $C(X)$ be the lattice of all continuous functions $f\colon X \to \mathbb{R}$. A set $A \subseteq C(X)$ is *uniformly dense* if, for any $f \in C(X)$ and any $\varepsilon > 0$, there is $g \in A$ such that $|f(x) - g(x)| < \varepsilon$ for all $x \in X$. The well-known Stone-Weierstrass Theorem says that if X is a compact Hausdorff space and $A \subseteq C(X)$ is an algebra of functions containing all constant functions, then A is uniformly dense if and only if it separates points.

There exists a (compact) space X and a lattice automorphism of $C(X)$ that does not preserve uniformly dense subsets. Hence, one cannot define uniform density of subsets of $C(X)$ in the language of lattices, without additional assumptions on X.

On the other hand, Kaplansky [1] showed that if X, Y are compact Hausdorff spaces and the lattices $C(X)$, $C(Y)$ are isomorphic, then the spaces X, Y are homeomorphic. In [2], he showed that if X is compact and metrizable then each lattice automorphism of $C(X)$ is continuous under uniform metric, hence it does preserve uniformly dense subsets.

We discuss the definability of the uniform density of subsets of $C(X)$ in the language of lattices, on a compact metric space X.

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A study on dimensions of continuous mappings

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In Dimension Theory, there are meanings of dimensions for topological spaces that have been studied extensively. There are also mapping theorems establishing relationships between the dimensions of the domain and range of a continuous mapping. Simultaneously, notions of dimensions for continuous mappings have been investigated. For a given continuous mapping, its dimension is usually defined via the dimension of inverses of points. This talk has a double goal. Firstly, a review-study on dimensions of continuous mappings between topological spaces is given. We present meanings of dimensions which have been studied for continuous mappings and basic results which have been proved for them. Also, in this talk, we introduce and investigate a different notion of covering dimension for continuous mappings between topological spaces, which is closer to the classical definition of the Lebesgue covering dimension of a space. We also discuss results concerning continuous mappings between metric spaces and this new dimension. Moreover, we present open questions, which will be the tool for future investigations, and proposals of new dimensions of continuous mappings for further studies.

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Quasicontinuity and the topology of uniform convergence on compacta

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The notion of quasicontinuity is a classical one. It has found many applications in the study of topological groups, in the study of continuity points of separately continuous mappings and in characterizations of minimal usco and minimal cusco maps [1]. The aim of the talk is to present some recent results on quasicontinuous mappings. Let $Q(X, \mathbb{R})$ be the space of real-valued quasicontinuous mappingss on a topological space X. If X is an uncountable Polish space, then the topology of uniform convergence on compacta on $Q(X, \mathbb{R})$ is seen to behave like a metric topology in the sense that the weight, netweight, density, Lindelof number and cellularity are all equal for this topology and they are equal to $2^c = |Q(X, \mathbb{R})|$. We use results from [2] and [3].

Keywords: quasicontinuous mapping, topology of uniform convergence on compacta, cardinal invariants

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Simple density and summable ideals on $\mathbb{N}$ seen through densities on reals

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In this talk we give an overview about the results of research obtained jointly with M. Filipczak and T. Filipczak on relationship between the ideal of density zero subsets of $\mathbb{N}$ or its generalization - ideals of simple density and a classical dispersion point (or f-dispersion) point of a special "testing" subset of $\mathbb{R}$. We also consider in this context summable ideals and O'Malley density. The results obtained provide a deeper insight into the concept of the indicated ideals.

Keywords: simple density ideals, summable ideals, density, O'Malley points.

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Some properties of remainders of uniformly continuous mappings

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Recently, many concepts and statements of uniform topology have been extended from the case of spaces to the case of uniformly continuous mappings. In this case, uniform space is understood as the simplest uniformly continuous mapping of this uniform space into a one-point space.

The research carried out revealed large uniform analogues of continuous mappings and made many it possible to transfer many basic statements of the uniform topology of spaces to mappings.

