Decomposing the domains of continuous real functions

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Introduction

- let $\mathcal{G}$ be a fixed family of functions
- given a family of functions $\mathcal{F}$, consider all sets X such that

$$(\forall f \in \mathcal{F})(\exists g \in \mathcal{G})\ f \restriction X = g \restriction X$$

- given a family of sets $\mathcal{X}$, consider all functions f such that

$$(\forall X \in \mathcal{X})(\exists g \in \mathcal{G})\ f \restriction X = g \restriction X$$

- we want do study the above described relation between families of functions and families of sets
- we restrict ourselves to continuous real functions and closed sets of reals

For $f, g\colon \mathbb{R} \to \mathbb{R}$, denote $E_{f,g} = \{x \in \mathbb{R}\colon f(x) = g(x)\}$.

- if f, g are continuous then $E_{f,g}$ is closed
- for any closed set $E \subseteq \mathbb{R}$ there exist continuous f, g such that $E_{f,g} = E$
- for any closed set $E \subseteq \mathbb{R}$ and any continuous function $f\colon E \to \mathbb{R}$ there exists a continuous extension $g\colon \mathbb{R} \to \mathbb{R}$

Denote $CL(X)$ the family closed subsets of X, $C(X, Y)$ the family of all continuous functions from X to Y.

Let $\mathcal{G}$ be a fixed family of real continuous functions.
Define a binary relation between continuous functions and closed sets:

$$(f, E) \in R_{\mathcal{G}} \quad \iff \quad (\exists g \in \mathcal{G})\ f \restriction E = g \restriction E$$

Possible tasks/questions:

- describe the family $\mathcal{E}_f = \{E \in CL(\mathbb{R}) \colon (f, E) \in R_{\mathcal{G}}\}$
- characterize all families of the form $\mathcal{E}_f$
- given $\mathcal{E} \subseteq CL(\mathbb{R})$, does there exist $\mathcal{F} \subseteq C(\mathbb{R}, \mathbb{R})$ such that $\{E \in CL(\mathbb{R}) \colon (\forall f \in \mathcal{F})\ (f, E) \in R_{\mathcal{G}}\} = \mathcal{E}$?
- if yes, find the minimum size of such $\mathcal{F}$
- etc.

Galois connections

Let $\mathcal{G} \subseteq C(X, Y)$.
For $f \in C(X, Y)$, $E \in CL(X)$, define $R_{\mathcal{G}}$ as before:

$$(f, E) \in R_{\mathcal{G}} \iff (\exists g \in \mathcal{G})\ f \restriction E = g \restriction E$$

For $\mathcal{F} \subseteq C(X, Y)$, $\mathcal{E} \subseteq CL(X)$ denote:

$$E(\mathcal{F}) = \{E \in CL(X) \colon (\forall f \in \mathcal{F})\ (f, E) \in R_{\mathcal{G}}\}$$
$$F(\mathcal{E}) = \{f \in C(X, Y) \colon (\forall E \in \mathcal{E})\ (f, E) \in R_{\mathcal{G}}\}$$

Pair of mappings E, F forms an antitone Galois connection between ordered sets $(\mathcal{P}(C(\mathbb{R}, \mathbb{R})), \subseteq)$ and $(\mathcal{P}(CL(\mathbb{R})), \subseteq)$.

Recall:

$$(f, E) \in \mathbb{R}_{\mathcal{G}} \iff (\exists g \in \mathcal{G})\ f \restriction E = g \restriction E$$

$$E(\mathcal{F}) = \{E \in CL(X) \colon (\forall f \in \mathcal{F})\ (f, E) \in R_{\mathcal{G}}\}$$

$$F(\mathcal{E}) = \{f \in C(X, Y) \colon (\forall E \in \mathcal{E})\ (f, E) \in R_{\mathcal{G}}\}$$

- E, F are order-reversing
- $\mathcal{E} \subseteq E(\mathcal{F})$ iff $\mathcal{F} \subseteq F(\mathcal{E})$, for all $\mathcal{E} \subseteq CL(X)$, $\mathcal{F} \subseteq C(X, Y)$
- compound mappings EF, FE are closure operators
- $\mathcal{E} \subseteq CL(X)$ is closed iff $\mathcal{E} = E(\mathcal{F})$ for some $\mathcal{F} \subseteq C(X, Y)$
- $\mathcal{F} \subseteq C(X, Y)$ is closed iff $\mathcal{F} = F(\mathcal{E})$ for some $\mathcal{E} \subseteq CL(X)$
- families $\{E(\mathcal{F}) \colon \mathcal{F} \subseteq C(X, Y)\}$, $\{F(\mathcal{E}) \colon \mathcal{E} \subseteq CL(X)\}$ ordered by $\subseteq$ form dually isomorphic complete lattices

