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STATE COMPLEXITY OF CONCATENATION AND COMPLEMENTATION

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We investigate the state complexity of concatenation and the nondeterministic state complexity of complementation of regular languages. We show that the upper bounds on the state complexity of concatenation are also tight in the case that the first automaton has more than one accepting state. In the case of nondeterministic state complexity of complementation, we show that the entire range of complexities, up to the known upper bound can be produced.

Keywords: Regular languages; State complexity; Nondeterministic state complexity.

1. Introduction

Finite automata are one of the simplest computational models. Despite their simplicity, some challenging problems concerning finite automata are still open. For instance, we recall the question of how many states are sufficient and necessary for two-way deterministic finite automata to simulate two-way nondeterministic finite automata. The importance of this problem is underlined by its relation to the well-known open question whether or not DLOGSPACE equals NLOGSPACE [1, 25].

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Recently, a renewed interest in regular languages and finite automata can be observed. For a discussion, the reader may refer to [14, 29]. Some aspects of this area are now intensively investigated. One such aspect is the state complexity of regular languages and their operations.

The state complexity of a regular language is the number of states of its minimal deterministic finite automaton (DFA). The nondeterministic state complexity of a regular language is the number of states of a minimum state nondeterministic finite automaton (NFA) accepting the language. The state complexity (the nondeterministic state complexity) of an operation on regular languages represented by DFAs (NFAs, respectively) is the number of states that are sufficient and necessary in the worst case for a DFA (an NFA, respectively) to accept the language resulting from the operation.

Some early results on the state complexity of regular languages can be found in [19, 21, 20]. The state complexity of some operations on regular languages was investigated in [18, 2, 3]. Yu, Zhuang, and Salomaa [27] were the first to systematically study the complexity of regular language operations. Their paper was followed by several articles investigating the state complexity of finite language operations and unary language operations [4, 22, 23]. The nondeterministic state complexity of regular language operations was studied by Holzer and Kutrib in [10, 11, 12]. Further results on this topic are presented in [7, 5, 17] and state-of-the-art surveys for DFAs can be found in [31, 30].

In this paper, we investigate the state complexity of concatenation and the nondeterministic state complexity of complementation of regular languages. In the case of concatenation, we show that the upper bounds $m2^n - k2^{n-1}$ on the concatenation of an m-state DFA language and an n-state DFA language, where k is the number of accepting states in the m-state automaton, are tight for any integer k with $0 < k < m$. We prove the result for a binary alphabet. In the case of complementation, we show that for any positive integers n and α with $\log n \leq \alpha \leq 2^n$, there is an n-state NFA language such that minimal NFAs for its complement have exactly α states. To prove the result we use an alphabet the size of which grows exponentially with n. Then, we show that some special values of α (e.g., $\alpha = n - k + 2^k$, where $1 \leq k \leq n$, or $\alpha = 1 + n + k(k-1)/2$, where $2 \leq k \leq n$) can be reached by complementation of binary NFAs.

To prove the result on concatenation we show that a deterministic finite automaton is minimal. To obtain the result on complementation we use a fooling-set lower-bound technique known from communication complexity theory [13], cf. also [2, 3, 8].

The paper consists of five sections, including this introduction. The next section contains basic definitions and notations used throughout the paper. In Section 3, we present our result on concatenation. Section 4 deals with the problem of which kind of relations between the sizes of minimal NFAs for regular languages and minimal NFAs for their complements are possible. In this section, we give a complete solution for a growing alphabet and some partial solutions for a binary alphabet. The last section contains concluding remarks and open problems.

2. Preliminaries

In this section, we recall some basic definitions and notations. For further details, the reader may refer to [26, 28].

Let Σ be an alphabet and Σ^* the set of all strings over the alphabet Σ including the empty string ε. The length of a string w is denoted by $|w|$ and the number of occurrences of a symbol a in a string w by $\#_a(w)$. The power-set of a finite set A is denoted by 2^A and its maximum by $\max A$.

A *deterministic finite automaton* (DFA) is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$, where Q is a finite set of states, Σ is a finite input alphabet, $\delta : Q \times \Sigma \to Q$ is the transition function, $q_0 \in Q$ is the initial state, and $F \subseteq Q$ is the set of accepting states. In this paper, all DFAs are assumed to be complete, i.e., the next state $\delta(q, a)$ is defined for any state q in Q and any symbol a in Σ. The transition function δ is extended to a function from $Q \times \Sigma^*$ to Q in the natural way. A string w in Σ^* is accepted by the DFA M if the state $\delta(q_0, w)$ is an accepting state of the DFA M.

A *nondeterministic finite automaton* (NFA) is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$, where Q, Σ, q_0, and F are as above, and $\delta : Q \times \Sigma \to 2^Q$ is the transition function which can be naturally extended to the domain $Q \times \Sigma^*$. A string w in Σ^* is accepted by the NFA M if the set $\delta(q_0, w)$ contains an accepting state of the NFA M.

The *language accepted by* a finite automaton M, denoted $L(M)$, is the set of all strings accepted by the automaton M. Two automata are said to be *equivalent* if they accept the same language.

A DFA (an NFA) M is called *minimal* if all DFAs (all NFAs, respectively) that are equivalent to M have at least as many states as M. By a well-known result, each regular language has a unique minimal DFA, up to isomorphism. However, the same result does not hold for minimal NFAs.

Any nondeterministic finite automaton $M = (Q, \Sigma, \delta, q_0, F)$ can be converted to an equivalent deterministic finite automaton $M' = (2^Q, \Sigma, \delta', q_0', F')$ using an algorithm known as the "subset construction" [24] in the following way. Every state of the DFA M' is a subset of the state set Q. The initial state of the DFA M' is $\{q_0\}$. A state R in 2^Q is an accepting state of the DFA M' if it contains an accepting state of the NFA M. The transition function δ' is defined by $\delta'(R, a) = \bigcup_{r \in R} \delta(r, a)$ for any state R in 2^Q and any symbol a in Σ. The DFA M' need not be minimal since some of its states may be unreachable or equivalent.

3. Concatenation

We start our investigation with concatenation operation. The state complexity of concatenation of regular languages represented by deterministic finite automata was studied by Yu *et al.* [27]. They showed that $m2^n - k2^{n-1}$ states are sufficient for a DFA to accept the concatenation of an m-state DFA language and an n-state DFA language, where k is the number of the accepting states in the m-state DFA. In the case of $n = 1$, the upper bound m was shown to be tight, even for a unary alphabet. In the case of $m = 1$ and $n \geq 2$, the worst case $2^n - 2^{n-1}$ was given by the concatenation of two binary languages. Otherwise, the upper bound $m2^n - 2^{n-1}$

was shown to be tight for a binary alphabet in [17]. In the case of unary languages, the upper bound on concatenation is mn and it is known to be tight if m and n are relatively prime [27]. The unary case when m and n are not necessarily relatively prime was studied by Pighizzini and Shallit in [22, 23]. In this case, the tight bounds are given by the number of states in the noncyclic and in the cyclic parts of the resulting automata. The next theorem shows that the upper bounds $m2^n - k2^{n-1}$ are also tight if the first automaton has k accepting states, where $0 < k < m$. The witness languages are defined over a binary alphabet.