The method of transferring results from spaces to mappings is universal and not simple, but it allows many results to be generalized. Therefore, the problem of extending some concepts and statements concerning spaces to mappings has not yet been completely solved.

Let $f : (X, U) \to (Y, V)$ be uniformly continuous mapping of a uniform space (X, U) to a uniform space (Y, V). A uniformly continuous mapping $\hat{f} : (\hat{X}, \hat{U}) \to (Y, V)$ of uniform space $(\hat{X}, \hat{U})$ to a uniform space (Y, V) is called the completion of the mapping f if the following conditions hold:

1. The uniform space (X, U) is a dense uniform subspace of the uniform space $(\hat{X}, \hat{U})$;

2. $f = \hat{f}|_X$;

3. The mapping $\hat{f}$ is complete.

Let $\hat{f} : (\hat{X}, \hat{U}) \to (Y, V)$ be a completion of the mapping f. We denote by $\hat{f}|_{\hat{X}\setminus X} : (\hat{X}\setminus X, \hat{U}_{\hat{X}\setminus X}) \to (Y, V)$ the remainder of the mapping f.

Let P and Q be properties of uniformly continuous mappings. The problem naturally arises: If a uniformly continuous mapping f has property P, then under what necessary and sufficient conditions does its remainder $\hat{f}|_{\hat{X}\setminus X}$ have property Q?

In this work necessary and sufficient conditions are found for its remainder to have the precompact, complete and uniformly perfect properties.

Keywords: Uniformly continuous mapping, uniformly perfect, complete, precompact, remainder.

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Certain variants of regular and rapid variations

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Some results on regular and rapid variations [1] will be presented. Especially, we consider translationally regularly and rapidly varying functions and sequences [2] and their relations with selection principles, games and fixed point theory, as well as with rates of divergence. Theorems of Bojanić-Galambos-Seneta type [3] will be also presented. An equivalence relation related to rapid variation will be considered.

Keywords: Regular (translational) variation, rapid variation, r-equivalence, index function.

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On continuity in generalized topology

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In many fields of mathematics, especially in theory of real functions, there are considered different kinds of generalized continuity. The aim of our talk is unification of properties of sets of points of these generalized continuities. The useful tool for this purpose is the notion of generalized topology introduce by Á. Császár in 2002 in [3]. It turned out that many of previously considered types of generalized continuities may be equivalently defined as a continuity in a generalized topology. In the talk we characterize sets of points of continuity for functions defined on a generalized topological space. Every generalized topology is associated with some topology. Frequently sets of points of generalized continuity can be described in terms of this topology associated with generalized topology. Many interesting results can be proved for such types of continuities.

In our considerations important role plays a generalized topology $\mathcal{Q}_{\mathcal{T}}$, which defines quasicontinuity in topological space $(X, \mathcal{T})$. We show that for every topological space $(X, \mathcal{T})$ if Γ is a generalized topology in X such that $\mathcal{T}$ is associated with Γ then Γ is contained in $\mathcal{Q}_{\mathcal{T}}$.

The last part of the talk is devoted to research into path continuity with respect to a generalized topology Γ and a topology $\mathcal{T}$ which is associated with Γ. We present conditions for equivalence of path continuity and continuity in generalized topology.

Keywords: points of continuity, generalized topology, generalized continuity, path continuity, resolvability.

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On permutations preserving density

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In the 71st problem included in the Scottish Book, Stanisław Ulam asks about the characteristics of permutations that preserve the density of subsets of natural numbers. Partial answers to this question will be presented.

