Operations on Boolean and Alternating Finite Automata

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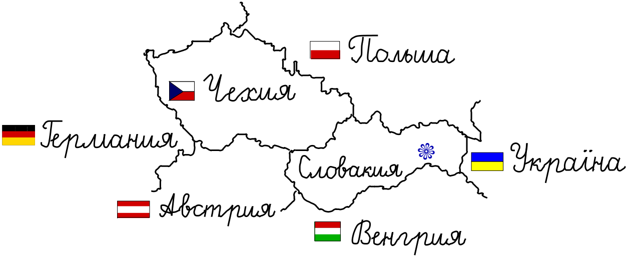

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- One of the easternmost Gothic cathedrals in Europe
- The oldest coat of arms given to a city
- The oldest marathon in Europe
- ...

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Why Alternating and Boolean Automata?

Motivation and History

- Referee report in 2007 on a paper on AFAs
- Fellah, Jürgensen, Yu:
  Constructions for alternating finite automata (1990)

> In Theorem 9.3 we show that $2^{m_1}+m_2+1$ states suffice for an AFA to accept the concatenation of two languages accepted by AFA with m_1 and m_2 states, respectively. We conjecture that this number of states is actually necessary in the worst case, but have no proof.

  - Jirásková (CSR 2012): concatenation $\geq 2^m + n$
  - Krajňáková (master thesis 2016): square $\geq 2^n + n + 1$
  - Hospodár (NCMA 2016): concatenation $\geq 2^m + n + 1$
- Leiss: Succinct representation of regular languages by boolean automata (1981)

Outline

1. Deterministic, Nondeterministic, Alternating Automata
2. Boolean Automata
3. Brzozowski-Leiss Lemma (BFA/AFA for L vs. DFA for L^R)
4. The Complexity of Operations on AFA and BFA Languages
   - Lower Bounds
   - Upper Bounds
5. Summary and Open Problems

$$A = (Q, \Sigma, \delta, s, F)$$

- $Q = \{q_1, \ldots, q_n\}$ is a non-empty finite set of states
- Σ is an input alphabet
- $s \in Q$ is the starting state
- $F \subseteq Q$ is the set of final states
- δ is the transition function that maps $Q \times \Sigma$ to
  - a single state in DFA
  - a set (disjunction) of states in NFA
  - a boolean function of states in AFA

DFA, NFA, AFA: Examples

$$A = (\{q_1, q_2\}, \{a, b\}, \delta, q_1, \{q_2\})$$

Example (Deterministic Finite Automaton)

δ	a	b
q_1	q_2	q_2
q_2	q_2	q_1

computation on *bba*:

$q_1 \xrightarrow{b} q_2 \xrightarrow{b} q_1 \xrightarrow{a} q_2 \in F$

$bba \in L(A)$

Example (Nondeterministic Finite Automaton)

δ	a	b
q_1	q_2	$q_1 \vee q_2$
q_2	q_1	q_1

$q_1 \xrightarrow{b} q_1 \vee q_2 \xrightarrow{b} (q_1 \vee q_2) \vee q_1 \xrightarrow{a}$

$\xrightarrow{a} q_2 \vee q_1 \vee q_2$ where $q_2 \in F$

$bba \in L(A)$

Example (Alternating Finite Automaton)

δ	a	b
q_1	q_2	$q_1 \wedge \overline{q_2}$
q_2	q_1	$\overline{q_1}$

$q_1 \xrightarrow{b} q_1 \wedge \overline{q_2} \xrightarrow{b} (q_1 \wedge \overline{q_2}) \wedge \overline{\overline{q_1}} \xrightarrow{a}$

$\xrightarrow{a} (q_2 \wedge \overline{q_1}) \wedge \overline{\overline{q_2}} = q_2 \wedge \overline{q_1}$

this function gives 1 in finality vector (0,1)

$bba \in L(A)$

Boolean Finite Automata

Example (Alternating Finite Automaton)

$$A = (\{q_1, q_2\}, \{a, b\}, \delta, q_1, \{q_2\})$$

δ	a	b
q_1	q_2	$q_1 \wedge \overline{q_2}$
q_2	q_1	$\overline{q_1}$

$q_1 \xrightarrow{b} q_1 \wedge \overline{q_2} \xrightarrow{b} (q_1 \wedge \overline{q_2}) \wedge \overline{\overline{q_1}} \xrightarrow{a}$
$\xrightarrow{a} (q_2 \wedge \overline{q_1}) \wedge \overline{\overline{q_2}} = q_2 \wedge \overline{q_1}$

this function gives 1 in finality vector (0,1)

