A Galois connection related to restrictions of continuous real functions

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Abstract

Given a family of continuous real functions $\mathcal{G}$, let $R_{\mathcal{G}}$ be a binary relation defined as follows: a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is in the relation with a closed set $E \subseteq \mathbb{R}$ if and only if there exists $g \in \mathcal{G}$ such that $f \restriction E = g \restriction E$. We consider a Galois connection between families of continuous functions and hereditary families of closed sets of reals naturally associated to $R_{\mathcal{G}}$. We study complete lattices determined by this connection and prove several results showing the dependence of the properties of these lattices on the properties of $\mathcal{G}$. In some special cases we obtain exact description of these lattices.

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1. Introduction and statement of main results

Let X and Y be topological spaces. Denote by $CL(X)$ the family of all closed subsets of X, and by $C(X, Y)$ the family of all continuous functions $f\colon X \to Y$. Given a fixed family $\mathcal{G} \subseteq C(X, Y)$, let $R_{\mathcal{G}}$ be the binary relation defined by

$$R_{\mathcal{G}} = \{(f, E) \in C(X, Y) \times CL(X)\colon (\exists g \in \mathcal{G})\, f \restriction E = g \restriction E\}.$$

For $\mathcal{F} \subseteq C(X, Y)$ and $\mathcal{E} \subseteq CL(X)$ denote

$$E_{\mathcal{G}}(\mathcal{F}) = \{E \in CL(X)\colon (\forall f \in \mathcal{F})\, (f, E) \in R_{\mathcal{G}}\},$$
$$F_{\mathcal{G}}(\mathcal{E}) = \{f \in C(X, Y)\colon (\forall E \in \mathcal{E})\, (f, E) \in R_{\mathcal{G}}\}.$$

The mappings

$$E_{\mathcal{G}}\colon \mathcal{P}(C(X, Y)) \to \mathcal{P}(CL(X)) \quad \text{and} \quad F_{\mathcal{G}}\colon \mathcal{P}(CL(X)) \to \mathcal{P}(C(X, Y))$$

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form a Galois connection between the partially ordered sets $(\mathcal{P}(C(X,Y)),\subseteq)$ and $(\mathcal{P}(CL(X)),\subseteq)$. This means that $E_{\mathcal{G}}$ and $F_{\mathcal{G}}$ are inclusion-reversing mappings such that for any $\mathcal{F}\subseteq C(X,Y)$ and $\mathcal{E}\subseteq CL(X)$ one has

$$\mathcal{E}\subseteq E_{\mathcal{G}}(\mathcal{F}) \quad \text{if and only if} \quad F_{\mathcal{G}}(\mathcal{E})\supseteq\mathcal{F}.$$

The compond mappings

$$E_{\mathcal{G}}F_{\mathcal{G}}\colon \mathcal{P}(CL(X))\to\mathcal{P}(CL(X)) \quad \text{and} \quad F_{\mathcal{G}}E_{\mathcal{G}}\colon \mathcal{P}(C(X,Y))\to\mathcal{P}(C(X,Y))$$

are closure operators on $\mathcal{P}(CL(X))$ and $\mathcal{P}(C(X,Y))$, respectively. Hence, for any $\mathcal{E}\subseteq CL(X)$, $\mathcal{E}\subseteq E_{\mathcal{G}}F_{\mathcal{G}}(\mathcal{E})$ and $E_{\mathcal{G}}F_{\mathcal{G}}(\mathcal{E})=\mathcal{E}$ if and only if $\mathcal{E}=E_{\mathcal{G}}(\mathcal{F})$ for some $\mathcal{F}\subseteq C(X,Y)$. Similarly, for any $\mathcal{F}\subseteq C(X,Y)$, $\mathcal{F}\subseteq F_{\mathcal{G}}E_{\mathcal{G}}(\mathcal{F})$ and $F_{\mathcal{G}}E_{\mathcal{G}}(\mathcal{F})=\mathcal{F}$ if and only if $\mathcal{F}=F_{\mathcal{G}}(\mathcal{E})$ for some $\mathcal{E}\subseteq CL(X)$. Let us denote

$$\begin{aligned}\mathcal{K}_{\mathcal{G}}&=\{\mathcal{E}\subseteq CL(X)\colon E_{\mathcal{G}}F_{\mathcal{G}}(\mathcal{E})=\mathcal{E}\}=\{E_{\mathcal{G}}(\mathcal{F})\colon \mathcal{F}\subseteq C(X,Y)\},\\ \mathcal{L}_{\mathcal{G}}&=\{\mathcal{F}\subseteq C(X,Y)\colon F_{\mathcal{G}}E_{\mathcal{G}}(\mathcal{F})=\mathcal{F}\}=\{F_{\mathcal{G}}(\mathcal{E})\colon \mathcal{E}\subseteq CL(X)\}\end{aligned}$$

the classes of all closed families with respect to the closure operators associated with the relation $R_{\mathcal{G}}$. The families $\mathcal{K}_{\mathcal{G}}$ and $\mathcal{L}_{\mathcal{G}}$ are in a one-to-one correspondence and, when ordered by inclusion, form dually isomorphic complete lattices. In fact, the mapping $\mathcal{E}\mapsto F_{\mathcal{G}}(\mathcal{E})$ is an isomorphism $(\mathcal{K}_{\mathcal{G}},\subseteq)\to(\mathcal{L}_{\mathcal{G}},\supseteq)$ and its inverse is the mapping $\mathcal{F}\mapsto E_{\mathcal{G}}(\mathcal{F})$. Moreover, the infimum in both lattices coincides with the set-theoretic intersection. Let us note that in a complete lattice there exist the least and the greatest elements.