Theorem 1 *For any integers m, n, k such that $m \geq 2, n \geq 2$, and $0 < k < m$, there exist a binary DFA A of m states and k accepting states, and a binary DFA B of n states such that any DFA accepting the language $L(A)L(B)$ needs at least $m2^n - k2^{n-1}$ states.*

Proof. Let m, n, and k be arbitrary but fixed integers such that $m \geq 2, n \geq 2$, and $0 < k < m$. Let $\Sigma = \{a, b\}$.

Define an m-state DFA $A = (Q_A, \Sigma, \delta_A, q_0, F_A)$, where $Q_A = \{q_0, q_1, \dots, q_{m-1}\}$, $F_A = \{q_{m-k}, q_{m-k+1}, \dots, q_{m-1}\}$, and for any $i \in \{0, 1, \dots, m-1\}$,

$$\delta_A(q_i, X) = \begin{cases} q_{(i+1) \bmod m}, & \text{if } X = a, \\ q_i, & \text{if } X = b. \end{cases}$$

Define an n-state DFA $B = (Q_B, \Sigma, \delta_B, 0, F_B)$, where $Q_B = \{0, 1, \dots, n-1\}$, $F_B = \{n-1\}$, and for any $i \in \{0, 1, \dots, n-1\}$,

$$\delta_B(i, X) = \begin{cases} (i+1) \bmod n, & \text{if } X = a, \\ 0, & \text{if } i = 0 \text{ and } X = b, \\ (i+1) \bmod n, & \text{if } i > 0 \text{ and } X = b. \end{cases}$$

The DFA A and B are shown in Fig. 1 and Fig. 2, respectively.

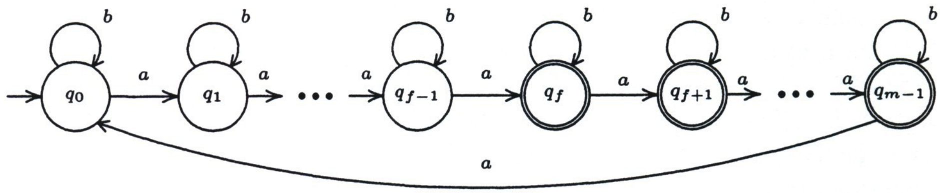

Fig. 1. The deterministic finite automaton A; $f = m - k$.

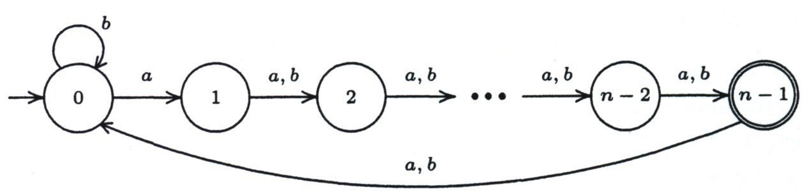

Fig. 2. The deterministic finite automaton B.

We first describe an NFA accepting the language $L(A)L(B)$, then we construct an equivalent DFA, and show that the DFA has at least $m2^n - k2^{n-1}$ reachable states no two of which are equivalent.

Consider the NFA $C = (Q, \Sigma, \delta, q_0, F)$, where $Q = Q_A \cup Q_B$, $F = \{n-1\}$, and for any $q \in Q$ and any $X \in \Sigma$,

$$\delta(q, X) = \begin{cases} \{\delta_A(q, X)\}, & \text{if } q \in Q_A \setminus F_A, \\ \{\delta_A(q, X), \delta_B(0, X)\}, & \text{if } q \in F_A, \\ \{\delta_B(q, X)\}, & \text{if } q \in Q_B, \end{cases}$$

see Fig. 3.

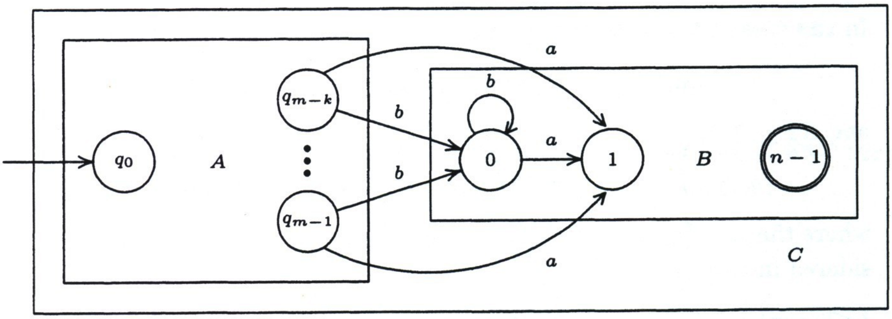

Fig. 3. The nondeterministic finite automaton C.

Clearly, the NFA C accepts the language $L(A)L(B)$. Let $C' = (2^Q, \Sigma, \delta', \{q_0\}, F')$ be the DFA obtained from the NFA C by the subset construction. Let $\mathcal{R}$ be the following system of sets:

$$\mathcal{R} = \{\{q\} \cup S \mid q \in Q_A \setminus F_A \text{ and } S \subseteq Q_B\} \cup \{\{q\} \cup S \mid q \in F_A, S \subseteq Q_B \text{ and } 0 \in S\},$$

i.e., any set in $\mathcal{R}$ consists of exactly one state of Q_A and some states of Q_B, and if a set in $\mathcal{R}$ contains a state of F_A, then it also contains state 0. There are $m2^n - k2^{n-1}$ sets in $\mathcal{R}$. To prove the theorem it is sufficient to show that (I) any set in $\mathcal{R}$ is a reachable state of the DFA C' and (II) no two different states in $\mathcal{R}$ are equivalent.

We prove (I) by induction on the size of sets. The singletons $\{q_0\}, \{q_1\}, \ldots, \{q_{m-k-1}\}$ are reachable since $\{q_i\} = \delta'(\{q_0\}, a^i)$ for $i = 0, 1, \ldots, m-k-1$. Let $1 \le s \le n$ and assume that any set in $\mathcal{R}$ of size s is a reachable state of the DFA C'. Using this assumption we prove that any set $\{q_i, j_1, j_2, \ldots, j_s\}$, where
$0 \le j_1 < j_2 < \cdots < j_s \le n-1$ if $0 \le i \le m-k-1$, and
$0 = j_1 < j_2 < \cdots < j_s \le n-1$ if $m-k \le i \le m-1$,
is a reachable state of the DFA C'. There are four cases:

(i) $m-k+1 \le i \le m-1$ and $j_1 = 0$. We prove this case by induction on i. For $i = m-k+1$, we have

$$\{q_{m-k+1}, 0, j_2, \ldots, j_s\} = \delta'(\{q_{m-k-1}, j_2 - 1, \ldots, j_s - 1\}, aab^{n-1}),$$

where the latter set is reachable by induction on s. Next, since

$$\{q_{i+1}, 0, j_2, \ldots, j_s\} = \delta'(\{q_i, 0, j_2, \ldots, j_s\}, ab^{n-1})$$

for $i = m-k+1, m-k+2, \ldots, m-2$, we are ready in this case.