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A base modulo an ideal

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Given ideals $\mathcal{I}, \mathcal{J}$ on the same set X we define the cofinality-like cardinal invariant

$$\mathrm{cof}^{\mathcal{J}}(\mathcal{I}) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{I} \wedge \mathcal{B} \text{ is cofinal in } (\mathcal{I}, \subseteq^{\mathcal{J}})\}.$$

The invariant describes the smallest subfamilies of $\mathcal{I}$ that cover all elements of $\mathcal{I}$ modulo $\mathcal{J}$. Besides its affinity with classical cofinality, this invariant is very closely related to ideal convergence. E.g., the invariant describes the smallest possible cardinality of families ensuring that ideal uniform convergence is stronger than ideal quasi-normal convergence, see [1, 2]. We shall present basic properties of this invariant and we will find "values" of $\mathrm{cof}^{\mathcal{J}}(\mathcal{I})$ for pairs of particular critical ideals on ω as well as on the unit interval.

Keywords: P-ideal, cofinality, base.

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Hyperconnectedness, Resolvability and Submaxibility of Ideal Topological Spaces and Soft Ideal Topological Spaces

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The talk deals with some soft notions defined in soft ideal topological space. It proved that soft ideal hyperconnectedness, soft ideal hyperconnectednes modulo ideal and soft ideal semi-hyperconnectedness introduced in [1] are equivalent. It is shown that some results are valid even without an assumption that a given soft ideal is soft codense, and a few examples are found when this assumption cannot be omitted. Some connections between a soft ideal semi-irresolvable space, a soft ideal submaximal space and general Volterra space are given. In the article a significant correspondence between soft topology and general topology in shown.

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Algebraically measure regular families of functions in product spaces

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Our investigations are based on the following result.

Theorem 1. For each $j \in \{1, 2\}$, let $(X_j, \mathcal{A}_j, \mu_j)$ be a σ-finite measure space. Suppose that $f_j : X_j \to \mathbb{C}$ $(j = 1, 2)$ fulfill

$$f_1(x)f_2(y) = 0 \tag{3}$$

for all $(x, y) \in D$, where $D \subseteq X_1 \times X_2$ is a $\mu_1 \otimes \mu_2$ measurable subset with positive measure (here $\mu_1 \otimes \mu_2$ denotes the Lebesgue completion of the product measure $\mu_1 \times \mu_2$). Then there exist an index $j \in \{1, 2\}$ and $A_j \in \mathcal{A}_j$ such that $\mu_j(A_j) > 0$ and $f_j(x) = 0$ for every $x \in A_j$.

We call a family of functions *algebraically measure regular* if every member of this family that vanishes on a set of positive measure must be identically equal to zero. Now we can formulate a corollary of the previous theorem to products related to such families of functions.

Theorem 2. Let For each $j \in \{1, 2\}$, let $(X_j, \mathcal{A}_j, \mu_j)$ be a σ-finite measure space and let $\mathcal{F}_j$ denote an algebraically measure regular family of functions $f : X_j \to \mathbb{C}$. Let $f_j \in \mathcal{F}_j$ $(j = 1, 2)$ such that

$$f_1(x)f_2(y) = 0 \tag{4}$$

holds for all $(x, y) \in D$, where $D \subseteq X_1 \times X_2$ is a $\mu_1 \otimes \mu_2$ measurable subset with positive measure. Then either f_1 or f_2 is identically equal to zero.

According to Székelyhidi's result [1, Theorem 2], if G is a locally compact Abelian group which is generated by any neighborhood of zero, then the family of all generalized polynomials $p : G \to \mathbb{C}$ constitutes an algebraically measure regular family of functions with respect to the Haar measure. So we present the application of Theorem 2 to generalized polynomials as well.

Keywords: product measure, zeros of functions, generalized polynomials.

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Spaces of minimal usco and minimal cusco maps as Fréchet topological vector spaces

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The concepts of minimal usco and minimal cusco maps have significant applications across various areas of mathematical analysis, including optimization, functional analysis, and the study of differentiability of Lipschitz functions [1]. Understanding the topological properties of the spaces of these maps is thus crucial.

The topology of uniform convergence on compact sets is one of the key topologies used in function spaces. In [2], we identified conditions under which the spaces of minimal usco and minimal cusco maps, equipped with this topology, form topological vector spaces.