For a history of Galois connections and their applications we refer the reader to [4]. For more on their relations to complete lattices and formal concept analysis see [2]. Let us note that Galois connections occur naturally in various settings; for some examples related to analysis and topology see [6] or [8]. For applications of Galois connections in the theory of cardinal characteristics see [1]. Our study of restrictions of continuous functions was loosely motivated by classical notions of Kronecker and Dirichlet sets from harmonic analysis, see [7] and [5].

In the present paper we deal with the case $X=Y=\mathbb{R}$. Our aim is to analyze the structure of the lattices $\mathcal{K}_{\mathcal{G}}$ and $\mathcal{L}_{\mathcal{G}}$ for certain simple families $\mathcal{G}\subseteq C(\mathbb{R},\mathbb{R})$. In Section 2 we characterize the elements of the lattices $\mathcal{K}_{\mathcal{G}}$ and $\mathcal{L}_{\mathcal{G}}$. We prove that every element of $\mathcal{K}_{\mathcal{G}}$ is a hereditary family of closed sets and that each hereditary family of closed sets is the least element of some lattice $\mathcal{K}_{\mathcal{G}}$. We also find a family $\mathcal{G}$ such that $\mathcal{K}_{\mathcal{G}}$ is the lattice of all nonempty hereditary families of closed sets. In Sections 3–5 we describe the lattice $\mathcal{K}_{\mathcal{G}}$ for three families $\mathcal{G}$ determined by a single continuous function g: the singleton $\{g\}$, the family of all functions f such that $f(x)<g(x)$ for all x, and the family of all functions f satisfying $f(x)\neq g(x)$ for all x. In each case we characterize all families that yield the same lattice $\mathcal{K}_{\mathcal{G}}$.

1.1. Notation and terminology.

For $\mathcal{F}\subseteq C(\mathbb{R},\mathbb{R})$ and $E\subseteq\mathbb{R}$ we denote $\mathcal{F}\restriction E=\{f\restriction E: f\in\mathcal{F}\}$. For $x\in\mathbb{R}$ we also denote $\mathcal{F}[x]=\{f(x): f\in\mathcal{F}\}$. If $\mathcal{H}\subseteq C(E,\mathbb{R})$, we denote $[\mathcal{H}]=\{f\in C(\mathbb{R},\mathbb{R})\colon f\restriction E\in\mathcal{H}\}$. We write $[h]$ instead of $[\{h\}]$ for $h\in C(E,\mathbb{R})$.

For $E \subseteq \mathbb{R}$ let $CL(E)$ denote the family of all closed subsets of E. To avoid ambiguity, we use notation $CL(E)$ only if E is closed; otherwise the family of all subsets of E that are closed in $\mathbb{R}$ is expressed by the term $CL(\mathbb{R}) \cap \mathcal{P}(E)$. Denote $Eq_{f,g} = \{x \in \mathbb{R} : f(x) = g(x)\}$ for $f, g \in C(\mathbb{R}, \mathbb{R})$. Then for any $\mathcal{F} \subseteq C(\mathbb{R}, \mathbb{R})$ and $\mathcal{E} \subseteq CL(\mathbb{R})$ we have $E_{\mathcal{G}}(\mathcal{F}) = \bigcap_{f\in\mathcal{F}} \bigcup_{g\in\mathcal{G}} CL(Eq_{f,g})$ and $F_{\mathcal{G}}(\mathcal{E}) = \bigcap_{E\in\mathcal{E}} \bigcup_{g\in\mathcal{G}} [g \restriction E]$.

We shall identify a function $f : \mathbb{R} \to \mathbb{R}$ and its graph $\{(x, y) \in \mathbb{R}^2 : f(x) = y\}$. We say that a family $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ is *complete* if $g \in \mathcal{G}$ holds for every $g \in C(\mathbb{R}, \mathbb{R})$ satisfying $g \subseteq \bigcup \mathcal{G}$. A family $\mathcal{G}$ is *connected* if for any $f, g \in \mathcal{G}$ and $x \neq y$ there exists $h \in \mathcal{G}$ such that $h(x) = f(x)$ and $h(y) = g(y)$.

Let $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$. For $f, g \in C(\mathbb{R}, \mathbb{R}^*)$, define inequalities $f < g$ and $f \leq g$ by $(\forall x \in \mathbb{R})\, f(x) < g(x)$ and $(\forall x \in \mathbb{R})\, f(x) \leq g(x)$, respectively. Further, let $(f, g) = \{h \in C(\mathbb{R}, \mathbb{R}) : f < h < g\}$ and $[f, g] = \{h \in C(\mathbb{R}, \mathbb{R}) : f \leq h \leq g\}$. If there is no ambiguity we denote the constant function $f : x \mapsto z \in C(\mathbb{R}, \mathbb{R}^*)$ simply by z.

Let $\mathcal{X}$ be a family of subsets of a topological space. We say that $\mathcal{X}$ is *separated* if for any distinct sets $X, Y \in \mathcal{X}$ one can find disjoint open sets U, V such that $X \subseteq U$, $Y \subseteq V$ and $(\forall Z \in \mathcal{X})(Z \subseteq U$ or $Z \subseteq V)$.

1.2. Main results

We will prove the following equalities and inclusions.

Theorem 1.1. *Let $g \in C(\mathbb{R}, \mathbb{R})$ and let $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ be nonempty.*

(1a) $\mathcal{K}_{\{g\}} = \{CL(E) : E \in CL(\mathbb{R})\}$.
(1b) $\mathcal{K}_{\{g\}} \subseteq \mathcal{K}_{\mathcal{G}}$ *if and only if* $\mathcal{G}[x] \neq \mathbb{R}$ *for every* $x \in \mathbb{R}$.
(1c) $\mathcal{K}_{\{g\}} \supseteq \mathcal{K}_{\mathcal{G}}$ *if and only if* $\mathcal{G} = [f, h]$ *for some* $f, h \in C(\mathbb{R}, \mathbb{R}^*)$ *such that* $f \leq h$, $-\infty < h$, *and* $f < \infty$.