(ii) $i = 0$. In the case of $k = 1$, we have

$$\{q_0, 0, j_2, \dots, j_s\} = \delta'(\{q_{m-2}, j_2 - 1, \dots, j_s - 1\}, aab^{n-1}),$$

and for $j_1 \geq 1$,

$$\{q_0, j_1, j_2, \dots, j_s\} = \delta'(\{q_{m-2}, j_2 - j_1 - 1, \dots, j_s - j_1 - 1\}, aab^{j_1 - 1}),$$

where the sets $\{q_{m-2}, j_2 - 1, \dots, j_s - 1\}$ and $\{q_{m-2}, j_2 - j_1 - 1, \dots, j_s - j_1 - 1\}$ of size s are reachable by induction.

In the case of $k \geq 2$, we have

$$\{q_0, 0, j_2, \dots, j_s\} = \delta'(\{q_{m-1}, 0, j_2, \dots, j_s\}, ab^{n-1}),$$

and for $j_1 \geq 1$,

$$\{q_0, j_1, j_2, \dots, j_s\} = \delta'(\{q_{m-1}, 0, j_2 - j_1, \dots, j_s - j_1\}, ab^{j_1 - 1}),$$

where the sets $\{q_{m-1}, 0, j_2, \dots, j_s\}$ and $\{q_{m-1}, 0, j_2 - j_1, \dots, j_s - j_1\}$ are considered in case (i).

(iii) $1 \leq i \leq m - k - 1$. Then we have

$$\{q_i, j_1, j_2, \dots, j_s\} = \delta'(\{q_0, (j_1 - i) \bmod n, \dots, (j_s - i) \bmod n\}, a^i),$$

where the latter set is considered in case (ii).

(iv) $i = m - k$ and $j_1 = 0$. Then we have

$$\{q_{m-k}, 0, j_2, \dots, j_s\} = \delta'(\{q_{m-k-1}, j_2 - 1, \dots, j_s - 1, n - 1\}, a),$$

where the latter set is considered in case (iii).

To prove (II) let $\{q_i\} \cup S$ and $\{q_j\} \cup T$ be two different states in the system $\mathcal{R}$ with $0 \leq i \leq j \leq m - 1$. There are two cases:

(i) $i = j$. Without loss of generality, there is a state l in Q_B such that $l \in S$ and $l \notin T$ (note that $l \geq 1$ if $m - k \leq i \leq m - 1$). Then, the string a^{n-1-l} is accepted by the DFA C' starting in state $\{q_i\} \cup S$ but it is not accepted by the DFA C' starting in state $\{q_j\} \cup T$.

(ii) $i < j$. We will consider two subcases:

(a) $j - i \leq k$. Let $v = a^{m-1-j}b^n aab^{n-2}$. Then

$$q_{m-j+i} \in \delta(q_i, a^{m-1-j}b^n a),$$

where $m - k \leq m - j + i \leq m - 1$. It follows that

$$n - 1 \in \delta(q_{m-j+i}, ab^{n-2}),$$

and so the string v is accepted by the DFA C' starting in state $\{q_i\} \cup S$. On the other hand, we have

$$\delta'(\{q_j\} \cup T, a^{m-1-j}b^n a) = \{q_0, 1\} \text{ and } \delta'(\{q_0, 1\}, ab^{n-2}) = \{q_1, 0\},$$

so the string v is not accepted by the DFA C' starting in state $\{q_j\} \cup T$.

(b) $j - i > k$. Let $w = a^{m-1-i-k}b^n aab^{n-2}$. Then

$$q_{m-k} \in \delta(q_i, a^{m-1-i-k}b^n a) \text{ and } n-1 \in \delta(q_{m-k}, ab^{n-2}).$$

It follows that the string w is accepted by the DFA C' starting in state $\{q_i\} \cup S$. On the other hand, we have

$$\delta'(\{q_j\} \cup T, a^{m-1-i-k}b^n a) = \{q_{(m-i-k+j) \bmod m}, 1\},$$

where $(m - i - k + j) \bmod m = j - i - k \leq m - k - 1$, and so

$$\delta'(\{q_{j-i-k}, 1\}, ab^{n-2}) = \{q_{j-i-k+1}, 0\}.$$

Thus the string w is not accepted by the DFA C' starting in state $\{q_j\} \cup T$ which completes our proof. □

4. Complementation

We now turn our attention to complementation operation. For DFAs, it is an efficient operation since to accept the complement we can simply exchange accepting and rejecting states. On the other hand, the complementation of NFAs is an expensive task. The upper bound on the size of an NFA accepting the complement of an n-state NFA language is 2^n and it is known to be tight for a binary alphabet [17]. For complementation of unary NFA languages, a crucial role is played by the function $F(n) = \max\{\text{lcm }(x_1, \ldots, x_k) \mid x_1 + \cdots + x_k = n\}$. It is known that $F(n) \in e^{\Theta(\sqrt{n \ln n})}$ and that $O(F(n))$ states suffice to simulate any unary n-state NFA by a DFA [6]. This means that $O(F(n))$ states are sufficient for an NFA to accept the complement of an n-state unary NFA language. The lower bound is known to be $F(n-1)$ in this case [11].

In this section, we deal with the question of which kind of relations between the nondeterministic complexity of a regular language and the nondeterministic complexity of its complement are possible. We provide a complete solution by showing that for any positive integers n and α with $\log n \leq \alpha \leq 2^n$, there exists an n-state NFA language such that minimal NFAs for its complement have exactly α states. However, to prove this result we use languages defined over an alphabet the size of which grows exponentially with n. In the second part of this section, we show that some special values of α can be reached by complementation of binary NFAs.

To obtain the above results we use a fooling-set lower-bound technique known from communication complexity theory [13], cf. also [2, 3, 8]. Although lower bounds based on fooling sets may sometimes be exponentially smaller than the true bounds [15, 16], for some regular languages the lower bounds are tight [2, 3, 8]. In this section, the technique helps us to obtain tight lower bounds. After defining a fooling set, we give the lemma from [2] describing a fooling-set lower-bound technique. For the sake of completeness, we recall its proof here. Then, we give an example.