$bba \in L(A)$

$A = (Q, \Sigma, \delta, g_s, F)$, where g_s is the initial function

Example (Boolean Finite Automaton)

$$A = (\{q_1, q_2\}, \{a, b\}, \delta, \overline{q_1}, \{q_2\})$$

δ	a	b
q_1	q_2	$q_1 \wedge \overline{q_2}$
q_2	q_1	$\overline{q_1}$

$\overline{q_1} \xrightarrow{bba} \overline{q_2 \wedge \overline{q_1}}$

this function gives 0 in finality vector (0,1)

$bba \notin L(A)$

A. N. Maslov: Estimates of the number of states of finite automata (1970)

Доклады Академии наук СССР
1970. Том 194, № 6

УДК 519.95 МАТЕМАТИКА

А. Н. МАСЛОВ

ОЦЕНКИ ЧИСЛА СОСТОЯНИЙ КОНЕЧНЫХ АВТОМАТОВ

Известно, что если $T(A)$ и $T(B)$ представимы в автоматах A и B с m и n состояниями соответственно ($m \geqslant 1$, $n \geqslant 1$), то

1) $T(A) \cup T(B)$ представимо в автомате с $m \cdot n$ состояниями.

2) $T(A) \cdot T(B)$ представимо в автомате с $(m-1)\cdot 2^n + 2^{n-1}$ состояниями ($n \geqslant 3$),

3) $T(A)^*$ представимо в автомате $\frac{3}{4} \cdot 2^m - 1$ состояниями ($m \geqslant 2$).

Нами построены примеры автоматов над алфавитом $\Sigma = \{0, 1\}$, на которых эти оценки достигаются.

1. Объединение: A имеет состояния $\{S_0, \dots, S_{m-1}\}$ и переходы $S_{m-1}1 = S_0$, $S_i1 = S_{i+1}$ при $i \neq m-1$, $S_i0 = S_i$, S_{m-1} — заключительное состояние; B имеет состояния $\{P_0, \dots, P_{n-1}\}$ и переходы $P_i1 = P_i$, $P_{n-1}0 = P_0$, $P_i0 = P_{i+1}$ при $i \neq n-1$, P_{n-1} — заключительное состояние.

2. Произведение: B имеет состояния $\{P_0, \dots, P_{n-1}\}$ и переходы $P_{n-1}1 = P_{n-2}$, $P_{n-2}1 = P_{n-1}$, $P_i1 = P_i$ при $i < n-2$, $P_{n-1}0 = P_{n-1}$, $P_i0 = P_{i+1}$ при $i \neq n-1$, P_{n-1} — заключительное состояние; автомат A такой же, как для объединения.

3. Итерация: A имеет состояния $\{S_0, \dots, S_{m-1}\}$ и переходы $S_{m-1}1 = S_0$, $S_i1 = S_{i+1}$ при $i \neq m-1$, $S_00 = S_0$, $S_i0 = S_{i-1}$ при $i > 0$, S_{m-1} — заключительное состояние.

По A и B строим соответствующие автоматы, как в ([2], [4]), и находим необходимое число достижимых и различных состояний, что и доказывает минимальность ([3]).

General Formulation of the Problem

Maslov 1970:

Общая постановка задач такого рода: имеются события $T(A_i)$ ($1 \leqslant i \leqslant k$), представимые в автоматах A_i с n_i состояниями соответственно, и k — местная операция f над событиями, сохраняющая представимость в конечных автоматах. Каким может быть максимальное число состояний минимального автомата, представляющего $f(T(A_1), \ldots, T(A_k))$, при данных n_i?

"Given languages $L(A_i)$ $(1 \leq i \leq k)$
accepted by automata A_i with n_i states
and a k-ary regular operation f,
what is the maximal number of states
in the minimal automaton for $f(L(A_1), \ldots, L(A_k))$,
considered as a function of n_i's?"

In this paper:

- automata are boolean or alternating
- f: boolean operations, reversal, star, left and right quotients

Known Results

- Fellah, A., Jürgensen, H., Yu, S.: Constructions for alternating finite automata. Int. J. Comput. Math. 35(1-4), 117–132 (1990)
- Jirásková, G.: Descriptional complexity of operations on alternating and boolean automata. CSR 2012.
- Hospodár, M., Jirásková, G.: Concatenation on deterministic and alternating automata. NCMA 2016.
- Krajňáková, I., Jirásková, G.: Square on deterministic, alternating, and boolean finite automata. DCFS 2017.