(2a) $\mathcal{K}_{(g,\infty)} = \{CL(\mathbb{R}) \cap \mathcal{P}(X) : X \subseteq \mathbb{R}\}$.
(2b) $\mathcal{K}_{(g,\infty)} \subseteq \mathcal{K}_{\mathcal{G}}$ *if and only if for every* $x \in \mathbb{R}$ *there exists* $f \in C(\mathbb{R}, \mathbb{R})$ *such that* $E_{\mathcal{G}}(\{f\}) = \{E \in CL(\mathbb{R}) : x \notin E\}$.
(2c) $\mathcal{K}_{(g,\infty)} \supseteq \mathcal{K}_{\mathcal{G}}$ *if and only if* $\mathcal{G}$ *is complete and connected.*

(3a) $\mathcal{K}_{(-\infty,g)\cup(g,\infty)} = \big\{CL(\mathbb{R}) \cap \bigcup_{X\in\mathcal{X}} : \mathcal{X} \subseteq \mathcal{P}(\mathbb{R})$ *is separated*$\big\}$.
(3b) $\mathcal{K}_{(-\infty,g)\cup(g,\infty)} \subseteq \mathcal{K}_{\mathcal{G}}$ *if and only if* $CL(\mathbb{R}) \cap \mathcal{P}(\mathbb{R} \setminus \{x\}) \in \mathcal{K}_{\mathcal{G}}$ *for every* $x \in \mathbb{R}$ *and* $CL(\mathbb{R}) \cap (\mathcal{P}(U) \cup \mathcal{P}(\mathbb{R} \setminus \operatorname{cl} U)) \in \mathcal{K}_{\mathcal{G}}$ *for every regular open set* $U \subseteq \mathbb{R}$.
(3c) $\mathcal{K}_{(-\infty,g)\cup(g,\infty)} \supseteq \mathcal{K}_{\mathcal{G}}$ *if and only if* $\mathcal{G} = \bigcup_{i\in I} \mathcal{G}_i$ *for some linearly ordered set* $(I, <)$ *and an indexed system of complete connected families* $\{\mathcal{G}_i : i \in I\}$ *such that for every* $i \in I$ *there exist functions* $f_i, h_i \in C(\mathbb{R}, \mathbb{R}^*)$ *satisfying*

$$\bigcup_{j<i} \mathcal{G}_j \subseteq (-\infty, f_i), \quad \mathcal{G}_i \subseteq (f_i, h_i) \quad \textit{and} \quad \bigcup_{j>i} \mathcal{G}_j \subseteq (h_i, \infty).$$

First three statements are proved in Section 3 (Theorems 3.1, 3.2 and 3.3), statements (2a)–(2c) in Section 4 (Theorems 4.1, 4.2 and 4.5), and (3a)–(3c) in Section 5 (Theorems 5.1, 5.2 and 5.5).

2. The elements of lattices $\mathcal{K}_{\mathcal{G}}$ and $\mathcal{L}_{\mathcal{G}}$

We begin with two extremal cases.

Proposition 2.1. *Let $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$.*

(1) $\mathcal{K}_{\emptyset} = \{\emptyset, CL(\mathbb{R})\}$, $\mathcal{L}_{\emptyset} = \{\emptyset, C(\mathbb{R}, \mathbb{R})\}$.
(2) $\mathcal{K}_{C(\mathbb{R},\mathbb{R})} = \{CL(\mathbb{R})\}$, $\mathcal{L}_{C(\mathbb{R},\mathbb{R})} = \{C(\mathbb{R}, \mathbb{R})\}$.

If $\mathcal{G} \neq \emptyset$ then every family $\mathcal{E} \in \mathcal{K}_{\mathcal{G}}$ is nonempty because it contains $\emptyset$, and every $\mathcal{F} \in \mathcal{L}_{\mathcal{G}}$ is nonempty because $\mathcal{G} \subseteq \mathcal{F}$. Hence, $\emptyset \in \mathcal{K}_{\mathcal{G}}$ if and only if $\emptyset \in \mathcal{L}_{\mathcal{G}}$ if and only if $\mathcal{G} = \emptyset$.

Proposition 2.2. *Let $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$, $\mathcal{G} \neq \emptyset$.*

(1) *The least element of $\mathcal{K}_{\mathcal{G}}$ is $E_{\mathcal{G}}(C(\mathbb{R}, \mathbb{R})) = \{E \in CL(\mathbb{R}) \colon \mathcal{G} \restriction E = C(E, \mathbb{R})\}$.*
(2) *The least element of $\mathcal{L}_{\mathcal{G}}$ is $F_{\mathcal{G}}(CL(\mathbb{R})) = F_{\mathcal{G}}(\{\mathbb{R}\}) = \mathcal{G}$.*

It follows that the lattices $\mathcal{K}_{\mathcal{G}}$ and $\mathcal{L}_{\mathcal{G}}$ have at least two elements if and only if $\mathcal{G} \neq C(\mathbb{R}, \mathbb{R})$.

By Proposition 2.2 (2), every family $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ is the least element of $\mathcal{L}_{\mathcal{G}}$. Now we are going to characterize families $\mathcal{E} \subseteq CL(\mathbb{R})$ that can be least elements of lattices $\mathcal{K}_{\mathcal{G}}$.

We say that a family $\mathcal{E} \subseteq CL(\mathbb{R})$ is *hereditary* if for any $D, E \in CL(\mathbb{R})$, if $D \subseteq E$ and $E \in \mathcal{E}$ then $D \in \mathcal{E}$. We show that the elements of lattices $\mathcal{K}_{\mathcal{G}}$ are exactly hereditary subfamilies of $CL(\mathbb{R})$ and every hereditary family $\mathcal{E} \subseteq CL(\mathbb{R})$ is the least element of some lattice $\mathcal{K}_{\mathcal{G}}$.