Definition 1 *A set of pairs of strings* $\{(x_i, y_i) \mid i = 1, 2, \ldots, n\}$ *is said to be a fooling set for a regular language* L *if for any* i *and* j *in* $\{1, 2, \ldots, n\}$*,*
(1) the string $x_i y_i$ *is in the language* L*, and*
(2) if $i \neq j$*, then at least one of the strings* $x_i y_j$ *and* $x_j y_i$ *is not in the language* L*.*

Lemma 1 (Birget [2]) *Let a set of pairs* $\{(x_i, y_i) \mid i = 1, 2, \ldots, n\}$ *be a fooling set for a regular language* L*. Then any NFA for the language* L *needs at least* n *states.*

Proof. Let $M = (Q, \Sigma, \delta, q_0, F)$ be any NFA accepting the language L. Since $x_i y_i \in L$, there is a state p_i in Q such that $p_i \in \delta(q_0, x_i)$ and $\delta(p_i, y_i) \cap F \neq \emptyset$ (i.e., p_i is a state on an accepting computation on $x_i y_i$ that is reached after reading x_i). Assume that a fixed choice of p_i has been made for any i in $\{1, 2, \ldots, n\}$. We prove that $p_i \neq p_j$ for $i \neq j$. Suppose by contradiction that $p_i = p_j$ for some $i \neq j$. Then the NFA M accepts both strings $x_i y_j$ and $x_j y_i$ which contradicts the assumption that the set $\{(x_i, y_i) \mid 1 \leq i \leq n\}$ is a fooling set for the language L. Hence the NFA M has at least n states. □

Example 1 Let $n \geq 1$, let $L_n = \{w \in \{a, b\}^* \mid \#_a(w) \equiv 0 \bmod n\}$, and let $\mathcal{A}_n = \{(a^i, a^{n-i}) \mid i = 1, 2, \ldots, n\}$. Note that for any i and j in $\{1, 2, \ldots, n\}$,
(1) $a^i a^{n-i} = a^n$ and the string a^n is in the language L_n, and
(2) if $i < j$, then $a^i a^{n-j} = a^{n-(j-i)}$ and the string $a^{n-(j-i)}$ is not in the language L_n since $0 < n - (j - i) < n$.
Hence the set $\mathcal{A}_n$ is a fooling set for the language L_n, and so any NFA for the language L_n needs at least n states.

We start our investigation with two propositions.

Proposition 1 *For any* α *in* $\{1, 2\}$*, there is a 1-state NFA* D_α *such that minimal NFAs for the complement of the language* $L(D_\alpha)$ *have* α *states.*

Proof. Let $\Sigma = \{a, b\}$. Consider the following 1-state NFAs:
$D_1 = (\{s\}, \Sigma, \delta_1, s, \{s\})$ with $\delta_1(s, X) = \{s\}$ for any $X \in \Sigma$,
$D_2 = (\{s\}, \Sigma, \delta_2, s, \{s\})$ with $\delta_2(s, a) = \{s\}$ and $\delta_2(s, b) = \emptyset$.
The NFAs D_1 and D_2 do satisfy the proposition since the complement of the language $L(D_1)$ is the empty language, and the set of pairs of strings $\{(\varepsilon, b), (b, \varepsilon)\}$ is a fooling set for the complement of the language $L(D_2)$, and so minimal NFAs for the complement of the language $L(D_2)$ have 2 states. □

Proposition 2 *For any integer* $n \geq 2$*, there is a minimal NFA* N *of* n *states such that minimal NFAs for the complement of the language* $L(N)$ *have exactly* n *states.*

Proof. Let n be arbitrary but fixed integer with $n \geq 2$. Let $\Sigma = \{a, b\}$.
Define an n-state NFA $N = (Q, \Sigma, \delta, n, F)$, where $Q = \{1, 2, \ldots, n\}$, $F = \{2, 3, \ldots, n\}$, and for any $i \in Q$ and any $X \in \Sigma$,

$$\delta(i, X) = \begin{cases} \{2\}, & \text{if } i = 1 \\ \{i-1\}, & \text{if } i > 1 \text{ and } X = a, \\ \{1\}, & \text{if } i > 1 \text{ and } X = b, \end{cases}$$

see Fig. 4. We are going to show that (a) the NFA N is a minimal NFA for the language $L(N)$; (b) the language $L^c(N)$ is accepted by an n-state DFA; (c) any NFA for the language $L^c(N)$ needs at least n states. Then, the proposition follows.

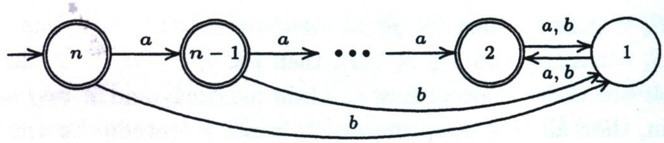

Fig. 4. The nondeterministic finite automaton N.

To prove (a) consider the set of pairs $\mathcal{A} = \{(a^{i-1}, a^{n-i}b) \mid i = 1, 2, \ldots, n\}$. The set $\mathcal{A}$ is a fooling set for the language $L(N)$ because for any i and j in $\{1, 2, \ldots, n\}$,
(1) $a^{i-1}a^{n-i}b \in L(N)$ since the string $a^{n-1}b$ is accepted by the NFA N, and
(2) if $i < j$, then $a^{i-1}a^{n-j}b \notin L(N)$ since any string $a^l b$ with $0 \leq l < n-1$ is not accepted by the NFA N.
By Lemma 1, any NFA for the language $L(N)$ needs at least n states and so the NFA N is minimal.

To prove (b) note that the NFA N is, in fact, deterministic, and so after exchanging the accepting and the rejecting states we obtain an n-state DFA for the language $L^c(N)$.

To prove (c) consider the set of pairs $\mathcal{B} = \{(a^{i-1}, a^{n-i}) \mid i = 1, 2, \ldots, n\}$. The set $\mathcal{B}$ is a fooling set for the language $L^c(N)$ because for any i and j in $\{1, 2, \ldots, n\}$,
(1) $a^{i-1}a^{n-i} \in L^c(N)$ since the string a^{n-1} is not accepted by the NFA N, and
(2) if $i < j$, then $a^{i-1}a^{n-j} \notin L^c(N)$ since any string a^l with $0 \leq l < n-1$ is accepted by the NFA N.
By Lemma 1, any NFA for the language $L^c(N)$ needs at least n states and our proof is complete. □

The following theorem is proved in [17].

Theorem 2 ([17]) *For any positive integer n, there exists a binary NFA M of n states such that any NFA for the complement of the language $L(M)$ needs at least 2^n states.*

In the next theorem, we show that the nondeterministic state complexity of complements of n-state NFA languages may be arbitrary between $n+1$ and $2^n - 1$.