	AFA (FJY90)	AFA (known)	$\|\Sigma\|$	BFA (known)	$\|\Sigma\|$
concatenation	$\leq 2^m + n + 1$	$2^m + n + 1$	2	$2^m + n$	2
square	$\leq 2^n + n + 1$	$2^n + n + 1$	2	$2^n + n$	2
star	$\leq 2^n + 1$	$2^n \leq \cdot \leq 2^n+1$	2	$2^n \leq \cdot \leq 2^n+1$	2
reversal	$\leq 2^n + 1$	$2^n \leq \cdot \leq 2^n+1$	2	2^n	2
complement	$\leq n$				
intersection	$\leq m + n + 1$	$m + n + 1$	2	$m + n$	2
union	$\leq m + n + 1$	$m + n + 1$	2	$m + n$	2

Known Results: BFA/AFA for L vs. DFA for L^R

Brzozowski, Leiss (1980); Fellah, Jürgensen, Yu (1990)

Lemma (BFA for L vs. DFA for L^R)

1. *If L is recognized by a BFA with n states, then L^R is recognized by a DFA with 2^n states.*
2. *If L is recognized by a DFA with 2^n states, then L^R is recognized by a BFA with n states.*

Lemma (AFA for L vs. DFA for L^R)

1. *If L is recognized by an AFA with n states, then L^R is recognized by a DFA with 2^n states and 2^{n-1} final states.*
2. *If L is recognized by a DFA with 2^n states and 2^{n-1} final states, then L^R is recognized by an AFA with n states.*

Lower Bound Method for AFAs

Lemma (AFA for L vs. DFA for L^R; BL80, FJY90)

1. *If L is recognized by an AFA with n states, then L^R is recognized by a DFA with 2^n states and 2^{n-1} final states.*
2. *If L is recognized by a DFA with with 2^n states and 2^{n-1} final states, then L^R is recognized by an AFA with n states.*

Example (Square on AFAs: Lower Bound $2^n + n + 1$)

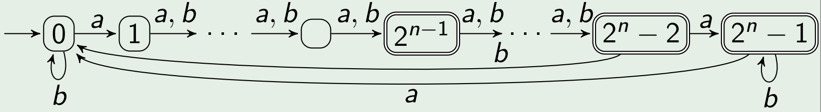
- start with K recognized by a 2^n-state DFA with 2^{n-1} final states that is hard for square on DFAs, that is, K^2 requires many states on DFAs
- take $L = K^R \overset{(2)}{\Rightarrow} L$ is recognized by an n-state AFA
- $(L^2)^R = (L^R)^2 = K^2$ requires $\frac{3}{4}2^{2^n}2^n$-state DFA
  $\overset{(1)}{\Rightarrow} L^2$ requires $(2^n + n)$-state AFA
- minimal DFA for $(L^2)^R$ has more then $\frac{1}{2}2^{2^n+n}$ final states
  $\overset{(1)}{\Rightarrow} L^2$ requires $(2^n + n + 1)$-state AFA

Lower Bounds for Operations on AFAs and BFAs

- Worst-case examples from the literature: star, reversal

Example (Palmovský, M.: RAIRO - Theor. Inform. Appl. (2016))

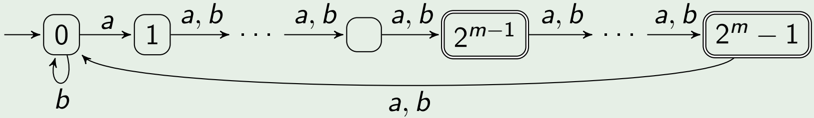
star



- Our new worst-case examples: quotients, boolean operations

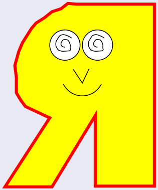
Example

right quotient by Σ^*



- All witnesses for AFAs work for BFAs.

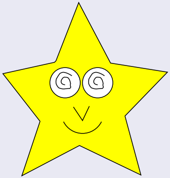
Reversal: 2^n



L is recognized by n-state BFA

$\overset{(1)}{\Rightarrow}$ L^R is recognized by 2^n-state DFA which is also an AFA.

Star: 2^n



L is recognized by n-state BFA

$\overset{(1)}{\Rightarrow}$ L^R is recognized by 2^n-state DFA

$\overset{Maslov70}{\Rightarrow}$ $(L^R)^*$ is recognized by $\frac{3}{4}2^{2^n}$-state DFA with $\frac{1}{4}2^{2^n}$ final states

$\Rightarrow$ $(L^*)^R = (L^R)^*$ is recognized by 2^{2^n}-state DFA with $\frac{1}{2}2^{2^n}$ final states

