Nondeterministic Complexity of L^k and L^+ on Subclasses of Convex Languages

Michal Hospodár

Mathematical Institute
Slovak Academy of Sciences
Košice, Slovakia

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Basic Notions

Regular Operations

- Concatenation:
  $KL = \{uv \mid u \in K,\ v \in L\}$
- k-th power:
  $L^k = LL^{k-1}$
  where $L^0 = \{\varepsilon\}$
- Kleene closure:
  $L^* = \bigcup_{i \geq 0} L^i$
- Positive closure:
  $L^+ = \bigcup_{i \geq 1} L^i$

Nondeterministic State Complexity

- of a language L, $\mathrm{nsc}(L)$, is the number of states in a minimal NFA for L
- of a unary operation $\circ$:

$$n \mapsto \max\{\mathrm{nsc}(L^\circ) \mid \mathrm{nsc}(L) \leq n\}$$

- of a unary operation $\circ$ on a class $\mathcal{C}$:

$$n \mapsto \max\{\mathrm{nsc}(L^\circ) \mid \mathrm{nsc}(L) \leq n \text{ and } L \in \mathcal{C}\}$$

Subclasses of Convex Languages

Prefix, Suffix, Factor, Subword

$w = uxv$

- u is a prefix of w
- x is a suffix of w
- v is a factor of w

$w = u_0 v_1 u_1 \cdots v_m u_m$

- $v_1 v_2 \cdots v_m$ is a subword of w

Ideal

- L is a right ideal if $L = L\Sigma^*$
- left, two-sided, all-sided $L = \Sigma^* L$, $L = \Sigma^* L \Sigma^*$, $L = L \sqcup\!\sqcup\, \Sigma^*$

Free, Closed, Convex

- L is prefix-free if $w \in L$ $\Rightarrow$ no proper prefix of w is in L
- L is prefix-closed if $w \in L$ $\Rightarrow$ every prefix of w is in L
- L is prefix-convex if $u, w \in L$ and $u \leq_p w$ $\Rightarrow$ v with $u \leq_p v \leq_p w$ is in L

suffix, factor, subword analogously

- every prefix-free, -closed, and right ideal language is also prefix-convex
- suffix (left), factor (two-sided), subword (all-sided) analogously

Known Results on (Deterministic) State Complexity

- Han et al.:
  - State Complexity of Prefix-Free Regular Languages (2006)
  - State Complexity of Basic Operations on Suffix-Free Regular Languages (TCS 2009)
- Jirásková et al.:
  - State Complexity of Intersection and Union of Suffix-Free Languages and Descriptional Complexity (NCMA 2009)
  - Complexity in Prefix-Free Regular Languages (DCFS 2010)
  - Basic Operations on Binary Suffix-Free Languages (2011)
  - Prefix-free languages: Left and right quotient and reversal (TCS 2016)
- Brzozowski et al.:
  - Complexity in Convex Languages (LATA 2010)
  - Quotient Complexity of Ideal Languages (TCS 2013)
  - Quotient Complexity of Closed Languages (ToCS 2014)
  - Quotient Complexity of Bifix-, Factor-, and Subword-Free Regular Languages (Acta Cybernetica 2014)

Known Results on (Deterministic) State Complexity

	$K \cap L$	$K \cup L$	KL	L^*	L^R
right ideal	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
left ideal	✓ ✓	✓, 4	✓ ✓	✓ ✓	✓, 3
two-sided ideal	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓, 3
all-sided ideal	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓, 2n
prefix-free	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
suffix-free	✓ ✓	✓ ✓	✓ ✓	✓, 3	✓ ✓
factor-free	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓, 3
subword-free	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
prefix-closed	✓ ✓	✓ ✓	✓, 3	✓, 3	✓ ✓
suffix-closed	✓ ✓	✓, 4	✓, 3	✓ ✓	✓, 3
factor-closed	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓, 3
subword-closed	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓, 2n
prefix-convex	✓ ✓	✓ ✓			
suffix-convex	✓ ✓	✓ ✓			
factor-convex	✓ ✓	✓ ✓			
subword-convex	✓ ✓	✓ ✓			

Motivation and History

- Holzer, Kutrib (IJFCS 2003):
  definition of NSC, basic operations on regular languages
- Han, Salomaa, Wood (FI 2009): prefix-free
- Han, Salomaa (DCFS 2010): suffix-free
- Jirásková, Krausová (DCFS 2010): prefix-free
- Jirásková, Olejár (NCMA 2009): boolean op. on suffix-free
- Jirásková, Mlynárčik (DCFS 2014):
  complement on prefix-free, suffix-free, non-returning
- Mlynárčik (DCFS 2015): complement on free and ideal
- Hospodár, Jirásková, Mlynárčik (CIAA 2016): closed, ideal
- Hospodár, Jirásková, Mlynárčik (CIAA 2017): free, convex