Theorem 3 *For any integers n and α with $3 \leq n+1 \leq \alpha \leq 2^n - 1$, there exists a minimal NFA M of n states such that minimal NFAs for the complement of the language $L(M)$ have exactly α states.*

Proof. Let n and α be arbitrary but fixed integers such that $3 \leq n+1 \leq \alpha \leq 2^n - 1$. Then α can be expressed as $\alpha = n+k$ for an integer k with $1 \leq k \leq 2^n - 1 - n$. Let

$$\Sigma = \{a, b\} \cup \{c_1, c_2, \ldots, c_k\} \cup \{d_1, d_2, \ldots, d_k\}$$

be a $(2k+2)$-letter alphabet. We are going to define a minimal n-state NFA M over the alphabet Σ such that minimal NFAs for the language $L^c(M)$ have exactly $n+k$ states. To this aim let

$$S_1, S_2, \ldots, S_{2^n-1-n}$$

be a sequence of subsets of the set $\{1, 2, \ldots, n\}$ that contain at least two elements and are ordered in such a way that for any i and j in $\{1, 2, \ldots, 2^n - 1 - n\}$, the following two conditions hold:

(1) if $\max S_i < \max S_j$, then $i < j$;
(2) if $\max S_i = \max S_j$ and $1 \in S_i \setminus S_j$, then $i < j$,
i.e., the subsets are ordered according to their maxima, and if two sets have the same maximum, then all sets that contain the state 1 precede the sets that do not contain the state 1. Clearly, there are several such orderings, we choose one of them. Note that $S_1 = \{1,2\}$. For example, the subsets of $\{1,2,3,4\}$ that contain at least two elements could be ordered as follows: $S_1 = \{1,2\}$, $S_2 = \{1,3\}, S_3 = \{1,2,3\}$, $S_4 = \{2,3\}$, $S_5 = \{1,4\}$, $S_6 = \{1,2,4\}$, $S_7 = \{1,3,4\}, S_8 = \{1,2,3,4\}$, $S_9 = \{2,4\}$, $S_{10} = \{3,4\}$, $S_{11} = \{2,3,4\}$.

Define an n-state NFA $M = (Q, \Sigma, \delta, n, F)$, where $Q = \{1,2,\ldots,n\}$, $F = \{2,3,\ldots,n\}$, and for any $i \in Q$ and any $j \in \{1,2,\ldots,k\}$,

$$\delta(i,X) = \begin{cases} \{1,2\}, & \text{if } i = 1 \text{ and } X = a, \\ \{i-1\}, & \text{if } i > 1 \text{ and } X = a, \\ \{2\}, & \text{if } i = 1 \text{ and } X = b, \\ \{1\}, & \text{if } i > 1 \text{ and } X = b, \\ S_j, & \text{if } i = 1 \text{ and } X = c_j, \\ \{1\}, & \text{if } i > 1 \text{ and } X = c_j, \\ \{1\}, & \text{if } i \in S_j \text{ and } X = d_j, \\ \{2\}, & \text{if } i \notin S_j \text{ and } X = d_j, \end{cases}$$

see Fig. 5.

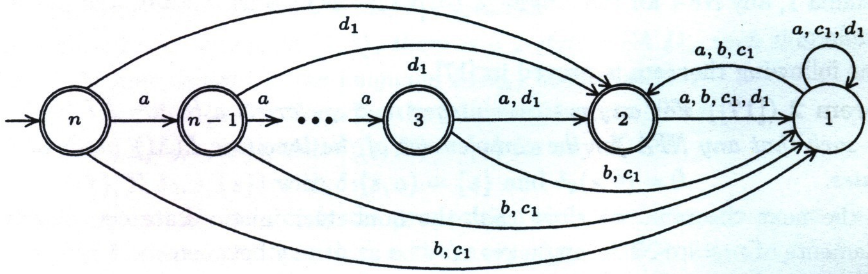

Fig. 5. Transitions on reading a, b, c_1, d_1 in the NFA M.

We will show that
(a) the NFA M is a minimal NFA for the language $L(M)$;
(b) the language $L^c(M)$ can be accepted by an $(n+k)$-state DFA;
(c) any NFA for the language $L^c(M)$ needs at least $n+k$ states.
Then, the theorem follows immediately.

To prove (a) consider the following set of pairs of strings

$$\mathcal{A} = \{(a^{i-1}, a^{n-i}b) \mid i = 1,2,\ldots,n\}.$$

The set $\mathcal{A}$ is a fooling set for $L(M)$ because for any i and j in $\{1,2,\ldots,n\}$,
(1) $a^{i-1}a^{n-i}b = a^{n-1}b$ and the string $a^{n-1}b$ is in the language $L(M)$ since it is accepted by the NFA M, and
(2) if $i < j$, then the string $a^{i-1}a^{n-j}b$ is not in the language $L(M)$ since for any l with $0 \leq l < n-1$, the string $a^l b$ is not accepted by the NFA M.

By Lemma 1, any NFA for the language $L(M)$ needs at least n states and so the NFA M is minimal.

To prove (b) let $M' = (2^Q, \Sigma, \delta', \{n\}, F')$ be the DFA obtained from the NFA M by the subset construction. Let $\mathcal{R}$ be the following system of sets

$$\mathcal{R} = \{\{1\}, \{2\}, \ldots, \{n\}, S_1, S_2, \ldots, S_k\}.$$

Note that the initial state $\{n\}$ of the DFA M' and the state $S_1 = \{1, 2\}$ belong to the system $\mathcal{R}$. We are going to prove that any set in $\mathcal{R}$ is a reachable state of the DFA M' and no other states are reachable in the DFA M'. Clearly, any set of the system $\mathcal{R}$ is reachable since we have $\{i\} = \delta'(\{n\}, a^{n-i})$ for $i = 1, 2, \ldots, n$, and $S_j = \delta'(\{1\}, c_j)$ for $j = 1, 2, \ldots, k$. To prove that no other subset of the set Q is a reachable state of the DFA M' it is sufficient to show that for any state R in $\mathcal{R}$ and any symbol X in Σ, the state $\delta'(R, X)$ is also in the system $\mathcal{R}$. There are three cases:

(i) $R = \{1\}$. Then we have $(j = 1, 2, \ldots, k)$:

$$\delta'(\{1\}, X) = \begin{cases} \{1,2\}, & \text{if } X = a, \\ \{2\}, & \text{if } X = b, \\ S_j, & \text{if } X = c_j, \\ \{1\}, & \text{if } 1 \in S_j \text{ and } X = d_j, \\ \{2\}, & \text{if } 1 \notin S_j \text{ and } X = d_j. \end{cases}$$

Since all sets on the right are in the system $\mathcal{R}$, we are ready in this case.

(ii) $R = \{i\}$ for an $i \neq 1$. Then for any X in Σ, the set $\delta'(\{i\}, X)$ is a singleton set and so is in $\mathcal{R}$.

(iii) $R = S_j$ for a j in $\{1, 2, \ldots, k\}$. Then the set $\delta'(S_j, a)$ is a subset of the set $\{1, 2, \ldots, \max S_k - 1\}$ or equals $\{1, 2\}$. Since the sets $S_1, S_2, \ldots, S_k$ are ordered according to their maxima, any subset of $\{1, 2, \ldots, \max S_k - 1\}$ is in the system $\mathcal{R}$. Next, the set $\delta'(S_j, b)$ is equal either to $\{1\}$ or to $\{1, 2\}$, and the set $\delta'(S_j, d_l)$, $l = 1, 2, \ldots, k$, is equal either to $\{1\}$, or to $\{2\}$, or to $\{1, 2\}$. Finally, the set $\delta'(S_j, c_l)$, $l = 1, 2, \ldots, k$, is equal either to $\{1\}$ or to $S_l \cup \{1\}$. Since the set $S_l \cup \{1\}$ precedes the set S_l or equals S_l, we are ready in this case.