Proposition 2.3. *Let $\mathcal{E} \subseteq CL(\mathbb{R})$. The following conditions are equivalent.*

(1) *There exists $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ such that $\mathcal{E} \in \mathcal{K}_{\mathcal{G}}$.*
(2) *$\mathcal{E}$ is hereditary.*

Proof. (1) $\Rightarrow$ (2). If $\mathcal{E} \in \mathcal{K}_{\mathcal{G}}$ then $\mathcal{E} = E_{\mathcal{G}}(\mathcal{F})$ for some $\mathcal{F} \subseteq C(\mathbb{R}, \mathbb{R})$. It follows from the definition that the family $E_{\mathcal{G}}(\mathcal{F})$ is hereditary.

(2) $\Rightarrow$ (1). Fix $f \in C(\mathbb{R}, \mathbb{R})$. For every $E \in \mathcal{E}$, fix $g_E \in C(\mathbb{R}, \mathbb{R})$ such that $Eq_{f, g_E} = E$ and let $\mathcal{G} = \{g_E \colon E \in \mathcal{E}\}$. Then $\mathcal{E} = E_{\mathcal{G}}(\{f\}) \in \mathcal{K}_{\mathcal{G}}$. $\square$

Lemma 2.4. *Let $I, J \subseteq \mathbb{R}$ be non-degenerate, bounded, closed intervals. For every $n \in \omega$, let $x_n \in \operatorname{int} I$ be distinct and $A_n \subseteq J$ be dense in J. Then there exists an increasing bijection $f \colon I \to J$ such that $(\forall n \in \omega)\, f(x_n) \in A_n$.*

Proof. Let us note that any increasing bijection from I to J is necessarily continuous. Define a sequence of increasing bijections $f_n \colon I \to J$ by induction as follows. Let f_0 be linear. For every n, let $a_0, \ldots, a_{n+1}$ be the increasing enumeration of the set $\{a, b\} \cup \{x_j \colon j < n\}$, where $I = [a, b]$. Assume that $f_n \colon I \to J$ is an increasing bijection which is linear on each interval $[a_j, a_{j+1}]$ and moreover the function $f_n - \frac{1}{2} f_0$ is strictly inceasing. For every $j \leq n$, if $x_n \notin (a_j, a_{j+1})$ then let $f_{n+1}(x) = f_n(x)$ for every $x \in [a_j, a_{j+1}]$. If $x_n \in (a_j, a_{j+1})$, let f_{n+1} be defined linearly on intervals $[a_j, x_n]$ and $[x_n, a_{j+1}]$, where $f_{n+1}(a_j) = f_n(a_j)$,

$f_{n+1}(a_{j+1}) = f_n(a_{j+1})$, and $f_{n+1}(x_n) \in A_n$ is chosen so that $f_{n+1} - \frac{1}{2}f_0$ is strictly increasing and for every $x \in (a_j, a_{j+1})$, $|f_{n+1}(x) - f_n(x)| < 2^{-n}$. We obtain a uniformly convergent sequence of increasing bijections $f_n\colon I \to J$. Its limit $f\colon I \to J$ is continuous and surjective. Function $f - \frac{1}{2}f_0$, being a limit of a sequence of strictly increasing functions, is non-decreasing, hence f is strictly increasing. For every n we have $f(x_n) = f_{n+1}(x_n) \in A_n$. $\square$

Theorem 2.5. *Let $\mathcal{E} \subseteq CL(\mathbb{R})$ be hereditary. Then there exists $\mathcal{G} \subseteq C(\mathbb{R},\mathbb{R})$ such that $\mathcal{E}$ is the least element of $\mathcal{K}_{\mathcal{G}}$.*

Proof. Fix disjoint countable dense sets $D_0, D_1 \subseteq \mathbb{R}$ and $h\colon \mathbb{R} \to \{0,1\}$ such that both $h^{-1}[\{0\}]$ and $h^{-1}[\{1\}]$ are dense. Given a hereditary family $\mathcal{E} \subseteq CL(\mathbb{R})$, let $\mathcal{G}$ be the family of all functions $g \in C(\mathbb{R},\mathbb{R})$ such that $E_g \in \mathcal{E}$, where $E_g = \operatorname{cl}\left\{x \in \mathbb{R}\colon g(x) \in D_{h(x)}\right\}$. We prove that $\mathcal{E} = E_{\mathcal{G}}(C(\mathbb{R},\mathbb{R}))$.

Let $E \in \mathcal{E}$ and $f \in C(\mathbb{R},\mathbb{R})$ be arbitrary. We will find $g \in [f\restriction E]$ such that $E_g \subseteq E$. Without a loss of generality we may assume that the complement of E is a disjoint union of bounded open intervals and that the values of f at the endpoints of each of these intervals are different. We can accomplish this by adding to E an unbounded discrete set $Z \subseteq \mathbb{R} \setminus E$ dividing each interval adjacent to E, and suitably modifying the values of f outside E to ensure that $f(z) \notin D_{h(z)}$ for $z \in Z$ and $f(a) \neq f(b)$ for each interval $[a,b]$ adjacent to $E \cup Z$.