Known Results on NSC on Subclasses of Convex Languages

	$K \cap L$	$K \cup L$	KL	L^*	L^R	L^c
right ideal	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
left ideal	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
two-sided ideal	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
all-sided ideal	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓, 2^n
prefix-free	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
suffix-free	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
factor-free	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
subword-free	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓, 2^n
prefix-closed	✓ ✓	✓ ✓	✓, 3	✓ ✓	✓ ✓	✓ ✓
suffix-closed	✓ ✓	✓ ✓	✓, 3	✓ ✓	✓ ✓	✓ ✓
factor-closed	✓ ✓	✓ ✓	✓, 3	✓ ✓	✓ ✓	✓ ✓
subword-closed	✓ ✓	✓ ✓	✓, 3	✓ ✓	✓, $2n$	✓, 2^n
prefix-convex	✓ ✓	✓ ✓	✓, 3	✓ ✓	✓ ✓	✓ ✓
suffix-convex	✓ ✓	✓ ✓	✓, 3	✓ ✓	✓ ✓	✓, 5
factor-convex	✓ ✓	✓ ✓	✓, 3	✓ ✓	✓ ✓	
subword-convex	✓ ✓	✓ ✓	✓, 3	✓ ✓	✓, $2n$	

The Aims of This Paper

	$K \cap L$	$K \cup L$	KL	L^*	L^R	L^c	L^k	L^+
right ideal	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓		
left ideal	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓		
two-sided ideal	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓		
all-sided ideal	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓, 2^n		
prefix-free	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓		
suffix-free	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓		
factor-free	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓		
subword-free	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓, 2^n		
prefix-closed	✓ ✓	✓ ✓	✓, 3	✓ ✓	✓ ✓	✓ ✓		
suffix-closed	✓ ✓	✓ ✓	✓, 3	✓ ✓	✓ ✓	✓ ✓		
factor-closed	✓ ✓	✓ ✓	✓, 3	✓ ✓	✓ ✓	✓ ✓		
subword-closed	✓ ✓	✓ ✓	✓, 3	✓ ✓	✓, $2n$	✓, 2^n		
prefix-convex	✓ ✓	✓ ✓	✓, 3	✓ ✓	✓ ✓	✓ ✓		
suffix-convex	✓ ✓	✓ ✓	✓, 3	✓ ✓	✓ ✓	✓, 5		
factor-convex	✓ ✓	✓ ✓	✓, 3	✓ ✓	✓ ✓			
subword-convex	✓ ✓	✓ ✓	✓, 3	✓ ✓	✓, $2n$			

Known Results for L^k and L^+

- Rampersad: The state complexity of L^2 and L^k (IPL 2006)
- Domaratzki, Okhotin: State complexity of power (TCS 2009)
- Holzer, Kutrib: Nondeterministic descriptional complexity of regular languages (IJFCS 2003)

Known results

(Deterministic) state complexity

	L^k	L^+
regular	$\Theta(n2^{(k-1)n})$, $\|\Sigma\| \geq 6$	$\frac{3}{4}2^n - 1$
unary regular	$k(n-1)+1$	$(n-1)^2$

Nondeterministic state complexity

	L^k	L^+
regular	kn, $\|\Sigma\| \geq 2$	n
unary regular	$k(n-1)+1 \leq \cdot \leq kn$	n

A Useful Lemma Used In Our Proof

Lemma 3.

Let $\{(X_i, Y_i) \mid i = 1, 2, \ldots, m\}$ be a set of pairs of subsets of the state set of an NFA A such that for each i in $\{1, 2, \ldots, m\}$

(1) X_i is reachable and Y_i is co-reachable in A,

(2) $i \in X_i \cap Y_i$, and

(3) $X_i \subseteq \{i, i+1, \ldots, n\}$ and $Y_i \subseteq \{1, 2, \ldots, i\}$.

Then every NFA for $L(A)$ has at least m states.

Proof.

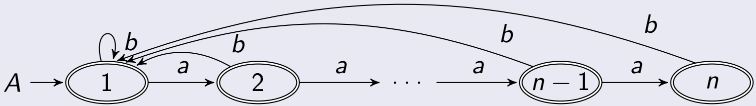
- X_i is reachable $\Rightarrow$ there is a string x_i such that $s \xrightarrow{x_i} X_i$
- Y_i is co-reachable $\Rightarrow$ there is a string y_i such that $Y_i \xrightarrow{y_i} acc$
- (2) and (3) $\Rightarrow X_i \cap Y_i = \{i\} \Rightarrow x_i y_i \in L(A)$
- $i > j$ and (3) $\Rightarrow X_i \cap Y_j = \emptyset \Rightarrow x_i y_j \notin L(A)$

$\Rightarrow$ the set $\{(x_i, y_i) \mid i = 1, 2, \ldots, m\}$ is a fooling set for $L(A)$, so every NFA for $L(A)$ has at least m states □

The Most Interesting Result of This Paper

Theorem 5 (4).

There exists a binary factor-closed language L accepted by an n-state NFA such that every NFA for L^k has at least kn states.



