A Galois connection related to restrictions of continuous real functions

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Outline

1. Introduction
    - notation and terminology
    - Galois connection
    - lattices $\mathcal{K}_{\mathcal{G}}$, $\mathcal{L}_{\mathcal{G}}$

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2. Elements of lattices $\mathcal{K}_{\mathcal{G}}$, $\mathcal{L}_{\mathcal{G}}$
    - bottom elements of lattices $\mathcal{K}_{\mathcal{G}}$
    - lattice $\mathcal{K}_{\mathcal{G}}$ containing all elements of all lattices $\mathcal{K}_{\mathcal{G}'}$

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   - bottom elements of lattices $\mathcal{K}_\mathcal{G}$
   - lattice $\mathcal{K}_\mathcal{G}$ containing all elements of all lattices $\mathcal{K}_{\mathcal{G}'}$
3. Lattices $\mathcal{K}_\mathcal{G}$ for $\mathcal{G}$ determined by a single continuous function
   - $\mathcal{G} = \{g\}$
   - $\mathcal{G} = (-\infty, g)$
   - $\mathcal{G} = (-\infty, g) \cup (g, \infty)$

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- we study the above described relation between families of functions and families of sets
- we restrict ourselves to continuous real functions and closed sets of reals

Notation

Let X, Y be topological spaces, $Z \subseteq X$.

- $C(X, Y)$: the family of all continuous functions from X to Y
- $CL_X(Z)$: the family of all subsets of Z closed in X
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Let $\mathcal{F}$ be a family of functions, E be a set. Denote:

- $\mathcal{F} \restriction E = \{f \restriction E \colon f \in \mathcal{F}\}$

Galois connection

Let $\mathcal{G} \subseteq C(\mathbb{R},\mathbb{R})$. For $\mathcal{E} \subseteq CL(\mathbb{R})$ and $\mathcal{F} \subseteq C(\mathbb{R},\mathbb{R})$ denote:

- $E_{\mathcal{G}}(\mathcal{F}) = \{E \in CL(\mathbb{R}) \colon (\forall f \in \mathcal{F})\ f \restriction E \in \mathcal{G} \restriction E\}$
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Maps $E_{\mathcal{G}} \colon \mathcal{P}(C(\mathbb{R}, \mathbb{R})) \to \mathcal{P}(CL(\mathbb{R}))$, $F_{\mathcal{G}} \colon \mathcal{P}(CL(\mathbb{R})) \to \mathcal{P}(C(\mathbb{R}, \mathbb{R}))$ form a Galois connection between partial orders $(\mathcal{P}(C(\mathbb{R}, \mathbb{R})), \subseteq)$ and $(\mathcal{P}(CL(\mathbb{R}, \mathbb{R})), \subseteq)$.

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Then $(\mathcal{K}_{\mathcal{G}}, \subseteq)$ and $(\mathcal{L}_{\mathcal{G}}, \subseteq)$ are dually isomorphic complete lattices in which $\bigwedge$ coincides with $\bigcap$.

Elements of lattices $\mathcal{K}_{\mathcal{G}}, \mathcal{L}_{\mathcal{G}}$

The least elements of lattices $\mathcal{K}_{\mathcal{G}}, \mathcal{L}_{\mathcal{G}}$

Fact

For every $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$*,* $\bigwedge \mathcal{L}_{\mathcal{G}} = \mathcal{F}_{\mathcal{G}}(\{\mathbb{R}\}) = \mathcal{G}$*.*

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Any $\mathcal{F} \subseteq C(\mathbb{R}, \mathbb{R})$ is the least element of some lattice $\mathcal{L}_{\mathcal{G}}$.

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Theorem

Let $\mathcal{E} \subseteq CL(\mathbb{R})$. The following conditions are equivalent.

1. *There exists $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ such that $\mathcal{E} \in \mathcal{K}_{\mathcal{G}}$.*
2. *There exists $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ such that $\mathcal{E} = \bigwedge \mathcal{K}_{\mathcal{G}}$.*
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To prove 3 $\Rightarrow$ 2 we have to "fool" the Intermediate Value Theorem.

Lemma

Let $a < b$, $u < v$. There exists an increasing (hence continuous) surjection $f\colon [a, b] \to [u, v]$ such that for every $x \in (a, b)$, $f(x) \notin \mathbb{Q} + \sqrt{2}\chi_{\mathbb{Q}}(x)$.