Let $[a,b]$ be a closed interval adjacent to E. Assume that $f(a) < f(b)$. Let $I = [f(a), f(b)]$, $J = [a,b]$, $\{x_n : n \in \omega\} = (D_0 \cup D_1) \cap \operatorname{int} I$, and for every n, let $A_n = J \cap h^{-1}[\{i\}]$ where $i \in \{0,1\}$ is such that $x_n \notin D_i$. Let $g_J\colon I \to J$ be the increasing bijection obtained in Lemma 2.4. Its inverse $g_J^{-1}\colon [a,b] \to [f(a), f(b)]$ is an increasing bijection as well and for every $x \in (a,b)$ we have $g_J^{-1}(x) \notin D_{h(x)}$. Similarly, if $f(b) < f(a)$ then there exists a decreasing bijection $g_J^{-1}\colon [a,b] \to [f(b), f(a)]$ such that $g_J^{-1}(x) \notin D_{h(x)}$ for all $x \in (a,b)$. Define $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x), & \text{if } x \in E, \\ g_J^{-1}(x) & \text{if } J \text{ is a closed interval adjacent to } E \text{ and } x \in \operatorname{int} J. \end{cases}$$

Obviously, g is continuous and $E_g \subseteq E$. Since $E \in \mathcal{E}$ and $\mathcal{E}$ is hereditary, we have $E_g \in \mathcal{E}$, hence $g \in \mathcal{G}$.

We have shown that for every $E \in \mathcal{E}$ and $f \in C(\mathbb{R},\mathbb{R})$ there exists $g \in \mathcal{G}$ such that $g\restriction E = f\restriction E$, hence $\mathcal{E} \subseteq E_{\mathcal{G}}(C(\mathbb{R},\mathbb{R}))$. To prove the opposite inclusion, let us take $E \in CL(\mathbb{R}) \setminus \mathcal{E}$. Let $\{x_n\colon n \in \omega\}$ be a countable dense subset of E, and for every n, let $A_n = D_{h(x_n)}$. By repeated use of Lemma 2.4 we can find $f \in C(\mathbb{R},\mathbb{R})$ such that $f(x_n) \in A_n$ for all n. Since $\mathcal{E}$ is hereditary and $E \notin \mathcal{E}$, for every $g \in \mathcal{G}$ we have $E \nsubseteq E_g$, hence there exists n such that $x_n \in E \setminus E_g$. We have $f(x_n) = D_{h(x_n)}$ and $g(x_n) \notin D_{h(x_n)}$, hence $f\restriction E \neq g\restriction E$. It follows that $f\restriction E \notin \mathcal{G}\restriction E$, thus $E \notin E_{\mathcal{G}}(C(\mathbb{R},\mathbb{R}))$. $\square$

Theorem 2.6. *There exists $\mathcal{G} \subseteq C(\mathbb{R},\mathbb{R})$ such that $\mathcal{K}_{\mathcal{G}}$ contains every nonempty hereditary family $\mathcal{E} \subseteq CL(\mathbb{R})$.*

Proof. Let $\{E_\alpha : \alpha < 2^\omega\}$ be a one-to-one enumeration of all nonempty closed subsets of $\mathbb{R}$. For each $\alpha < 2^\omega$ fix some $x_\alpha \in E_\alpha$. Using transfinite induction for $\alpha < 2^\omega$ we define $y_\alpha \in \mathbb{R}$ and a sequence of functions $\{g_{\alpha,n} : n \in \omega\} \subseteq C(\mathbb{R},\mathbb{R})$.

We proceed as follows. If y_β and $g_{\beta,n}$ are defined for all $\beta < \alpha$ and $n \in \omega$, find $y_\alpha \notin \{y_\beta : \beta < \alpha\} \cup \{g_{\beta,n}(x_\alpha) : \beta < \alpha,\ n \in \omega\}$. Let $\{I_{\alpha,n} : n \in \omega\}$ be the family of all nonempty open intervals with rational endpoints having nonempty intersection with E_α. For every n, there exists a function $g_{\alpha,n} \in C(\mathbb{R},\mathbb{R})$ such that $g_{\alpha,n}(x) = y_\alpha$ if and only if $x \notin I_{\alpha,n}$, and $g_{\alpha,n}(x_\beta) \neq y_\beta$ for all $\beta < \alpha$.

Let $\mathcal{G} = \{g_{\alpha,n} : \alpha < 2^\omega,\ n \in \omega\}$. For every $\alpha < 2^\omega$, let f_α be the constant function with value y_α. We show that $f_\alpha \restriction E_\alpha \notin \mathcal{G} \restriction E_\alpha$. If $g \in \mathcal{G}$ then $g = g_{\beta,n}$ for some $\beta < 2^\omega$ and $n \in \omega$. If $\beta < \alpha$ then $g_{\beta,n}(x_\alpha) \neq y_\alpha$ by the definition of y_α. Since $x_\alpha \in E_\alpha$, we have $f_\alpha \restriction E_\alpha \neq g \restriction E_\alpha$. If $\beta = \alpha$ then there exists $x \in E_\alpha \cap I_{\alpha,n}$. We have $g_{\alpha,n}(x) \neq y_\alpha$, hence $f_\alpha \restriction E_\alpha \neq g \restriction E_\alpha$. Finally, if $\beta > \alpha$ then $g_{\beta,n}(x_\alpha) \neq y_\alpha$ by the definition of $g_{\beta,n}$. Again, $f_\alpha \restriction E_\alpha \neq g \restriction E_\alpha$.