Proof idea: lower bound kn for the k-th power

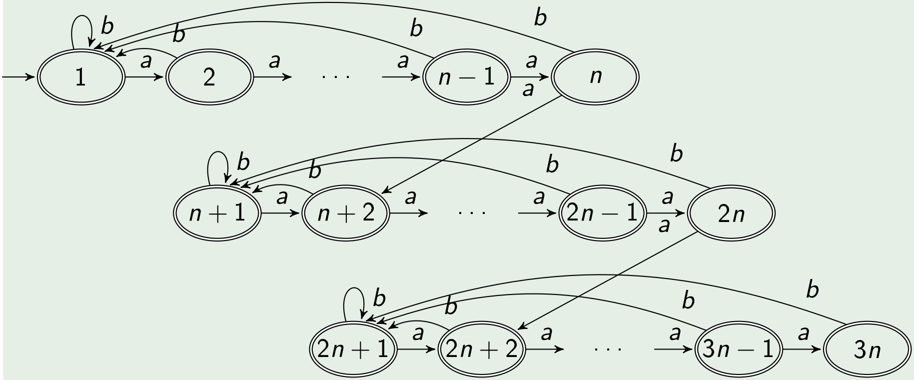
- the minimal partial DFA D for L^k has kn states and by Lemma 3, it is a minimal NFA for L^k
  - we show that in D, for every i with $1 \le i \le kn$, every set $\{i\}$ is reachable, and every set $\{1, 2, \ldots, i\}$ is co-reachable
  - using these pairs, we get a fooling set for L^k of size kn
- language L works also as a witness for concatenation

□

The Most Interesting Result of This Paper

Example ($k = 3$)

The minimal partial DFA D for L^3



NSC of L^k and L^+ on Subclasses of Convex Languages

	L^k	$\|\Sigma\|$	L^+	$\|\Sigma\|$
right ideal	$k(n-1)+1$,	1	n,	1
left ideal	$k(n-1)+1$,	1	n,	1
two-sided ideal	$k(n-1)+1$,	1	n,	1
all-sided ideal	$k(n-1)+1$,	1	n,	1
prefix-free	$k(n-1)+1$,	1	n,	1
suffix-free	$k(n-1)+1$,	1	n,	1
factor-free	$k(n-1)+1$,	1	n,	1
subword-free	$k(n-1)+1$,	1	n,	1
prefix-closed	kn,	2	n,	2
suffix-closed	kn,	2	n,	2
factor-closed	kn,	2	1,	1
subword-closed	kn,	3	1,	1
prefix-convex	kn,	2	n,	1
suffix-convex	kn,	2	n,	1
factor-convex	kn,	2	n,	1
subword-convex	kn,	3	n,	1

Summary – NSC on Subclasses of Convex Languages

	$K \cap L$	$K \cup L$	KL	L^*	L^R	L^c	L^k	L^+
right ideal	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
left ideal	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
two-sided ideal	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
all-sided ideal	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓, 2^n	✓ ✓	✓ ✓
prefix-free	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
suffix-free	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
factor-free	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
subword-free	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓, 2^n	✓ ✓	✓ ✓
prefix-closed	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
suffix-closed	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
factor-closed	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
subword-closed	✓ ✓	✓ ✓	✓, 3	✓ ✓	✓, $2n$	✓, 2^n	✓, 3	✓ ✓
prefix-convex	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓
suffix-convex	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓, 5	✓ ✓	✓ ✓
factor-convex	✓ ✓	✓ ✓	✓ ✓	✓ ✓	✓ ✓		✓ ✓	✓ ✓
subword-convex	✓ ✓	✓ ✓	✓, 3	✓ ✓	✓, $2n$		✓, 3	✓ ✓

Open Problems

From this paper

- Complexity of L^k on binary subword-closed and subword-convex languages

From our older papers

- Complexity of L^c: unknown on factor-convex and subword-convex
- Smaller alphabets ?
  - L^c all-sided ideal, subword-free, subword-closed, suffix-convex
  - KL
  - L^R subword-closed, subword-convex

ありがとう

Danke	Ďakujem	Kiitos
Paldies	Köszönöm	ขอบคุณ
Спаси́бо	Obrigado	Grazie

Summary and Open Problems

	L^k	$\lvert\Sigma\rvert$	L^+	$\lvert\Sigma\rvert$
right ideal	$k(n-1)+1$,	1	n,	1
left ideal	$k(n-1)+1$,	1	n,	1
two-sided ideal	$k(n-1)+1$,	1	n,	1
all-sided ideal	$k(n-1)+1$,	1	n,	1
prefix-free	$k(n-1)+1$,	1	n,	1
suffix-free	$k(n-1)+1$,	1	n,	1
factor-free	$k(n-1)+1$,	1	n,	1
subword-free	$k(n-1)+1$,	1	n,	1
prefix-closed	kn,	2	n,	2
suffix-closed	kn,	2	n,	2
factor-closed	kn,	2	1,	1
subword-closed	kn,	3	1,	1
prefix-convex	kn,	2	n,	1
suffix-convex	kn,	2	n,	1
factor-convex	kn,	2	n,	1
subword-convex	kn,	3	n,	1

Open problems

- Complexity of L^c: factor-convex and subword-convex
- Smaller alphabets:
  - L^c all-sided ideal, subword-free, subword-closed, suffix-convex
  - KL
  - L^k
  - L^R subword-closed, subword-convex