Thus we have shown that the DFA M' obtained from the NFA M by the subset construction has exactly $n + k$ reachable states. After exchanging the accepting and the rejecting states of the DFA M' we obtain an $(n + k)$−state DFA for the language $L^c(M)$ which proves (b).

To prove (c) consider the following sets of pairs of strings

$$\mathcal{B} = \{(a^{i-1}, a^{n-i}) \mid i = 1, 2, \ldots, n\},$$

$$\mathcal{C} = \{(a^{n-1}c_j, d_j) \mid j = 1, 2, \ldots, k\}.$$

We will show that the set $\mathcal{B} \cup \mathcal{C}$ is a fooling set for the language $L^c(M)$.

(1) For any $i \in \{1, 2, \ldots, n\}$, the string $a^{i-1}a^{n-i}$ is in the language $L^c(M)$ since the string a^{n-1} is not accepted by the NFA M.
For any $j \in \{1, 2, \ldots, k\}$, the string $a^{n-1}c_jd_j$ is in the language $L^c(M)$ since

$$\delta(n, a^{n-1}) = \{1\}, \quad \delta(\{1\}, c_j) = S_j, \quad \delta(S_j, d_j) = \{1\}, \text{ and } 1 \notin F,$$

and so the string $a^{n-1}c_jd_j$ is not accepted by the NFA M.

(2) If $1 \le i < s \le n$, then the string $a^{i-1}a^{n-s}$ is not in the language $L^c(M)$ since the NFA M accepts any string a^l with $0 \le l < n-1$.
Next, if $1 \le j, t \le k$ and $j \ne t$, then, w.l.o.g., there is a state p in Q such that $p \in S_j$ and $p \notin S_t$. Thus,

$$p \in \delta(n, a^{n-1}c_j) \text{ and } 2 \in \delta(p, d_t),$$

and so the string $a^{n-1}c_jd_t$ is accepted by the NFA M, i.e., is not in the language $L^c(M)$.
Finally, if $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, k\}$, then the string $a^{n-1}c_ja^{n-i}$ is not in the language $L^c(M)$ since $\delta(n, a^{n-1}c_j) = S_j$, the size of the set S_j is at least two, and the string a^{n-i} is not accepted by the NFA M starting in state $n - i + 1$ but it is accepted by M starting in any other state.

Thus the set $\mathcal{B} \cup \mathcal{C}$ is a fooling set for the language $L^c(M)$. By Lemma 1, any NFA for the language $L^c(M)$ needs at least $n + k$ states which completes our proof. □

Corollary 1 *For any positive integers r and s with $\log r \le s \le r$, there exists a minimal NFA E of r states such that minimal NFAs for the complement of the language $L(E)$ have s states.*

Proof. Let r and s be arbitrary but fixed positive integers with $\log r \le s \le r$. Then we have

$$s \le r \le 2^s,$$

and by the above results, there is a minimal s-state NFA S such that a minimal NFA, say R, for the language $L^c(S)$ has r states. Set $E = R$. Then the NFA E is a minimal r-state NFA for the language $L^c(S)$, and minimal NFAs for the complement of the language $L^c(S)$, i.e., for $L^c(E)$, have s states. □

Hence, we have shown the following result.

Theorem 4 *For any positive integers n and α with $\log n \le \alpha \le 2^n$, there exists a minimal NFA M of n states such that minimal NFAs for the complement of the language $L(M)$ have exactly α states.*

To prove the above result, we used NFAs defined over an alphabet the size of which grows exponentially with n. The next theorem shows that some special values of α can be reached by complementation of binary NFAs. Note that for $k = n$, the NFA described in the theorem is similar as in [17].

Theorem 5 *For any integers n and k such that $1 \le k \le n$, there exists a minimal binary NFA M of n states such that minimal NFAs for the complement of the language $L(M)$ have exactly $n - k + 2^k$ states.*

Proof. Let n and k be arbitrary but fixed integers with $1 \leq k \leq n$. Let $\Sigma = \{a, b\}$.

Define an n-state NFA $M = (Q, \Sigma, \delta, n, F)$, where $Q = \{1, 2, \ldots, n\}$, $F = \{1\}$, and for any $i \in Q$,

$$\delta(i, X) = \begin{cases} \emptyset, & \text{if } i = 1 \text{ and } X = a, \\ \{i-1\}, & \text{if } i > 1 \text{ and } X = a, \\ \{1, 2, \ldots, k\}, & \text{if } i = 1 \text{ and } X = b, \\ \{k, i-1\}, & \text{if } 2 \leq i \leq k \text{ and } X = b, \\ \{i-1\}, & \text{if } i > k \text{ and } X = b. \end{cases}$$

The NFA M is shown in Fig. 6.

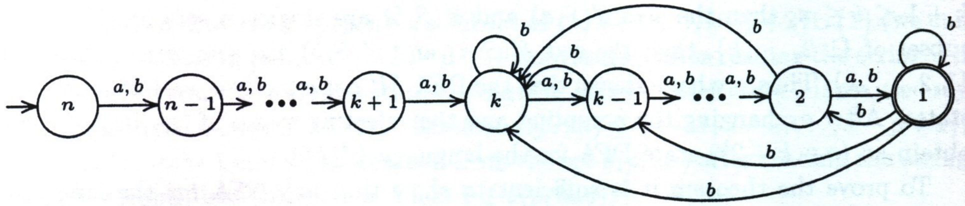

Fig. 6. The nondeterministic finite automaton M.

To prove that the NFA M is minimal let us consider the set of pairs of strings

$$\{(a^{i-1}, a^{n-i}) \mid i = 1, 2, \ldots, n\}.$$

It is a fooling set for the language $L(M)$ since for any i and j in $\{1, 2, \ldots, n\}$
(1) $a^{i-1}a^{n-i} = a^{n-1}$ and the string a^{n-1} is in the language $L(M)$,
(2) if $i < j$, then $a^{i-1}a^{n-j} = a^{n-1-(j-i)}$ and the string $a^{n-1-(j-i)}$ is not in the language $L(M)$ since $n - 1 - (j - i) < n - 1$.
By Lemma 1, any NFA for the language $L(M)$ needs at least n states and so the NFA M is minimal.

Now, let $M' = (2^Q, \Sigma, \delta', \{n\}, F')$ be the DFA obtained from the NFA M by the subset construction. We are going to show that the DFA M' has $n-k+2^k$ reachable states which follows that the language $L^c(M)$ is accepted by an $(n-k+2^k)$-state DFA (obtained from the DFA M' by exchanging the accepting and the rejecting states). Let $\mathcal{R}$ be the following system of sets

$$\mathcal{R} = \{\{i\} \mid k+1 \leq i \leq n\} \cup \{S \mid S \subseteq \{1, 2, \ldots, k\}\}.$$

The system $\mathcal{R}$ contains $n-k+2^k$ sets. We will show that any set in $\mathcal{R}$ is a reachable state of the DFA M' and no other states are reachable in the DFA M'.