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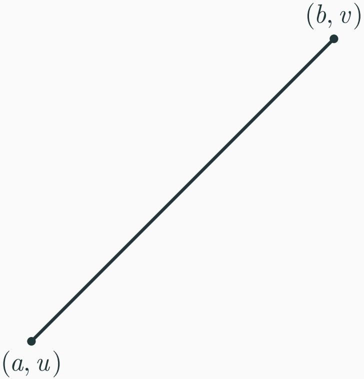

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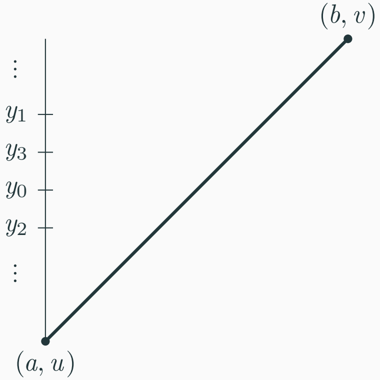

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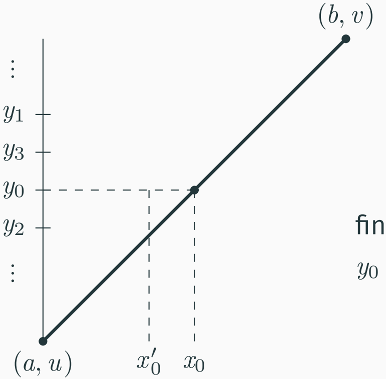

find x_0' near x_0 such that $y_0 \notin \mathbb{Q} + \sqrt{2}\chi_{\mathbb{Q}}(x_0')$

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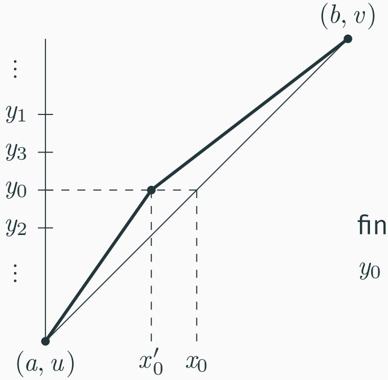

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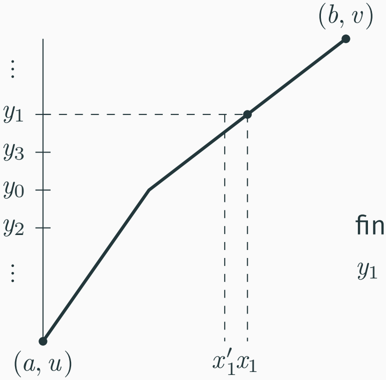

find x_1' near x_1 such that
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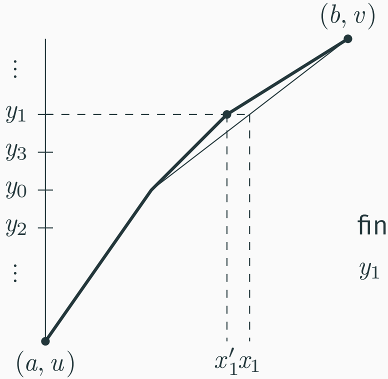

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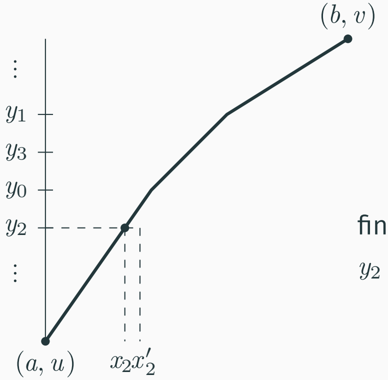

find x_2' near x_2 such that
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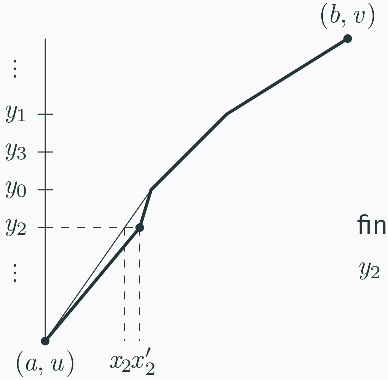

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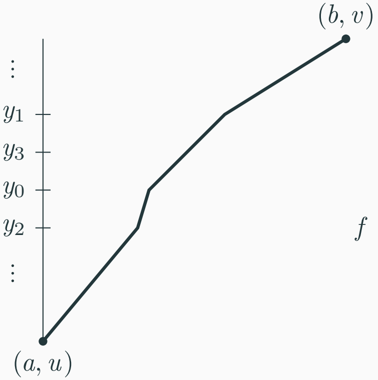

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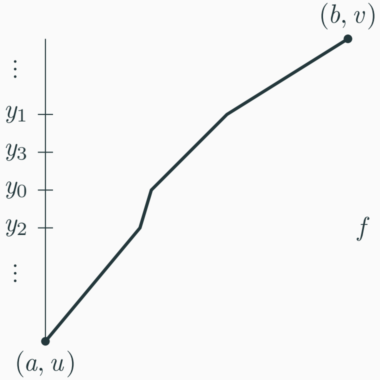

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q.e.d.