We show that spaces of these maps from a locally compact space to a Fréchet space, equipped with the topology of uniform convergence on compacta, are isomorphic as topological vector spaces. Moreover, if the domain space is also hemicompact, both spaces are Fréchet.

Keywords: minimal usco, minimal cusco, uniform convergence on compacta, topological vector space, locally convex topological vector space, isomorphism, Fréchet space.

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Local and global properties of spaces of minimal usco maps

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We explore the relationship between local and global properties of spaces of minimal usco maps endowed with the topology of uniform convergence on compact sets. Specifically, for any locally compact space X and metric space Y, we characterize the space of minimal usco maps from X to Y satisfying one of the following properties: (i) compactness, (ii) local compactness, (iii) σ-compactness, (iv) local σ-compactness, (v) metrizability, (vi) the countable chain condition (ccc), and (vii) local ccc, with the last two properties assuming Y is separable and non-discrete.

Our findings extend and complement results by Ľubica Holá and Dušan Holý.

Keywords: usco maps, compact, σ-compact, metrizable, ccc.

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The notion of Yao's neighborhoods from a topological point of view

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In this presentation, we investigate some problems concerning Yao's neighborhoods [1] defined for a generalized approximation space (X, ρ), where ρ is an arbitrary relation on X. There are two types of such neighborhoods, namely $N_r(x) = \{y \in X : x\rho y\}$ and $N_l(x) = \{y \in X : y\rho x\}$. Based on this notion, the operators $N_r, N_l : \mathcal{P}(X) \to \mathcal{P}(X)$ was defined as follows:

$N_r(A) = \bigcup\{N_r(a) : a \in A\}$ and

$N_l(A) = \bigcup\{N_l(a) : a \in A\}$.

In the paper [2], it has been investigated the families of subsets $A \subset X$ that satisfy on of the following conditions $N_r(A) \subset A$, $N_l(A) \subset A$, $A \subset N_r(A)$, $A \subset N_l(A)$.

• The condition $N_r(A) \subset A$ $(N_l(A) \subset A)$ define an Alexandrov topology $\mathcal{T}_r$ $(\mathcal{T}_l)$ as the family of
all closed subsets in $(X, \mathcal{T}_r)$ $((X, \mathcal{T}_l))$.

• The condition $A \subset N_r(A)$ $(A \subset N_l(A))$ define a family $\mathcal{U}_r$ $(\mathcal{U}_l)$, that satisfies the requirements
of supra-topology.

The assumption that the relation ρ is serial (inverse serial), implies $\mathcal{T}_r \subset \mathcal{U}_l$ $(\mathcal{T}_l \subset \mathcal{U}_r)$.

In a topological space (X, τ), we have the following properties:

• The condition $\overline{Int\overline{A}} \subset A$ define the family of all closed subsets in the topological space
in (X, τ_α), where τ_α is the family of all α-open subsets in topological space (X, τ).

• The condition $A \subset \overline{Int\overline{A}}$ define the family of all β-open subsets in the topological space
in (X, τ_α).

Comparing the presented facts of the spaces $(X, \mathcal{T}_l)$, (X, τ_α) begs the question:

Does a topology τ exist such that $\tau_\alpha = \mathcal{T}_l$ and $\mathcal{U}_r$ is the family of all β-open subsets in the topological space in $(X, \mathcal{T}_l)$ i.e., $N_r(A) = \overline{Int\overline{A}}$ for $A \subset X$?

The second part of this presentation concerns the relationships between the topologies $\mathcal{T}_r$, $\mathcal{T}_l$ and the families $\mathcal{U}_r$, $\mathcal{U}_l$.

The family $\mathcal{U}_r$ (resp. $\mathcal{U}_l$) designates a symmetrical pair (T_r, T'_r) $((T_l, T'_l))$ of Alexandrov topologies.

It easy to see that $\tau_r \subset T_l \cap T'_r$, $\tau_l \subset T_r \cap T'_l$, so we have the following questions. What assumptions about the relationship imply these equalities $\tau_r = T'_r$ and $\tau_l = T'_l$?