Clearly, any singleton set $\{i\}$, where $k+1 \leq i \leq n$, is reachable since $\{i\} = \delta'(\{n\}, a^{n-i})$.

Next, we show that any subset of $\{1, 2, \ldots, k\}$ is a reachable state of the DFA M'. We prove this by induction on the size of sets. The empty set and the singletons $\{1\}, \{2\}, \ldots, \{k\}$ are reachable because we have

$$\emptyset = \delta'(\{n\}, a^n) \text{ and } \{i\} = \delta'(\{n\}, a^{n-i}) \text{ for } i = 1, 2, \ldots, k.$$

Let $2 \leq m \leq k$ and assume that any subset of $\{1, 2, \ldots, k\}$ of size $m-1$ is a reachable state of the DFA M'. Let $\{i_1, i_2, \ldots, i_m\}$, where $k \geq i_1 > i_2 > \cdots > i_m \geq 1$, be a subset of size m. Then we have

$$\{i_1, i_2, \ldots, i_m\} = \delta'(\{k + i_2 - i_1 + 1, k + i_3 - i_1 + 1, \ldots, k + i_m - i_1 + 1\}, ba^{k-i_1}),$$

where the latter set is reachable by induction (note that $k \geq k + i_j - i_1 + 1 \geq 2$ for $j = 2, 3, \ldots, m$). It follows that the set $\{i_1, i_2, \ldots, i_m\}$ is reachable. Hence any set of the system $\mathcal{R}$ is a reachable state of the DFA M'.

To prove that no other subset of the state set Q is reachable in the DFA M' it is sufficient to show that for any set S in $\mathcal{R}$ and any symbol X in Σ the set $\delta'(S, X)$ is in the system $\mathcal{R}$. This is easy to see because if S is a singleton $\{i\}$, where $k + 1 \leq i \leq n$, then the sets $\delta'(S, a)$ and $\delta'(S, b)$ are singleton sets and if S is a subset of $\{1, 2, \ldots, k\}$, then the sets $\delta'(S, a)$ and $\delta'(S, b)$ are also some subsets of $\{1, 2, \ldots, k\}$. Thus, we have shown that the DFA M' has exactly $n-k+2^k$ reachable states. After exchanging the accepting and the rejecting states of the DFA M' we obtain an $(n - k + 2^k)$-state DFA for the language $L^c(M)$.

To prove the theorem it is sufficient to show that any NFA for the language $L^c(M)$ needs at least $n - k + 2^k$ states. We will do this in the following way. For any set S in the system $\mathcal{R}$ we will define a pair of strings (x_S, y_S) such that the string $x_S y_S$ is in the language $L^c(M)$ and, moreover, if S and T are two different sets in $\mathcal{R}$, then at least one of the strings $x_S y_T$ and $x_T y_S$ is not in the language $L^c(M)$. Then, the set of pairs $\{(x_S, y_S) \mid S \in \mathcal{R}\}$ will be a fooling set for the language $L^c(M)$ of size $n - k + 2^k$ which will follow that any NFA for the language $L^c(M)$ needs at $n - k + 2^k$ states.

Let S be a set in the system $\mathcal{R}$. Define the pair (x_S, y_S) as follows.

If $S = \emptyset$, let $x_S = a^n$ and $y_S = b^{n-1}$.

If $S = \{i\}$, where $k + 1 \leq i \leq n$, let $x_S = b^{n-i}$ and $y_S = b^{i-2}$.

If S is a nonempty subset of $\{1, 2, \ldots, k\}$, let x_S be an arbitrary string such that $\delta(n, x_S) = S$ (since any set in $\mathcal{R}$ is a reachable state of the DFA M', such string x_S must exist). Now, let us define the string y_S.

If $S = \{1, 2, \ldots, k\}$, let $y_S = a^k$.

If $S = \{2, 3 \ldots, k\}$, let $y_S = \varepsilon$.

Otherwise, let $y_S = y_1 y_2 \cdots y_{l-1}$, where l is *the greatest* number in $\{1, 2, \ldots, k\}$ such that $l \notin S$, and for any $i = 1, 2, \ldots, l - 1$,

$$y_i = \begin{cases} a, & \text{if } i \in S, \\ b, & \text{if } i \notin S. \end{cases}$$

First, let us prove the following claim.

Claim 1 *For any subset S of $\{1, 2, \ldots, k\}$, and any state p in $\{1, 2, \ldots, k\}$,*
(a) if $p \in S$, then $1 \notin \delta(p, y_S)$,
(b) if $p \notin S$, then $1 \in \delta(p, y_S)$,
i.e., the string y_S is not accepted by the NFA M starting in any state of S, but it is accepted by M starting in any state of $\{1, 2, \ldots, k\} \setminus S$.

Proof of Claim 1. It is easy to see that the claim holds if $S = \{1, 2, \ldots, k\}$, or $S = \{2, 3, \ldots, k\}$, or $S = \emptyset$.

Otherwise, $y_S = y_1 y_2 \cdots y_{l-1}$, where $l \notin S, \{l+1, l+2, \ldots, k\} \subseteq S$, and for any $i \in \{1, 2, \ldots, l-1\}, y_i = a$ if $i \in S$ and $y_i = b$ if $i \notin S$.

To prove (a) let p be any state such that $p \in S$. There are two cases:

(i) $p > l$. Then the accepting state 1 cannot be reached from state p after reading the string y_S of length less than l. Hence $1 \notin \delta(p, y_S)$.

(ii) $p < l$. Then $y_S = uav$ for some strings u and v such that

$$|u| = p - 1 \text{ and } |v| = l - 1 - p.$$

It follows that the set $\delta(p, u)$ is a subset of $\{k, k-1, \ldots, k-p+2\} \cup \{1\}$ (we can reach a state in $\{k, k-1, \ldots, k-p+2\}$ from state p after reading the string u if we use a transition from a state i to state k, and we reach state 1 if we use only transitions from i to $i-1$). Therefore $\delta(p, ua) \subseteq \{k-1, k-2, \ldots, k-p+1\}$, and so state 1 cannot be reached from the set $\delta(p, ua)$ after reading the string v of length less than $k - p$. Thus $1 \notin \delta(p, uav)$.

To prove (b) let p be any state in $\{1, 2, \ldots, k\} \setminus S$. There are two cases:

(i) $p = l$. Then state 1 can be reached from state p after reading any string of length $l - 1$, so $1 \in \delta(p, y_S)$.

