Nondeterministic Complexity of Operations on Free and Convex Languages

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CIAA 2017
Marne-la-Vallée, France, 27th June 2017

Outline

1. Preliminaries
   - Finite Automata
   - Lower-Bound Methods for NFAs
2. Free languages
3. Convex languages

Finite Automata

Definition (NFA)

Nondeterministic finite automaton (NFA) is a quintuple $A = (Q, \Sigma, \delta, s, F)$

- exactly one initial state s
- transition function $\delta : Q \times \Sigma \to 2^Q$

Definition (nsc)

The nondeterministic state complexity of L is the number of states of some minimal NFA for L. We use the denotation $\mathrm{nsc}(L)$.

Example

- $\delta(0, a) = \{0, 1\}$
- $L_{3a} = \{w \in \{a, b\}^* \mid w$ has an a in the 3rd position from the end$\}$
- $\mathrm{nsc}(L_{3a}) \leq 4$

If more initial states are allowed, we use the denotation NNFA

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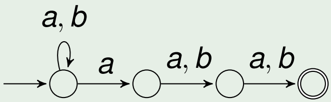

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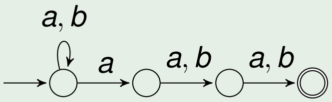

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Prefix-, Suffix-, Factor-, Subword-Free Languages

Definition

$w = uxv$

- u is a prefix of w
- v is a suffix of w
- x is a factor of w

$w = u_0 v_1 u_1 v_2 u_2 \cdots v_m u_m$

- $v_1 v_2 \cdots v_m$ is a subword of w

Example

$w =$ CONFERENCE

- CONFER is a prefix of w
- RENCE is a suffix of w
- FERENC is a factor of w
- CERN is a subword of w

Definition

- L is prefix-free iff $w \in L \Rightarrow$ no prefix of w is in L
- suffix-, factor-, subword-free defined analogously

Example

- $\{\varepsilon, FR, FRANCE\}$ is not prefix-free
- $\{FRANCE, PARIS\}$ is prefix-free

Properties of Free Languages

- L is prefix-free $\Rightarrow$ no out-transition from any final state
- L is suffix-free $\Rightarrow$ no in-transition to the initial state

Lemma (Sufficient conditions for an incomplete DFA to accept suffix-free language)

- *no in-transition to the initial state,*
- *single final state,*
- *no two transitions on the same symbol to any state*

Inclusions for classes of languages:
Prefix-free $\cup$ suffix-free = bifix-free
Bifix-free $\supsetneq$ factor-free $\supsetneq$ subword-free

Convex languages

Definition

- L is prefix-convex iff $u, uvw \in L \Rightarrow uv \in L$
- suffix-, factor-, subword-convex defined analogously

Every prefix-free, prefix-closed, and right ideal language is prefix-convex;
inclusions for suffix-, factor-, subword-convex languages hold analogously

Lemma (Property of Prefix-Convex Languages)

Let $D = (Q, \Sigma, \delta, s, F)$ be a DFA. *If for each final state q and each symbol a in Σ, the state $\delta(q, a)$ is final or dead, then $L(D)$ is prefix-convex.*

Why Free and Convex Languages?

Motivation and History

- Holzer, Kutrib (2003) (NFA), nsc(L) introduced
- Han, Salomaa, Wood (2009): prefix-free (DFA, NFA)
- Han, Salomaa (2010): suffix-free (DFA, NFA)
- Brzozowski et al. (2010, 2017): convex (DFA)
- P.M. (DCFS 2015): free, ideal (complement)
- M.H., G.J., P.M. (CIAA 2016): closed, ideal (NFA)

Fooling-Set Lower-Bound Method for NFAs

Definition (Fooling Set)

A set of pairs of strings $\{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$ is called a fooling set for a language L if for all i, j in $\{1, 2, \dots, n\}$,

(F1) $x_i y_i \in L$, and

(F2) if $i \neq j$, then $x_i y_j \notin L$ or $x_j y_i \notin L$.

Lemma (Birget, 1992)

Let $\mathcal{F}$ be a fooling set for a language L. Then every NNFA *for L has at least $|\mathcal{F}|$ states.*

If we insist on having a single initial state, we use very useful modification of fooling-set method.

