Square, Power, Positive Closure, and Complementation on Star-Free Languages

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Basic Notions

Regular Operations

- Concatenation:
  $KL = \{uv \mid u \in K,\ v \in L\}$
- k-th power: $L^k = LL^{k-1}$
  where $L^0 = \{\varepsilon\}$
- Square (second power): L^2
- Kleene closure:
  $L^* = \bigcup_{i \geq 0} L^i$
- Positive closure:
  $L^+ = \bigcup_{i \geq 1} L^i$
- Complementation:
  $L^c = \Sigma^* \setminus L$

State Complexity

- of a language L, $\mathrm{sc}(L)$,
  is the number of states
  in the minimal DFA for L
- of a unary operation $\circ$:
  $n \mapsto \max\{\mathrm{sc}(L^\circ) \mid \mathrm{sc}(L) \leq n\}$
- of a unary operation $\circ$ on a class $\mathcal{C}$:

$$n \mapsto \max\{\mathrm{sc}(L^\circ) \mid \mathrm{sc}(L) \leq n \text{ and } L \in \mathcal{C}\}$$

Nondeterministic State Complexity

$\mathrm{nsc}(L)$: defined analogously using NFA
with a unique initial state

Star-Free Languages

Definition: A language L is star-free iff

L is finite or
L is a complement, union, or concatenation of star-free languages

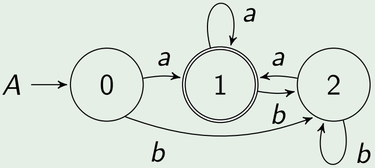
Equivalent conditions:

- for L exists an expression with $\bullet$, $+$, $()^c$, without $()^*$
- the minimal DFA for L is permutation-free (no string performs a non-trivial permutation on any subset of the state set)

Example

$L = \{a, b\}^* a$ is star-free:

- L is a concatenation of co-finite $\{a, b\}^*$ and finite $\{a\}$
- $L = \emptyset^c a$
- $L = L(A)$ and A is permutation-free



Known Results on SC and NSC of Operations

Operations on Star-Free Languages

$\cup$, $\cap$, $\bullet$, $()^*$, $()^R$, $()^c$

- Brzozowski, Liu (2012): on DFAs
- Holzer, Kutrib, Meckel (2012): on NFAs

– square was not considered

$()^2$, $()^k$, $()^+$ on Regular Lang.

- Holzer, Kutrib (2003): $()^+$ and other operations on NFAs
- Rampersad (2006): $()^2$ and $()^k$ on DFAs
- Domaratzki, Okhotin (2009): $()^k$ on NFAs

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Square on Other Classes (DFAs)

- Čevorová (2015, 2016): $()^2$ on free, ideal, and closed languages

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Here: star-free languages

- $()^2$ on DFAs
- $()^k$ on unary DFAs, NFAs
- $()^+$ on DFAs and NFAs
- $()^c$ on unary NFAs

Operations on Star-Free Languages – Known and New

Known Results on Complexity of Operations on Star-Free Languages

DFA

	general	unary
$\cup$	mn	$\max\{m,n\}$
$\cap$	mn	$\max\{m,n\}$
$\bullet$	$m2^n - 2^{n-1}$	$m+n-1$
$()^*$	$(3/4)2^n$	$n^2 - 7n + 13$
$()^R$	2^n	n
$()^c$	n	n
$()^2$	??	?
$()^+$	?	?

NFA

	general	unary
$\cup$	$m+n+1$	$m+n(+1)$
$\cap$	mn	$O(mn)$
$\bullet$	$m+n$	$m+n(-1)$
$()^*$	$n+1$	$n+1$
$()^R$	$n+1$	n
$()^c$	2^n	??
$()^k$	?	??
$()^+$	?	?

This paper:

- all open problems marked with ? are solved
- non-trivial upper and lower bounds will be given for ??
- lower bounds will be given for ?? – upper bounds are trivial

Monotonic Transformations

$$\text{DFA } A = (Q, \Sigma, \cdot, s, F)$$

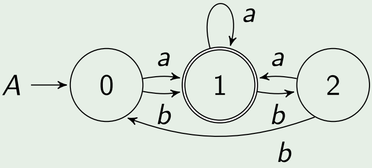
- each symbol of Σ induces a transformation on Q

Definition

A transformation a is monotonic if for every p, q in Q, we have

$$p < q \implies p \cdot a \leq q \cdot a$$

Example



- a is monotonic since $i \cdot a = 1$ for each i
- b is not monotonic since $0 \cdot b = 1$, but $2 \cdot b = 0$

The number of all monotonic transformations on n states is $\binom{2n-1}{n-1}$ [Howie 1971]

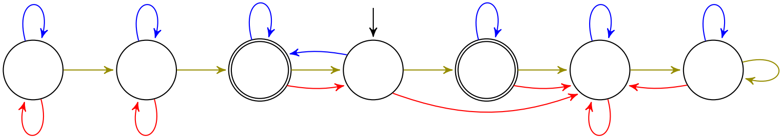
Claim

If all transformations in Σ are monotonic, then $L(A)$ is star-free

Proof.