Let $\mathcal{E}$ be a nonempty hereditary family of closed subsets of $\mathbb{R}$. Then $\emptyset \in \mathcal{E}$, hence each $E \in CL(\mathbb{R}) \setminus \mathcal{E}$ is nonempty. Denote $\mathcal{F} = \{f_\alpha : E_\alpha \in CL(\mathbb{R}) \setminus \mathcal{E}\}$. If $E \in \mathcal{E}$ and $f \in \mathcal{F}$ then $f = f_\alpha$ for some $\alpha < 2^\omega$ such that $E_\alpha \in CL(\mathbb{R}) \setminus \mathcal{E}$, hence $E_\alpha \nsubseteq E$. There exists n such that $E \subseteq \mathbb{R} \setminus I_{\alpha,n}$. By the definition of $g_{\alpha,n}$ we have $f_\alpha \restriction E = g_{\alpha,n} \restriction E$, hence $f_\alpha \restriction E \in \mathcal{G} \restriction E$. It follows that $E \in E_{\mathcal{G}}(\mathcal{F})$, and we conclude that $\mathcal{E} \subseteq E_{\mathcal{G}}(\mathcal{F})$. Conversely, if $E \in CL(R) \setminus \mathcal{E}$ then $E = E_\alpha$ for some $\alpha < 2^\omega$. Since $f_\alpha \restriction E_\alpha \notin \mathcal{G} \restriction E_\alpha$ and $f_\alpha \in \mathcal{F}$, we have $E \notin E_{\mathcal{G}}(\mathcal{F})$. It follows that $\mathcal{E} = E_{\mathcal{G}}(\mathcal{F})$, hence $\mathcal{E} \in \mathcal{K}_{\mathcal{G}}$. $\square$

3. Results for family $\mathcal{G} = \{g\}$

We show that if $\mathcal{G}$ is a singleton then the lattice $\mathcal{K}_{\mathcal{G}}$ is isomorphic to the complete lattice $(CL(\mathbb{R}), \subseteq)$ of all closed subsets of $\mathbb{R}$. Let us note that in $(CL(\mathbb{R}), \subseteq)$ we have $\bigwedge \mathcal{E} = \bigcap \mathcal{E}$ and $\bigvee \mathcal{E} = \mathrm{cl}\left(\bigcup \mathcal{E}\right)$, for any $\mathcal{E} \subseteq CL(\mathbb{R})$.

Theorem 3.1. *Let $\mathcal{G} = \{g\}$, $g \in C(\mathbb{R},\mathbb{R})$. Then $\mathcal{K}_{\mathcal{G}} = \{CL(E) : E \in CL(\mathbb{R})\}$ and $\mathcal{L}_{\mathcal{G}} = \{[g \restriction E] : E \in CL(\mathbb{R})\}$.*

Proof. It is clear that $(f, E) \in R_{\mathcal{G}}$ if and only if $E \subseteq Eq_{f,g}$, for any $f \in C(\mathbb{R},\mathbb{R})$ and $E \in CL(\mathbb{R})$. Hence, for any $\mathcal{F} \subseteq C(\mathbb{R},\mathbb{R})$ we have

$$E_{\mathcal{G}}(\mathcal{F}) = \bigcap_{f \in \mathcal{F}} CL(Eq_{f,g}) = CL\left(\textstyle\bigcap_{f \in \mathcal{F}} Eq_{f,g}\right).$$

Since for any set $E \in CL(\mathbb{R})$ there exists $f \in C(\mathbb{R},\mathbb{R})$ such that $E = Eq_{f,g}$, we obtain that $\mathcal{K}_{\mathcal{G}} = \{E_{\mathcal{G}}(\mathcal{F}) : \mathcal{F} \subseteq C(\mathbb{R},\mathbb{R})\} = \{CL(E) : E \in CL(\mathbb{R})\}$. For any $\mathcal{E} \subseteq CL(\mathbb{R})$ we also have

$$F_{\mathcal{G}}(\mathcal{E}) = \bigcap_{E \in \mathcal{E}} [g \restriction E] = \left[g \restriction \mathrm{cl}\left(\bigcup \mathcal{E}\right)\right],$$

hence $\mathcal{L}_{\mathcal{G}} = \{F_{\mathcal{G}}(\mathcal{E}) : \mathcal{E} \subseteq CL(\mathbb{R})\} = \{[g \restriction E] : E \in CL(\mathbb{R})\}$. $\square$

It can be easily seen that if $\mathcal{G} = \{g\}$ then each element $\mathcal{E} \in \mathcal{K}_\mathcal{G}$ can be generated by a family consisting of a single function f: if $\mathcal{E} \in \mathcal{K}_\mathcal{G}$ then there exists $E \in CL(\mathbb{R})$ such that $\mathcal{E} = CL(E)$, for any $f \in C(\mathbb{R}, \mathbb{R})$ satisfying $Eq_{f,g} = E$ we then have $\mathcal{E} = E_\mathcal{G}(\{f\})$. Similarly, each $\mathcal{F} \in \mathcal{L}_\mathcal{G}$ can be expressed as $F_\mathcal{G}(\{E\})$ for some $E \in CL(\mathbb{R})$. Moreover, this set E is unique; if $D, E \in CL(\mathbb{R})$ are distinct then $F_\mathcal{G}(\{D\}) \neq F_\mathcal{G}(\{E\})$ by the normality of $\mathbb{R}$.

The next two results allows us to characterize families $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ for which the lattice $\mathcal{K}_\mathcal{G}$ is the same as in Theorem 3.1.

Theorem 3.2. *Let $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ be nonempty. The following conditions are equivalent.*

(1) $\{CL(E) \colon E \in CL(\mathbb{R})\} \subseteq \mathcal{K}_\mathcal{G}$.
(2) *The least element of $\mathcal{K}_\mathcal{G}$ is $\{\emptyset\}$.*
(3) *For each $x \in \mathbb{R}$, $\mathcal{G}[x] \neq \mathbb{R}$.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3). If $E_\mathcal{G}(C(\mathbb{R}, \mathbb{R})) = \{\emptyset\}$ then for each nonempty $E \in CL(\mathbb{R})$ there exists $f \in C(\mathbb{R}, \mathbb{R})$ such that $f \restriction E \notin \mathcal{G} \restriction E$. In particular, for every $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$ such that if $f(x) = y$ then $f \restriction \{x\} \notin \mathcal{G} \restriction \{x\}$, hence $y \notin \mathcal{G}[x]$.