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Measurable solutions of an alternative functional equation

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Let $I_1\,, I_2$ be nonempty, open intervals of the real line, and let $J := \frac{1}{2}(I_1 + I_2)$. The solutions of the functional equation

$$\varphi\left(\frac{x+y}{2}\right)(\psi_1(x) - \psi_2(y)) = 0 \quad \text{(for all } x \in I_1 \text{ and } y \in I_2) \qquad (5)$$

where the functions $\psi_1 : I_1 \longrightarrow \mathbb{R}$, $\psi_2 : I_2 \longrightarrow \mathbb{R}$ and $\varphi : J \longrightarrow \mathbb{R}$ are unknown, were investigated by T. Kiss [2]. It has been established that if $\varphi^{-1}(0)$ is closed then the nontrivial solutions of (5) are constant on some open subintervals of their domain.

During the Problems and Remarks session of the 59th International Symposium on Functional Equations, T. Kiss proposed the following question (see[1]). Does the mentioned characterization of the solutions of (5) remain valid when some different (weaker) kind of regularity condition is assumed for φ instead of the closedness of $\varphi^{-1}(0)$? In particular, Kiss was concerned about the cases when φ has the Darboux property or φ is of the class Baire 1. This is motivated by the fact that in certain applications (such as the invariance problem of generalized weighted quasi-arithmetic means) the functions appearing in (5) are constructed from derivatives, for which the set of zeros might not be closed.

In our talk we will show the existence of such nontrivial solutions $(\psi_1\,, \psi_2\,, \varphi)$ of (5) that the functions are Darboux (or Baire 1), yet neither of them is constant on any open subinterval. On the other hand, we will show that if φ is Lebesgue measurable then an analogous version of the known characterization theorem for the solutions holds. Hence, if φ is supposed to be the derivative of a differentiable function, then (5) has exactly the same solutions as described in [2, Theorem 6.], which is desired for the applications.

Keywords: alternative functional equation, Darboux property, Lebesgue measurable function.

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Topologies generated by symmetric porosity on normed spaces

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We consider the families of symmetrically porouscontinuous functions. We find maximal additive classes for these families. Furthermore, we define new families of topologies generated by the symmetric porosity, which are useful to studying maximal multiplicative classes for symmetrically porouscontinuous functions. Some relevant properties of defined topologies are considered.

The aim of our talk is to investigate the families of symmetrically porouscontinuous functions and some topologies generated by symmetric porosity on a normed spaces.

First we define the symmetric porosity in a normed space $(X, \| \|)$. Let $M \subset X$, $x \in X$ and $R > 0$. Then we define $s(x, R, M)$ as the supremum of the set of all $r > 0$ for which there exists $z \in X$ such that $B(z, r) \cup B(2x - z, r) \subset B(x, R) \setminus M$. The number $p^s(M, x) = 2 \limsup_{R \to 0^+} \frac{s(x,R,M)}{R}$ is called the symmetric porosity of M at x.

This definition generalizes the notion of symmetric porosity on the real line introduced in [5].

We say that the set M is symmetrically porous at $x \in X$ if $p^s(M, x) > 0$. The set M is called symmetrically porous if M is symmetrically porous at each point $x \in M$. We say that M is strongly symmetrically porous at x if $p^s(M, x) \geq 1$ and M is called strongly symmetrically porous if M is strongly symmetrically porous at each $x \in M$. Similarly as in the case of porosity, every strongly symmetrically porous set is symmetrically porous and every symmetrically porous set is nowhere dense. Moreover, none of reverse inclusions is true.

In [1] J. Borsík and J. Holos defined families of porouscontinuous functions $f : \mathbb{R} \to \mathbb{R}$. Applying their ideas and replacing standard porosity by symmetric porosity we transfer this concept for real functions defined on a normed space.