Lemma (Jirásková, Masopust, 2011)

- *$\mathcal{A}$, $\mathcal{B}$ - sets of pairs of strings*
- *u, v - two strings*
- *$\mathcal{A} \cup \mathcal{B}$, $\mathcal{A} \cup \{(\varepsilon, u)\}$, and $\mathcal{B} \cup \{(\varepsilon, v)\}$ are fooling sets for a language L.*

Then every NFA *with a single initial state for L has at least $|\mathcal{A}| + |\mathcal{B}| + 1$ states.*

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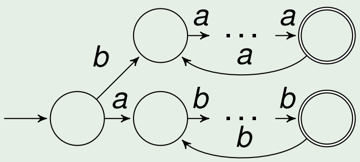
Other Lower-Bound Methods for NFAs

Lemma

Let A be an NNFA. *Let for each state q of A, the singleton set $\{q\}$ be reachable and co-reachable in A.*
Then A is minimal.

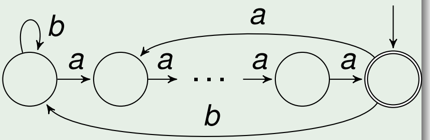
Example



Corollary

Let A be a trim NFA. *If both A and A^R are incomplete* DFAs, *then A and A^R are minimal* NFAs.

Example



We use these claims in the proofs of our results.

Complexity of operations on free languages

We examined the nondeterministic state complexity of the following operations:

Binary Operations

- union ($\cup$)
- intersection ($\cap$)
- concatenation ($\cdot$)

Unary Operations

- square (L^2)
- star (Kleene closure, L^*)
- reversal (L^R)
- complementation (L^c)

Known and New Results

	Prefix-free	$\|\Sigma\|$		Suffix-free	$\|\Sigma\|$	
$K \cap L$	$mn - (m + n - 2)$	2	[2]	$mn - (m + n - 2)$	2	[3]
$K \cup L$	$m + n$	2	[2]	$m + n - 1$	2	[3]
KL	$m + n - 1$	1	[2]	$m + n - 1$	1	[1]
L^2						
L^*	n	2	[2]	n	4	[1]
L^R	n	1	[2]	$n + 1$	3	[1]
L^c	2^{n-1}	3	[2]	2^{n-1}	3	[4]

Han, Salomaa 2010

Jirásková, Krausová 2010

Jirásková, Olejár 2009

Jirásková, Mlynárčik 2014

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$K \cup L$	$m + n$	2	[2]	$m + n - 1$	2	[3]
KL	$m + n - 1$	1	[2]	$m + n - 1$	1	[1]
L^2	$2n - 1$	1		$2n - 1$	1	
L^*	n	2	[2]	n	2	
L^R	n	1	[2]	$n + 1$	2	
L^c	2^{n-1}	3	[2]	2^{n-1}	3	[4]

Han, Salomaa 2010

Jirásková, Krausová 2010

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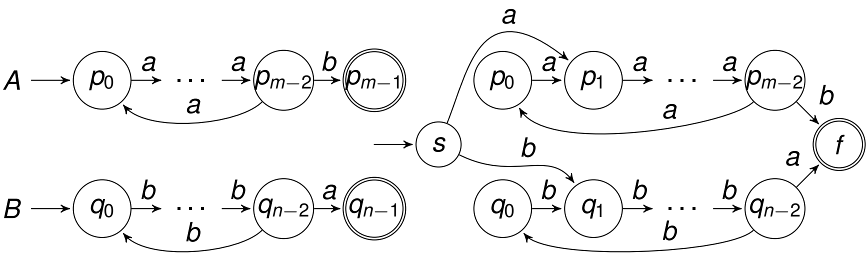
Jirásková, Mlynárčik 2014

New Results

	Factor-free	$\lvert\Sigma\rvert$	Subword-free	$\lvert\Sigma\rvert$
$K \cap L$	$mn - 2(m+n-3)$	2	$mn - 2(m+n-3)$	$m+n-5$
$K \cup L$	$m+n-2$	2	$m+n-2$	2
KL	$m+n-1$	1	$m+n-1$	1
L^2	$2n-1$	1	$2n-1$	1
L^*	$n-1$	1	$n-1$	1
L^R	n	1	n	1
L^c	$2^{n-2}+1$	3	$2^{n-2}+1$	2^{n-2}