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Corollary

Let $f \in C(\mathbb{R},\mathbb{R})$, $E \in CL(\mathbb{R})$. Then there exists $g \in C(\mathbb{R},\mathbb{R})$ such that $f \restriction E = g \restriction E$ and for all $x \notin E$, $g(x) \notin \mathbb{Q} + \sqrt{2}\chi_{\mathbb{Q}}(x)$.

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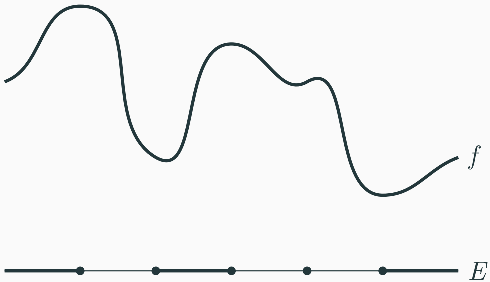

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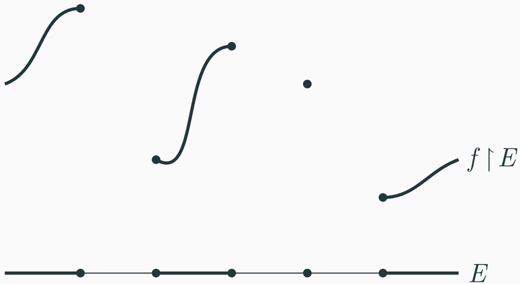

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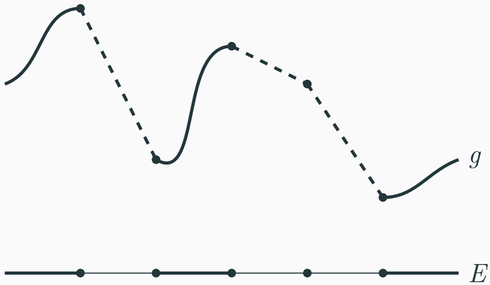

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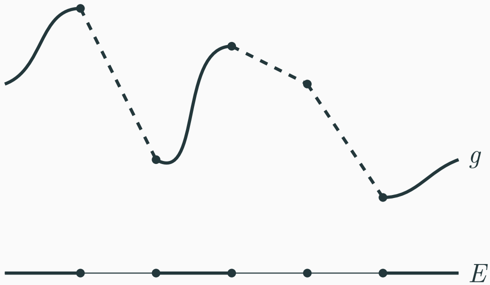

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Theorem

Let $\mathcal{E} \subseteq CL(\mathbb{R})$. The following conditions are equivalent.

1. *There exists $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ such that $\mathcal{E} \in \mathcal{K}_{\mathcal{G}}$.*
2. *There exists $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ such that $\mathcal{E} = \bigwedge \mathcal{K}_{\mathcal{G}}$.*
3. *$\mathcal{E}$ is hereditary.*

Sketch of the proof of $3 \Rightarrow 2$

Let $\mathcal{E} \subseteq CL(\mathbb{R})$ be hereditary. We have to find $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ such that $\mathcal{E} = E_{\mathcal{G}}(C(\mathbb{R}, \mathbb{R}))$.

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If $E \notin \mathcal{E}$ then let $\{x_n : n \in \omega\}$ be a countable dense subset of E. There exists $f \in C(\mathbb{R}, \mathbb{R})$ such that $f(x_n) \in \mathbb{Q} + \sqrt{2}\chi_{\mathbb{Q}}(x_n)$ for all n. Then $f \restriction E \notin \mathcal{G} \restriction E$. Hence, $E \notin E_{\mathcal{G}}(C(\mathbb{R}, \mathbb{R}))$.

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Lattice $\mathcal{K}_\mathcal{G}$ containing all nonempty hereditary families

Theorem

There exists $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ *such that* $\mathcal{E} \in \mathcal{K}_\mathcal{G}$ *for every nonempty hereditary family* $\mathcal{E} \subseteq CL(\mathbb{R})$.