(3) $\Rightarrow$ (1). Fix a function $h \in \mathcal{G}$. For every $E \in CL(\mathbb{R})$ and $x \notin E$ let us take $y \notin \mathcal{G}[x]$ and a function $f_x \in [h \restriction E]$ such that $f_x(x) = y$. Let $\mathcal{F} = \{f_x \colon x \notin E\}$. If $D \in E_\mathcal{G}(\mathcal{F})$ then for any $x \notin E$ we have $f_x \restriction D \in \mathcal{G} \restriction D$, hence $x \notin D$. It follows that $D \subseteq E$ and thus $E_\mathcal{G}(\mathcal{F}) \subseteq CL(E)$. The opposite inclusion is clear, hence we obtain $CL(E) = E_\mathcal{G}(\mathcal{F}) \in \mathcal{K}_\mathcal{G}$. □

Theorem 3.3. *Let $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ be nonempty. The following conditions are equivalent.*

(1) $\mathcal{K}_\mathcal{G} \subseteq \{CL(E) \colon E \in CL(\mathbb{R})\}$.
(2) *There exist $h_1, h_2 \in C(\mathbb{R}, \mathbb{R}^*)$ such that $\mathcal{G} = \{g \in C(\mathbb{R}, \mathbb{R}) \colon h_1 \leq g \leq h_2\}$.*

Proof. (1) $\Rightarrow$ (2). Denote $H = \bigcup \mathcal{G}$. Let us first show that H is a closed subset of $\mathbb{R}^2$. Assume that $(x, y) \in \operatorname{cl} H$. Since $\mathcal{G}$ is a nonempty family of continuous functions, there exists in H a sequence of points $\{(x_n, y_n) \colon n \in \omega\}$ converging to (x, y) and such that all x_n are distinct. Let $f \in C(\mathbb{R}, \mathbb{R})$ be such that $f(x_n) = y_n$ for every n and $f(x) = y$. Then $\{x_n\} \in E_\mathcal{G}(\{f\})$ for every n. Since $E_\mathcal{G}(\{f\}) \in \mathcal{K}_\mathcal{G}$, it follows from (1) that $\operatorname{cl}\{x_n \colon n \in \omega\} \in E_\mathcal{G}(\{f\})$, hence $\{x\} \in E_\mathcal{G}(\{f\})$ and so $(x, y) \in H$.

We show that $\mathcal{G}[x] = \{g(x) \colon g \in \mathcal{G}\}$ is connected, for every $x \in \mathbb{R}$. Otherwise we can find $g_1, g_2 \in \mathcal{G}$ and $y \notin \mathcal{G}[x]$ such that $g_1(x) < y < g_2(x)$. Since H is closed, there exist $a, b, c, d \in \mathbb{R}$ such that $x \in (a, b)$, $y \in (c, d)$, and $\big((a, b) \times (c, d)\big) \cap H = \emptyset$. Let $f \in C(\mathbb{R}, \mathbb{R})$ be such that $f(a) = g_1(a)$ and $f(b) = g_2(b)$. Then $\{a\}, \{b\} \in E_\mathcal{G}(\{f\})$, hence also $\{a, b\} \in E_\mathcal{G}(\{f\})$, and thus there exists $g \in \mathcal{G}$ such that $g(a) = g_1(a)$ and $g(b) = g_2(b)$. By Intermediate Value Theorem there is $z \in (a, b)$ such that $g(z) \in (c, d)$, which contradicts the assumption that $\big((a, b) \times (c, d)\big) \cap H$ is empty. So $\mathcal{G}[x]$ is a connected closed set, that is, a closed interval.

For every $x \in \mathbb{R}$ denote $h_1(x) = \inf \mathcal{G}[x]$ and $h_2(x) = \sup \mathcal{G}[x]$. We show that h_1, h_2 are continuous. If $y < h_1(x)$ then there exist $a, b, c, d \in \mathbb{R}$ such that $x \in (a, b)$, $y \in (c, d)$, and $\big((a, b)\times(c, d)\big)\cap H = \emptyset$. It follows $\big((a, b)\times(-\infty, d)\big)\cap H = \emptyset$, otherwise one could find a contradiction using Intermediate Value Theorem, as before. We can conclude that h_1 is lower semi-continuous. Since h_1 is the infimum of a family of continuous functions, it is also upper semi-continuous, and hence continuous. A similar argument shows the continuity of h_2.

It remained to show that $\mathcal{G} = \{g \in C(\mathbb{R}, \mathbb{R})\colon h_1 \le g \le h_2\}$. The inclusion from left to right is clear. If $g \in C(\mathbb{R}, \mathbb{R})$ is such that $h_1 \le g \le h_2$, then for every $x \in \mathbb{R}$ we have $g(x) \in \mathcal{G}[x]$, hence $\{x\} \in E_{\mathcal{G}}(\{g\})$. By (1), also $\mathbb{R} \in E_{\mathcal{G}}(\{g\})$, hence $g \in \mathcal{G}$.

(2) $\Rightarrow$ (1). Let $\mathcal{E} \in \mathcal{K}_{\mathcal{G}}$, that is, $\mathcal{E} = E_{\mathcal{G}}(\mathcal{F})$ for some $\mathcal{F} \subseteq C(\mathbb{R}, \mathbb{R})$, and let $E = \bigcup \mathcal{E}$. Then $E = \{x \in \mathbb{R}\colon \mathcal{F}[x] \subseteq \mathcal{G}[x]\}$. First, let us show that $E \in CL(\mathbb{R})$. If $x \in \operatorname{cl} E$ then there exists a sequence $\{x_n \colon n \in \omega\}$ in E such that $x_n \to x$. For any $y \in \mathcal{F}[x]$, let us take some $f \in \mathcal{F}$ such that $f(x) = y$. For every n we have $f(x_n) \in \mathcal{F}[x_n] \subseteq \mathcal{G}[x_n]$, hence $h_1(x_n) \le f(x_n) \le h_2(x_n)$. By the continuity of f, h_1, and h_2 we obtain that $h_1(x) \le f(x) \le h_2(x)$, hence $y \in \mathcal{G}[x]$. We have $\mathcal{F}[x] \subseteq \mathcal{G}[x]$, hence $x \in E$, and it follows that E is closed.