Our aim

Our aim is to study the structure of the lattices

$$\mathcal{K}_{\mathcal{G}} = \{\mathcal{E} \subseteq CL(\mathbb{R})\colon EF(\mathcal{E}) = \mathcal{E}\} = \{E(\mathcal{F})\colon \mathcal{F} \subseteq C(\mathbb{R}, \mathbb{R})\}$$
$$\mathcal{L}_{\mathcal{G}} = \{\mathcal{F} \subseteq C(\mathbb{R}, \mathbb{R})\colon FE(\mathcal{F}) = \mathcal{F}\} = \{F(\mathcal{E})\colon \mathcal{E} \subseteq CL(\mathbb{R})\}$$

in the case of various simple families $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$.

First results

Proposition

- $\mathcal{K}_\emptyset = \{\emptyset,\ CL(\mathbb{R})\}$, $\mathcal{L}_\emptyset = \{\emptyset,\ C(\mathbb{R},\mathbb{R})\}$
- $\mathcal{K}_{C(\mathbb{R},\mathbb{R})} = \{CL(\mathbb{R})\}$, $\mathcal{L}_{C(\mathbb{R},\mathbb{R})} = \{C(\mathbb{R},\mathbb{R})\}$
- *The top elements of* $\mathcal{K}_\mathcal{G}$ *and* $\mathcal{L}_\mathcal{G}$ *are* $CL(\mathbb{R})$ *and* $C(\mathbb{R},\mathbb{R})$*, resp.*
- *The least element of* $\mathcal{K}_\mathcal{G}$ *is* $E_\mathcal{G}(C(\mathbb{R},\mathbb{R})) = \{E \in CL(\mathbb{R}) \colon \mathcal{G} \restriction E = C(E,\mathbb{R})\}$.
- *The least element of* $\mathcal{L}_\mathcal{G}$ *is* $F_\mathcal{G}(CL(\mathbb{R})) = F_\mathcal{G}(\{\mathbb{R}\}) = \mathcal{G}$.

Corollary

- *If* $\mathcal{G} \neq C(\mathbb{R},\mathbb{R})$ *then* $|\mathcal{K}_\mathcal{G}| \geq 2$.
- *Any family* $\mathcal{F} \subseteq C(\mathbb{R},\mathbb{R})$ *is an element of some lattice* $\mathcal{L}_\mathcal{G}$.

Proposition

Let $\mathcal{E} \subseteq CL(\mathbb{R})$. The following conditions are equivalent.

1. *There exists $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ such that $\mathcal{E} \in \mathcal{K}_{\mathcal{G}}$.*
2. *$\mathcal{E}$ is hereditary, i.e., $CL(E) \subseteq \mathcal{E}$ for any $E \in \mathcal{E}$.*

Elements of lattices $\mathcal{K}_{\mathcal{G}}$ are exactly hereditary families of closed sets.

Problem

Is every hereditary family $\mathcal{E} \subseteq CL(\mathbb{R})$ the least element of some lattice $\mathcal{K}_{\mathcal{G}}$?

Proposition

If $\mathcal{G} = \{g\}$ *then* $\mathcal{K}_\mathcal{G} = \{CL(E)\colon E \in CL(\mathbb{R})\}$ *and* $\mathcal{L}_\mathcal{G} = \{[g \restriction E]\colon E \in CL(\mathbb{R})\}$*.*

Corollary

If $|\mathcal{G}| = 1$ *then* $\mathcal{K}_\mathcal{G}$ *is isomorphic to the lattice of all closed sets of reals.*

The same lattice $\mathcal{K}_\mathcal{G}$ is obtained, e.g., for $\mathcal{G} = C(\mathbb{R}, [0,1])$.