(*ii*) $p < l$. Then $y_S = ubv$ for some strings u and v such that

$$|u| = p - 1 \text{ and } |v| = l - 1 - p.$$

It follows that $1 \in \delta(p, u)$ and $1 \in \delta(l - p, v)$. Since $1 \leq l - p \leq k$, we have $l - p \in \delta(1, b)$. Hence $1 \in \delta(p, ubv)$ which completes the proof of Claim 1.

Now, we are ready to prove that the set of pairs of strings $\{(x_S, y_S) \mid S \in \mathcal{R}\}$ is a fooling set for the language $L^c(M)$. We need to show that
(1) for any set S in $\mathcal{R}$ the string $x_S y_S$ is in the language $L^c(M)$,
(2) if S and T are two different sets in $\mathcal{R}$, then at least one of the strings $x_S y_T$ and $x_T y_S$ is not in the language $L^c(M)$.

To prove (1) let S be any set in the system $\mathcal{R}$. We will consider three cases:

(i) $S = \emptyset$. Then $x_S y_S = a^n b^{n-1}$ and the string $a^n b^{n-1}$ is in the language $L^c(M)$ since it is not accepted by the NFA M.

(ii) $S = \{i\}$, where $k + 1 \leq i \leq n$. Then $x_S y_S = b^{n-i} b^{i-2} = b^{n-2}$ and the string b^{n-2} is in the language $L^c(M)$ since it is not accepted by the NFA M.

(iii) S is a nonempty subset of $\{1, 2, \ldots, k\}$. Then $\delta(n, x_S) = S$ and, by Claim 1(a), the string y_S is not accepted by the NFA M starting in any state of S. Hence the string $x_S y_S$ is in the language $L^c(M)$.

To prove (2) let S and T be two different sets in the system $\mathcal{R}$. We will consider four cases:

(i) $S = \emptyset$. Then $x_T y_S = x_T b^{n-1}$. The string $x_T b^{n-1}$ is not in the language $L^c(M)$ since $\delta(n, x_T) = T$, the set T is nonempty and the string b^{n-1} is accepted by the NFA M starting in any state of Q.

(ii) $S = \{i\}$ and $T = \{j\}$, where $k+1 \le i < j \le n$. Then we have

$$x_S y_T = b^{n-i} b^{j-2} = b^{n-2+j-i}.$$

Since $n-2+j-i \ge n-1$, the string $b^{n-2+j-i}$ is accepted by the NFA M, so the string $x_S y_T$ is not in the language $L^c(M)$.

(iii) $S = \{i\}$, where $k+1 \le i \le n$, and T is a nonempty subset of $\{1, 2, \ldots, k\}$. Then $x_T y_S = x_T b^{i-2}$ and $\delta(n, x_T) = T$. Since $i - 2 \ge k - 1$, the string b^{i-2} is accepted by the NFA M starting in any state of the nonempty set T. Hence the string $x_T y_S$ is not in the language $L^c(M)$.

(iv) S and T are two different nonempty subsets of $\{1, 2, \ldots, k\}$. Then, without loss of generality, there is a state p in $\{1, 2, \ldots, k\}$ such that $p \in S$ and $p \notin T$. Then, $\delta(n, x_S) = S$ and by Claim 1(b), the string y_T is accepted by the NFA M starting in state p. So the string $x_S y_T$ is not in the language $L^c(M)$.

Thus, we have shown that the set of pairs of strings $\{(x_S, y_S) \mid S \in \mathcal{R}\}$ is a fooling set for the language $L^c(M)$. By Lemma 1, any NFA for the language $L^c(M)$ needs at least $n - k + 2^k$ states and our proof is complete. □

In the same way as in the proof above, it can also be shown that the values $1 + n + k(k-1)/2$, where $2 \le k \le n$, can be reached by complementation of binary NFAs. To do this we can consider a binary NFA $D = (Q, \Sigma, \delta, n, F)$, where $Q = \{1, 2, \ldots, n\}$, $\Sigma = \{a, b\}$, $F = \{1\}$, and for any $i \in Q$,

$$\delta(i, X) = \begin{cases} \emptyset, & \text{if } i = 1 \text{ and } X = a, \\ \{i-1\}, & \text{if } i > 1 \text{ and } X = a, \\ \{1, 2, \ldots, k\}, & \text{if } i = 1 \text{ and } X = b, \\ \{k, k-1\}, & \text{if } i = k \text{ and } X = b, \\ \{i-1\}, & \text{if } i \notin \{1, k\} \text{ and } X = b. \end{cases}$$

The NFA D is shown in Fig. 7.

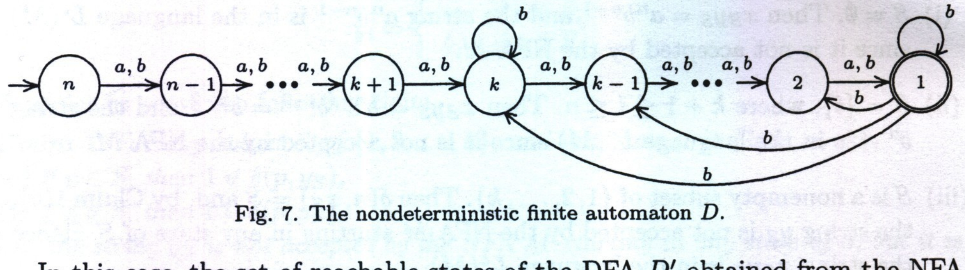

Fig. 7. The nondeterministic finite automaton D.

In this case, the set of reachable states of the DFA D' obtained from the NFA D by the subset construction consists of $1 + n + k(k-1)/2$ states:

$\emptyset, \{n\}, \{n-1\}, \ldots, \{1\},$
$\{k, k-1\}, \{k-1, k-2\}, \ldots, \{2, 1\},$
$\{k, k-1, k-2\}, \{k-1, k-2, k-3\}, \ldots, \{3, 2, 1\},$
$\vdots$
$\{k, k-1, \ldots, 2\}, \{k-2, k-3, \ldots, 1\},$
$\{k, k-1, \ldots, 1\}.$
The fooling set can be constructed in the same way as in the above proof.

5. Conclusions

In this paper, we obtained several results concerning the state complexity of concatenation and the nondeterministic state complexity of complementation of regular languages.

In the case of concatenation, we showed that the upper bounds $m2^n - k2^{n-1}$ on the concatenation of an m-state DFA language and an n-state DFA language, where k is the number of the accepting states in the m-state automaton, are tight for any integer k with $0 < k < m$. We proved the result for a binary alphabet.

In the case of complementation, we showed that for any positive integers n and α with $\log n \leq \alpha \leq 2^n$, there exists an n-state NFA language such that minimal NFAs for its complement have exactly α states. However, to prove the result we used languages defined over an alphabet the size of which grows exponentially with n. Then we showed that some special values of α can be reached by complementation of binary NFA languages. The problem whether the entire range of complexities can be produced by complementation of NFA languages over an alphabet of fixed size remains open. We conjecture that the result can be proved for an alphabet of a linear size.

Further investigations may concern the deterministic concatenation and the nondeterministic complementation of finite languages.

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