$\overset{(2)}{\Rightarrow}$ L^* is recognized by 2^n-state AFA

Summary and Open Problems

Summary

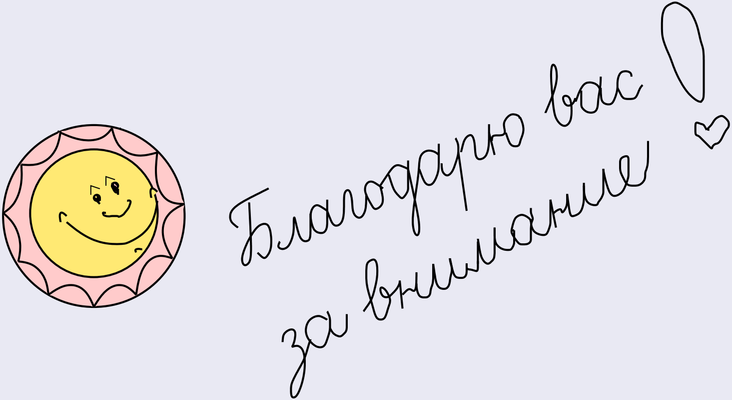
	AFA (FJY90)	AFA (known)	$\|\Sigma\|$	BFA (known)	$\|\Sigma\|$	AFA (new)	$\|\Sigma\|$	BFA (new)	$\|\Sigma\|$
concatenation	$\leq 2^m + n + 1$	$2^m + n + 1$	2	$2^m + n$	2				
square	$\leq 2^n + n + 1$	$2^n + n + 1$	2	$2^n + n$	2				
star	$\leq 2^n + 1$	$2^n \leq \cdot \leq 2^n+1$	2	$2^n \leq \cdot \leq 2^n+1$	2	2^n	2	2^n	2
reversal	$\leq 2^n + 1$	$2^n \leq \cdot \leq 2^n+1$	2	2^n	2	2^n	2		
complement	$\leq n$					n	1	n	1
intersection	$\leq m + n + 1$	$m + n + 1$	2	$m + n$	2			$m + n$	1
union	$\leq m + n + 1$	$m + n + 1$	2	$m + n$	2			$m + n$	1
difference						$m + n + 1$	2	$m + n$	1
symm. diff.						$m + n$	1	$m + n$	1
left quotient						$m + 1$	1	m	1
right quotient						$2^m + 1$	2	2^m	2

Known results are from our papers at CSR 2012 (star, reversal, union, intersection), NCMA 2016 (concatenation) and DCFS 2017 (square).

Open Problems

- Is binary alphabet for $\cap$, $\cup$, and $\setminus$ on AFAs optimal?
  - Unary DFA witnesses with half of states final are not known.
- Some other operations...
- AFAs with existential and universal states?

Thank You for Your Attention



BFA for $L \to$ DFA for L^R

δ	a	b
q_1	q_3	q_1
q_2	$q_1 \vee q_2$	q_3
q_3	$\neg q_1$	$q_1 \wedge \neg q_2$

Table: BFA $A = (\{q_1, q_2, q_3\}, \{a, b\}, \delta, g_s = q_1 \vee q_3, F = \{q_3\})$

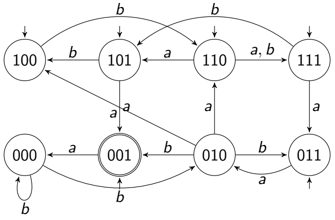


Figure: NFA N for language $L(A)$.

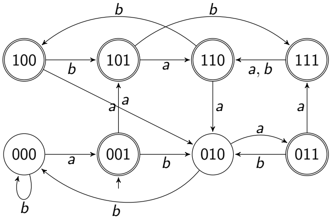
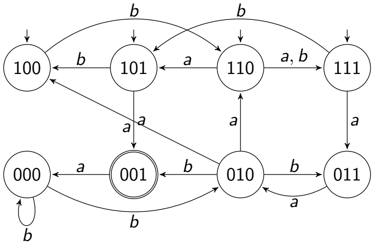


Figure: DFA N^R for language $L(A)^R$.

DFA for $L^R \rightarrow$ BFA for L



- $Q' = \{q_1, q_2, q_3\}$
- $F' = \{q_i \mid f_i = 1\} = \{q_3\}$
- $g_s = q_1 \vee q_3$
- δ' :

(q_1, q_2, q_3)	$\delta'(q_1, a),$	$\delta'(q_2, a)$	$\delta'(q_3, a)$	$\delta'(q_1, b)$	$\delta'(q_2, b)$	$\delta'(q_3, b)$
000	0	0	1	0	0	0
001	1	0	1	0	1	0
010	0	1	1	0	0	0
011	1	1	1	0	1	0
100	0	1	0	1	0	1
101	1	1	0	1	1	1
110	0	1	0	1	0	0
111	1	0	0	1	1	0

$$\big(\delta'(q_1, a), \delta'(q_2, a), \delta'(q_3, a)\big)(000) = (001)$$

In the original automaton N this corresponds to transition $000 \overset{a}{\leftarrow} 001$.