Problem

Characterize all families $\mathcal{G}$ *such that* $\mathcal{K}_\mathcal{G} = \{CL(E)\colon E \in CL(\mathbb{R})\}$*.*

Proposition

Let $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ *be nonempty. The following conditions are equivalent.*

1. $\mathcal{K}_\mathcal{G} \supseteq \{CL(E)\colon E \in CL(\mathbb{R})\}$*.*
2. *The least element of* $\mathcal{K}_\mathcal{G}$ *is* $\{\emptyset\}$*.*
3. *For each* $x \in \mathbb{R}$*,* $\{g(x)\colon g \in \mathcal{G}\} \neq \mathbb{R}$*.*

Proposition

If $1 \leq |C(\mathbb{R},\mathbb{R}) \setminus \mathcal{G}| < \mathfrak{c}$ *then*
$\mathcal{K}_{\mathcal{G}} = \{CL(\mathbb{R}) \setminus \{\mathbb{R}\},\ CL(\mathbb{R})\}$ *and* $\mathcal{L}_{\mathcal{G}} = \{\mathcal{G},\ C(\mathbb{R},\mathbb{R})\}$*.*

The same lattice $\mathcal{K}_{\mathcal{G}}$ is obtained, e.g., for the family $\mathcal{G}$ of all functions g for which there exist $a < b$ such that g is linear on $[a, b]$.

Problem

Characterize all families $\mathcal{G} \subseteq C(\mathbb{R},\mathbb{R})$ *such that* $|\mathcal{K}_{\mathcal{G}}| = 2$*.*

Restricting to constant functions

Let $\mathcal{G} = \mathsf{Const}$ be the family of all constant functions.

Proposition

Let $\mathcal{E} \in \mathcal{K}_{\mathsf{Const}}$. Then:

1. *$\mathcal{E}$ contains all singletons.*
2. *If $A, B \in \mathcal{E}$ and $A \cap B \neq \emptyset$ then $A \cup B \in \mathcal{E}$.*
3. *The family $\mathcal{M}_{\mathcal{E}}$ of all maximal elements of $\mathcal{E}$ is a decomposition of $\mathbb{R}$ into closed sets.*
4. *The equivalence relation corresponding to the decomposition $\mathcal{M}_{\mathcal{E}}$ has a closed graph, i.e., it is a closed subset of $\mathbb{R} \times \mathbb{R}$.*

Denote $\sim_{\mathcal{E}}$ the binary relation defined by

$$x \sim_{\mathcal{E}} y \quad \Longleftrightarrow \quad (\exists E \in \mathcal{E})\, \{x, y\} \subseteq E$$

.

Theorem

Let $\mathcal{E} \subseteq CL(\mathbb{R})$ be hereditary. The following conditions are equivalent.

1. $\mathcal{E} \in \mathcal{K}_{\mathsf{Const}}$.
2. $\sim_{\mathcal{E}}$ *is an equivalence relation on* $\mathbb{R}$ *with a closed graph.*

The proof of $2 \Rightarrow 1$ is based on the following theorems.

Theorem

If $\sim$ *is an equivalence relation on* $\mathbb{R}$ *with a closed graph then the quotient space* $\mathbb{R}/\sim$ *is Tychonoff.*

Theorem

Every Tychonoff space can be embedded into the Tychonoff cube $[0,1]^\kappa$ *for some cardinal* κ*.*

Corollary

Lattice $\mathcal{K}_{\mathsf{Const}}$ *is isomporphic to the lattice of equivalence relations on* $\mathbb{R}$ *having closed graphs, ordered by refinement.*

For $\mathcal{E} \in \mathcal{K}_{\mathsf{Const}}$, denote $\mu_{\mathcal{E}} = \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq C(\mathbb{R}, \mathbb{R})$ and $\mathcal{E} = E(\mathcal{F})\}$.

- $\mu_{\mathcal{E}}$ is the minimum size of a family of functions generating $\mathcal{E}$

Examples of various values of $\mu_{\mathcal{E}}$:

- $\mu_{C(\mathbb{R},\mathbb{R})} = 0$
- $\mu_{\mathcal{E}} = n$: $\mathcal{E} = E(\{f_1, \dots, f_n\})$ where $f_1, \dots, f_n$ are projections of a continuous surjection $f \colon \mathbb{R} \to \mathbb{R}^n$
- $\mu_{\mathcal{E}} = \omega$: $\mathcal{E} = E(\{f_n \colon n \in \omega\})$, where f_n are projections of a continuous surjection $f \colon \mathbb{R} \to \{x \in \mathbb{R}^\omega \colon (\exists m)(\forall n > m)\, x_n = 0\}$
- $\mu_{\mathcal{E}} = \mathfrak{d}$: $x \sim_{\mathcal{E}} y \iff x = y \vee \{x, y\} \subseteq \mathbb{Z}$
  $\mathbb{R}/{\sim_{\mathcal{E}}}$ is the real line with all integers glued together
  $\mathfrak{d}$ is the minimum size of the base of topology of $\mathbb{R}/{\sim_{\mathcal{E}}}$

Problem

Are there any other possible values of $\mu_{\mathcal{E}}$ for $\mathcal{E} \in \mathcal{K}_{\mathsf{Const}}$?