The results for complementation are from P.M., DCFS 2015

Union on Prefix-Free Languages



Let
$\mathcal{A} = \{(a^{m-1}, a^{m-2}b)\} \cup \{(a^i, a^{m-2-i}b) \mid 1 \leq i \leq m-2\}$
$\cup\{(a^{m-2}b, \varepsilon)\}$,
$\mathcal{B} = \{(b^{n-1}, b^{n-2}a)\} \cup \{(b^j, b^{n-2-j}a) \mid 1 \leq i \leq n-2\}$,
$u = b^{n-2}a$, and $v = a^{m-2}b$.
Using AB-Lemma, we show that every NFA for $K \cup L$ needs at least $m + n$ states. □

Our Results From CIAA 2016

Table shows nsc of operations on classes of convex languages

	Prefix-convex	$\|\Sigma\|$	Suffix-convex	$\|\Sigma\|$	Factor-convex	$\|\Sigma\|$	Subword-convex	$\|\Sigma\|$
$K \cap L$	mn	2	·	2	·	2	·	2
$K \cup L$	$m+n+1$	2	·	2	·	2	·	2
KL	$m+n$	2	·	3	·	3	·	3
L^2	$2n$	2	·	3	·	3	·	3
L^*	$n+1$	1	·	2	·	2	·	2
L^R	$n+1$	2	·	3	·	3	·	$2n-2$
L^c	2^n	2	$\geq 2^{n-1}+1$ $\leq 2^n$	2	·	2	·	2^n

All upper bounds are met by closed or ideal languages.

Complementation on Suffix-Convex Languages

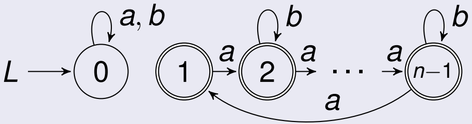
The value of nsc(L^c) on subclasses of suffix-convex languages

- suffix-closed: $2^{n-1} + 1$
- left ideal, suffix-free: 2^{n-1}

Lemma (Complementation)

If all subsets are reachable and co-reachable, then nsc$(L^c) = 2^n$

Suffix-convex witness: nsc$(L^c) = 2^n$



$0 \cdot c = \{0, 1, \ldots, n-1\}$,
$0 \cdot d = \{1, 2, \ldots, n-1\}$,
$q \cdot e = \{n-1\}$ for each state q of A

Proof Idea

- show that L^R is prefix-convex by Lemma (Property)
- use Lemma (Complementation)

Unary Case

- Unary free languages: $L = \{a^{n-1}\} \Leftrightarrow \text{nsc}(L) = n$
- Unary convex languages:
  - $L = \{a^i \mid i \geq k\} \Rightarrow \text{nsc}(L) = k + 1$
  - $L = \{a^i \mid k \leq i \leq \ell\} \Rightarrow \text{nsc}(L) = \ell + 1$

Unary	$K \cap L$	$K \cup L$	KL	L^2	L^*	L^c
free	n; $m = n$	$\max\{m, n\}$	$m + n - 1$	$2n - 1$	$n - 1$	$\Theta(\sqrt{n})$
convex	$\max\{m, n\}$	$\max\{m, n\}$	$m + n - 1$	$2n - 1$	$n - 1$	$n + 1$
regular	mn; $(m, n) = 1$	$m + n + 1$; $(m, n) = 1$	$\geq m + n - 1$ $\leq m + n$	$\geq 2n - 1$ $\leq 2n$	$n + 1$	$2^{\Theta(\sqrt{n \log n})}$

Summary and Open Problems

We have tight upper bounds for nondeterministic complexity of intersection, union, concatenation, square, star, and reversal on prefix-, suffix-, factor-, and subword-free and -convex languages.

Moreover, we have nondeterministic complexity 2^n of complementation on prefix-convex (CIAA 2016) and suffix-convex languages.

Open problems:

- complementation on factor-convex and subword-convex languages
- smaller alphabets for
  - intersection on subword-free languages
  - reversal on subword-convex languages
  - complementation on subword-closed, all-sided ideal, and subword-free languages (we still have exponential size of alphabets)

Thank You for Attention

Merci beaucoup pour votre attention