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Let $\{E_\alpha : \alpha < 2^\omega\}$ be an enumeration of all nonempty closed sets. For every $\alpha < 2^\omega$, define $x_\alpha, y_\alpha \in \mathbb{R}$ and $g_{\alpha,n} \in C(\mathbb{R}, \mathbb{R})$ so that for every interval I with $E_\alpha \cap I \neq \emptyset$ there is $n \in \omega$ such that $g_{\alpha,n}(x) = y_\alpha$ for all $x \notin I$ and $g_{\alpha,n}(x_\beta) \neq y_\beta$ for all $\beta < \alpha$.

Let $\mathcal{G} = \{g_{\alpha,n} : \alpha < 2^\omega,\ n \in \omega\}$. For every nonempty hereditary family $\mathcal{E} \subseteq CL(\mathbb{R})$, let $\mathcal{F}$ be the family consisting of all constant functions with values y_α where $E_\alpha \in CL(\mathbb{R}) \setminus \mathcal{E}$. Then $\mathcal{E} = E_{\mathcal{G}}(\mathcal{F})$. **q.e.d.**

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Problem

Can one define such $\mathcal{G}$ in some more constructive way?

Lattices $\mathcal{K}_{\mathcal{G}}$ for families $\mathcal{G}$ determined by a single function

Notation

Denote $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$.

For $f, g \in C(\mathbb{R}, \mathbb{R}^*)$, let us write:

- $f \leq g$ for $(\forall x \in \mathbb{R})\ f(x) \leq g(x)$,
- $f < g$ for $(\forall x \in \mathbb{R})\ f(x) < g(x)$.

Define intervals of functions:

- $(f, g) = \{h \in C(\mathbb{R}, \mathbb{R}) \colon f < h < g\}$,
- $[f, g] = \{h \in C(\mathbb{R}, \mathbb{R}) \colon f \leq h \leq g\}$,
- etc.

Lattice $\mathcal{K}_{\{g\}}$

Theorem

Let $g \in C(\mathbb{R}, \mathbb{R})$. *Then* $\mathcal{K}_{\{g\}} = \{CL(E) \colon E \in CL(\mathbb{R})\}$.

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We characterize inclusions $\mathcal{K}_{\mathcal{G}} \supseteq \{CL(E) \colon E \in CL(\mathbb{R})\}$ and $\mathcal{K}_{\mathcal{G}} \subseteq \{CL(E) \colon E \in CL(\mathbb{R})\}$ separately.

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Let $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$, $\mathcal{G} \neq \emptyset$*. The following conditions are equivalent.*

1. $\mathcal{K}_{\mathcal{G}} \supseteq \{CL(E) \colon E \in CL(\mathbb{R})\}$.
2. $\{\emptyset\} \in \mathcal{K}_{\mathcal{G}}$.
3. *For every* $x \in \mathbb{R}$, $\{g(x) \colon g \in \mathcal{G}\} \neq \mathbb{R}$.

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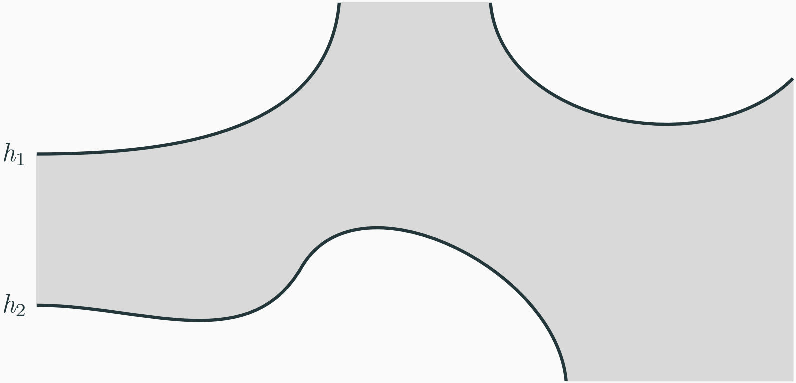
Lattice $\mathcal{K}_{\{g\}}$



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The family $\big\{CL\big(\mathbb{R} \setminus \{x\}\big) \colon x \in \mathbb{R}\big\}$ in condition 2 is minimal, that is, by excluding any set we obtain a strictly weaker condition.

Lattice $\mathcal{K}_{(-\infty,g)}$

We identify a function $f\colon \mathbb{R} \to \mathbb{R}$ and its graph $f \subseteq \mathbb{R}^2$.