Restricting to linear functions

Let $\mathcal{G} = \mathsf{Lin} = \{f_a \colon a \in \mathbb{R}\}$, where $f_a(x) = ax$ for all $x \in \mathbb{R}$.

Proposition

Let $\mathcal{E} \in \mathcal{K}_{\mathsf{Lin}}$. Then:

1. $\{x\} \in \mathcal{E}$ *for every* $x \neq 0$.
2. *If* $A, B \in \mathcal{E}$ *and* $(A \cap B) \setminus \{0\} \neq \emptyset$ *then* $A \cup B \in \mathcal{E}$.
3. *If* $\{0\} \in \mathcal{E}$ *then* $E \cup \{0\} \in \mathcal{E}$ *for every* $E \in \mathcal{E}$.
4. *If* $\{0\} \notin \mathcal{E}$ *then* $\mathcal{E} \cup \{E \cup \{0\} \colon E \in \mathcal{E}\} \in \mathcal{K}_{\mathsf{Lin}}$.

Problem

Does there exist $\mathcal{E} \in \mathcal{K}_{\mathsf{Lin}}$ *such that* $\{0\} \in \mathcal{E}$, 0 *is isolated in every* $E \in \mathcal{E}$, *and* $\{E \setminus \{0\} \colon E \in \mathcal{E}\} \notin \mathcal{K}_{\mathsf{Lin}}$?

Proposition

Let $\mathcal{E} \in \mathcal{K}_{\mathsf{Lin}}$. Then:

1. *If $\{0\} \notin \mathcal{E}$ then the family $\mathcal{M}_{\mathcal{E}}$ of all maximal elements of $\mathcal{E}$ is a decomposition of $\mathbb{R} \setminus \{0\}$ into closed sets. Moreover, $\mathcal{M}_{\mathcal{E}} \cup \{\{0\}\}$ is a decomposition of $\mathbb{R}$ with a closed graph.*
2. *If $\{0\} \in \mathcal{E}$ and 0 is isolated in every $E \in \mathcal{E}$ then the same holds true for the family $\mathcal{M}_{\mathcal{E}}$ of all maximal elements of $\{E \setminus \{0\} \colon E \in \mathcal{E}\}$.*

Theorem

Let $\mathcal{M}$ be a decomposition of $\mathbb{R}$ with a closed graph such that $\{0\} \in \mathcal{M}$. Then $\mathcal{E} \in \mathcal{K}_{\mathsf{Lin}}$, where

$$\mathcal{E} = \{E \in CL(\mathbb{R}) \colon 0 \in E \wedge (\exists F \in \mathcal{M})\, E \setminus \{0\} \subseteq F\}.$$

Restricting to polynomial functions

Let $\mathcal{G} = \mathsf{Pol}_n$ be the family of all polynomial functions of degree at most n.

Proposition

Let $\mathcal{E} \in \mathcal{K}_{\mathsf{Pol}_n}$. Then:

1. $E \in \mathcal{E}$ *for every* $E \subseteq \mathbb{R}$ *such that* $|E| \leq n+1$.
2. *If* $A, B \in \mathcal{E}$ *and* $|A \cap B| > n+1$ *then* $A \cup B \in \mathcal{E}$.
3. $(n+1)$*-ary relation* R *defined by*

$$(x_1, \ldots, x_{n+1}) \in R \quad \Longleftrightarrow \quad (\exists E \in \mathcal{E}) \, \{x_1, \ldots, x_{n+1}\} \subseteq E$$

is an $(n+1)$*-equivalence on* $\mathbb{R}$ *with a closed graph.*

Definition

An n-ary relation R is an n-equivalence on X if

1. $(x, \ldots, x) \in R$ for every $x \in X$,
2. $(x_1, \ldots, x_n) \in R \Rightarrow (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in R$ for every permutation σ,
3. if $x_1, \ldots, x_{n-1}$ are distinct and $(x_1, \ldots, x_{n-1}, y_i) \in R$ for every $i \in \{1, \ldots, n\}$, then $(y_1, \ldots, y_n) \in R$.

Problem

Let R be an $(n+1)$-equivalence on $\mathbb{R}$.
Denote $\mathcal{E}$ the family of all sets $E \in CL(\mathbb{R})$ such that
$(\forall x_1, \ldots, x_n \in E)\ (x_1, \ldots, x_n) \in R$. Is $\mathcal{E} \in \mathcal{K}_{\mathsf{Pol}_n}$?

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