Definition 1 ([4])**.** Let $(X, \| \|)$ be a normed space, $r \in (0, 1)$, $f : X \to \mathbb{R}$ and $x \in X$. The function f will be called:

- $\mathcal{P}^s_r$-continuous at x if there exists a set $A \subset X$ such that $x \in A$, $p^s(X \setminus A, x) > r$ and $f_{\restriction A}$ is continuous at x;

- $\mathcal{S}_r^s$-continuous at x if for each $\varepsilon > 0$ there exists a set $A \subset X$ such that $x \in A$, $p^s(X \setminus A, x) > r$ and $f(A) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$;
- $\mathcal{M}_r^s$-continuous at x if there exists a set $A \subset X$ such that $x \in A$, $p^s(X \setminus A, x) \geq r$ and $f_{\restriction A}$ is continuous at x;
- $\mathcal{N}_r^s$-continuous at x if for each $\varepsilon > 0$ there exists a set $A \subset X$ such that $x \in A$, $p^s(X \setminus A, x) \geq r$ and $f(A) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$.

It is easily seen that result of addition and multiplication of functions from discussed classes of functions, in general, need not belong to these classes. Therefore we studied the following similar notion.

Definition 2 ([2])**.** Let $\mathcal{F}$ be a family of real functions defined on a normed space $(X, \| \|)$. A set $\mathfrak{M}_a(\mathcal{F}) = \{g\colon X \to \mathbb{R}\colon \forall_{f \in \mathcal{F}}\,(f + g \in \mathcal{F})\}$ is called the maximal additive class for $\mathcal{F}$.

Definition 3 ([2])**.** Let $\mathcal{F}$ be a family of real functions defined on $(X, \| \|)$. A set $\mathfrak{M}_m(\mathcal{F}) = \{g\colon X \to \mathbb{R}\colon \forall_{f \in \mathcal{F}}\ (f \cdot g \in \mathcal{F})\}$ is called the maximal multiplicative class for $\mathcal{F}$.

We describe maximal additive classes and maximal multiplicative classes for symmetrically porouscontinuous functions.

Theorem 1 ([4])**.** *Let $(X, \| \|)$ be a normed space, $r \in (0, 1)$ and $A \subset X$. The family of sets $U \subset X$ satisfying condition:*

$$\forall_{x \in U} \forall_{E \subset X,\, p^s(X \setminus E, x) \geq r}\ \left(p^s\left(X \setminus \left[(E \cap U) \cup A\right], x\right) \geq r\right)$$

forms a topology. We will denote it by $\mathcal{T}_r^s(A)$. The topology $\mathcal{T}_r^s(A)$ is stronger than the initial topology generated by the norm.

Theorem 2 ([4])**.** *Let $(X, \| \|)$ be a normed space, $r \in (0, 1)$ and $A \subset X$. The family of sets $U \subset X$ satisfying condition:*

$$\forall_{x \in U} \forall_{E \subset X,\, p^s(X \setminus E, x) > r}\ \left(p^s\left(X \setminus \left[(E \cap U) \cup A\right], x\right) > r\right)$$

forms a topology. We will denote it by $\tau_r^s(A)$. The topology $\tau_r^s(A)$ is stronger than the initial topology generated by the norm.

Finally we compare $\mathcal{T}_r(A)$ with $\mathcal{T}_r^s(A)$ (and $\tau_r(A)$ with $\tau_r^s(A)$) for different A and $r \in (0, 1)$. Topologies $\mathcal{T}_r(A)$, $\tau_r(A)$ are defined in [3]. In their definitions the symmetric porosity is replacing by the usual porosity.

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On the concept of generalization of $\mathcal{I}$–density points

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This paper deals with essential generalization of $\mathcal{I}$–density points and $\mathcal{I}$–density topology. In particular, there is an example showing that this generalization of $\mathcal{I}$–density point yields the stronger concept of density point than the notion of $\mathcal{I}(\mathcal{J})$–density. Some properties of topologies generated by operators related to this essential generalization of density points are provided.

Keywords: density topology, generalization of density topology.

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