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- complete if for any $g \in C(\mathbb{R},\mathbb{R})$, if $g \subseteq \bigcup \mathcal{G}$ then $g \in \mathcal{G}$

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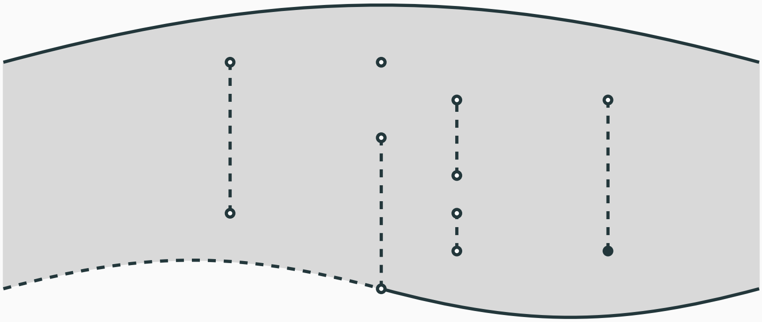

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Corollary

Let $g \in C(\mathbb{R},\mathbb{R})$. *Then* $\mathcal{K}_{(g,\infty)} = \mathcal{K}_{(-\infty,g)} = \{CL(X) \colon X \subseteq \mathbb{R}\}$.

Lattice $\mathcal{K}_{(-\infty,g)\cup(g,\infty)}$

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A family $\mathcal{X} \subseteq \mathcal{P}(\mathbb{R})$ is separated if for any distinct sets $X, Y \in \mathcal{X}$ there exist disjoint open sets $U, V \subseteq \mathbb{R}$ such that:

- $X \subseteq U \wedge Y \subseteq V$
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Lattice $\mathcal{K}_{(-\infty,g)\cup(g,\infty)}$

For any open set $U \subseteq \mathbb{R}$ denote $U' = \mathbb{R} \setminus \operatorname{cl} U$.

Fact

If U is a regular open set then U' is a regular open set and $U \cap U' = \emptyset$.

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2. (2a) *For every $x \in \mathbb{R}$, $CL\big(\mathbb{R} \setminus \{x\}\big) \in \mathcal{K}_{\mathcal{G}}$, and*
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Lattice $\mathcal{K}_{(-\infty,g)\cup(g,\infty)}$

For any open set $U \subseteq \mathbb{R}$ denote $U' = \mathbb{R} \setminus \operatorname{cl} U$.

Fact

If U is a regular open set then U' is a regular open set and $U \cap U' = \emptyset$.

Theorem

Let $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$, $\mathcal{G} \neq \emptyset$. The following conditions are equivalent.

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3. (3a) *For every $x \in \mathbb{R}$ there exists $f \in C(\mathbb{R}, \mathbb{R})$ such that $CL\big(\mathbb{R} \setminus \{x\}\big) = E_{\mathcal{G}}\big(\{f\}\big)$, and*
   (3b) *for any $x, y \in \mathbb{R}$ and any regular open set $U \subseteq \mathbb{R}$ such that $x \in U$ and $y \in U'$, there exists $f \in F_{\mathcal{G}}\big(CL(U) \cup CL(U')\big)$ such that $f \restriction \{x, y\} \notin \mathcal{G} \restriction \{x, y\}$.*

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   (c) *for every $i \in I$ there exist functions $g_i^-, g_i^+ \in C(\mathbb{R},\mathbb{R}^*)$ such that*

$$\bigcup_{j<i} \mathcal{G}_j \subseteq (-\infty, g_i^-), \quad \mathcal{G}_i \subseteq (g_i^-, g_i^+), \quad \text{and} \quad \bigcup_{j>i} \mathcal{G}_j \subseteq (g_i^+, \infty).$$

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Some open problems

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Fact

Let $g \in C(\mathbb{R}, \mathbb{R})$. Then $\mathcal{K}_{C(\mathbb{R},\mathbb{R})\setminus\{g\}} = \{CL(\mathbb{R}) \setminus \{\mathbb{R}\}, CL(\mathbb{R})\}$.

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Problem

Characterize families $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ such that $\mathcal{K}_{\mathcal{G}}$ satisfies one of these conditions:

- $\mathcal{K}_{\mathcal{G}} = \{CL(\mathbb{R}) \setminus \{\mathbb{R}\}, CL(\mathbb{R})\}$,
- $\mathcal{K}_{\mathcal{G}}$ *has exactly two elements,*
- $\mathcal{K}_{\mathcal{G}}$ *is finite,*
- $\mathcal{K}_{\mathcal{G}}$ *is linearly ordered.*