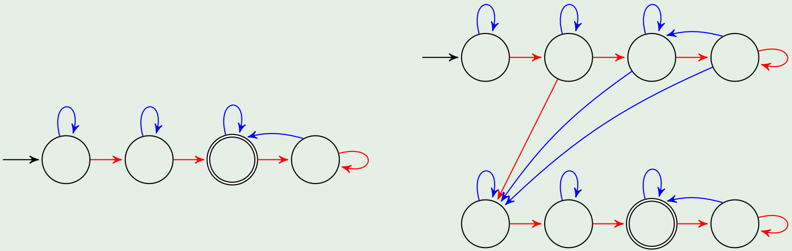
- The DFA A has a non-trivial permutation on $S \subseteq Q$ if there exists a string w such that $S \cdot w = S$, but it is not the identity
- Monotonic transformation cannot perform such permutation

$\Rightarrow$ If each transformation is monotonic,
then A is permutation-free, so $L(A)$ is star-free $\square$



- take two copies of A and make s_1 initial and F_2 final
- for every $q \xrightarrow{a} f$ with $f \in F_1$, add $q \xrightarrow{a} s_2$
- at the end "determinize" the resulting NFA
  to get the DFA B with $n2^n$ states of form (i, S)

Example



Square on Star-Free Languages

The DFA B for square has $n2^n$ states of form (i, S) where

- $i \in Q$ is a state in the first copy
- $S \subseteq Q$ is a subset of states in the second copy

Theorem

The state complexity of square on star-free languages has a lower bound $\sum_{i=0}^{n-1} g(n, i)$ and an upper bound $(n-1)2^n - 2(n-2)$, thus is in $\Theta(n2^n)$

Proof Idea for the Upper Bound

Since $L(A)$ is star-free,
no transformation of A performs a non-trivial permutation
$\Rightarrow$ some transformations are excluded
$\Rightarrow$ some states are unreachable
$\Rightarrow$ upper bound is $(n-1)2^n - 2(n-2)$

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Proof Idea for the Lower Bound:
Let A be the DFA with all monotonic transformations on n states
Since all transformations of A are monotonic, $L(A)$ is star-free
For each i, let $g(n,i)$ be the number of reachable subsets S in the DFA B
Then the total number of reachable states of B is $\sum_{i=0}^{n-1} g(n,i)$
We have $\sum_{i=0}^{n-1} g(n,i) \in \Omega(n2^n)$ □

Results of This Paper

What Was Known Before

	star-free	$\|\Sigma\|$	unary star-free
$\mathrm{sc}(L^2)$	??		?
$\mathrm{sc}(L^+)$	?		?
$\mathrm{nsc}(L^k)$	?		??
$\mathrm{nsc}(L^+)$	?		?
$\mathrm{nsc}(L^c)$	2^n [Holzer et al. 2012]	2	??

Our Results

	star-free	$\|\Sigma\|$	unary star-free
$\mathrm{sc}(L^2)$	$\Theta(n2^n)$	$4n$	$2n-1$
$\mathrm{sc}(L^+)$	$(3/4)2^n - 1$	4	$n^2 - 7n + 13$
$\mathrm{nsc}(L^k)$	kn	2	$k(n-1)+1 \leq \cdot \leq kn$
$\mathrm{nsc}(L^+)$	n	1	n
$\mathrm{nsc}(L^c)$			$(n-1)^2+1 \leq \cdot \leq n^2-2$

Conclusions and Open Problems

On star-free languages, we have found

- non-trivial upper and lower bound on the state complexity of the square operation (general case) $\Rightarrow \Theta(n2^n)$
- tight upper bounds on state complexity of L^+ (general and unary case) and L^k (unary case) and nondeterministic state complexity of L^k and L^+ (general case)
- upper and lower bounds on the nondeterministic state complexity of L^k and L^c (unary case)

Open Problems:

- tightness of square on DFAs and L^k and L^c on unary NFAs
- smaller alphabet for L^+ on DFAs?

Ďakujem za pozornosť

Danke	Grazie	Obrigado
Kiitos	Arigato	Děkuji
Köszönöm	Spasibo	Namaste