We show that for every $f \in \mathcal{F}$ there exists $g \in \mathcal{G}$ such that $f \restriction E = g \restriction E$. Fix some $f \in \mathcal{F}$. For every $x \in E$ we have $f(x) \in \mathcal{G}[x]$, hence $h_1(x) \le f(x) \le h_2(x)$. Let $g(x) = \min\{\max\{f(x), h_1(x)\}, h_2(x)\}$, for all $x \in \mathbb{R}$. Clearly, g is continuous and $g \restriction E = f \restriction E$. Since $\mathcal{G}$ is nonempty, we have $h_1 \le h_2$, and hence also $h_1 \le g \le h_2$. By (2), we have $g \in \mathcal{G}$. It follows that $E \in \mathcal{E}$, hence $\mathcal{E} = CL(E)$. $\square$

Now we can characterize those families $\mathcal{G}$ for which $\mathcal{K}_{\mathcal{G}} = \{CL(E) : E \in CL(\mathbb{R})\}$. Let us recall that for $f, h \in C(\mathbb{R}, \mathbb{R}^*)$ we denoted $[f, h] = \{g \in C(\mathbb{R}, \mathbb{R})\colon f \le g \le h\}$, where $f \le g$ if and only if $(\forall x \in \mathbb{R})\, f(x) \le g(x)$.

Corollary 3.4. *Let $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$. The following conditions are equivalent.*

(1) $\mathcal{K}_{\mathcal{G}} = \{CL(E)\colon E \in CL(\mathbb{R})\}$.

(2) *There exist $f, h \in C(\mathbb{R}, \mathbb{R}^*)$ such that $f \le h$, $f^{-1}[\mathbb{R}] \cup h^{-1}[\mathbb{R}] = \mathbb{R}$, and $\mathcal{G} = [f, h]$.*

It follows that the same lattice $\mathcal{K}_{\mathcal{G}}$ is obtained for families $\mathcal{G}$ of the form $(-\infty, g] = \{f \in C(\mathbb{R}, \mathbb{R})\colon f \le g\}$ and $[g, \infty) = \{f \in C(\mathbb{R}, \mathbb{R})\colon g \le f\}$.

Corollary 3.5. *Let $g \in C(\mathbb{R}, \mathbb{R})$. Then $\mathcal{K}_{(-\infty,g]} = \mathcal{K}_{[g,\infty)} = \{CL(E) : E \in CL(\mathbb{R})\}$.*

4. Results for family $\mathcal{G} = (g, \infty)$

Recall that for $f, h \in C(\mathbb{R}, \mathbb{R}^*)$, $(f, h) = \{g \in C(\mathbb{R}, \mathbb{R})\colon f < g < h\}$, where $f < g$ is a shorthand for $(\forall x \in \mathbb{R})\, f(x) < g(x)$.

Theorem 4.1. *Let $\mathcal{G} = (g, \infty)$ where $g \in C(\mathbb{R}, \mathbb{R})$. Then $\mathcal{K}_{\mathcal{G}} = \{CL(\mathbb{R}) \cap \mathcal{P}(X)\colon X \subseteq \mathbb{R}\}$.*

Proof. If $\mathcal{E} \in \mathcal{K}_\mathcal{G}$ then $\mathcal{E} = E_\mathcal{G}(\mathcal{F})$ for some $\mathcal{F} \subseteq C(\mathbb{R}, \mathbb{R})$. Denote $X = \bigcup \mathcal{E}$. Clearly $\mathcal{E} \subseteq CL(\mathbb{R}) \cap \mathcal{P}(X)$. If $E \in CL(\mathbb{R}) \cap \mathcal{P}(X)$ then for every $x \in E$ we have $x \in D$ for some $D \in \mathcal{E}$, hence $f(x) > g(x)$ for all $f \in \mathcal{F}$. For every $f \in \mathcal{F}$ there exists $f' \in \mathcal{G}$ such that $f' \restriction E = f \restriction E$; it suffices to take $f' = h' + g$ where $h' > 0$ is a continuous function such that $h' \restriction E = (f - g) \restriction E$, h' is linear on each bounded interval adjacent to E, and constant on unbounded adjacent intervals, if there are any. It follows that $E \in \mathcal{E}$, and we obtain $\mathcal{E} = CL(\mathbb{R}) \cap \mathcal{P}(X)$, hence $\mathcal{K}_\mathcal{G} \subseteq \{CL(\mathbb{R}) \cap \mathcal{P}(X) \colon X \subseteq \mathbb{R}\}$.

To prove the opposite, let $X \subseteq \mathbb{R}$. Denote $\mathcal{F} = \{f_a : a \in \mathbb{R} \setminus X\}$, where $f_a(x) = g(x) + |x - a|$ for all $x \in \mathbb{R}$. If $E \in E_\mathcal{G}(\mathcal{F})$ then $f_a(x) > g(x)$ for all $x \in E$ and $a \in \mathbb{R} \setminus X$, hence $E \subseteq X$. It follows that $E_\mathcal{G}(\mathcal{F}) \subseteq CL(\mathbb{R}) \cap \mathcal{P}(X)$. The opposite inclusion is clear since $\mathcal{F} \subseteq F_\mathcal{G}(CL(\mathbb{R}) \cap \mathcal{P}(X))$. We obtain that $CL(\mathbb{R}) \cap \mathcal{P}(X) = E_\mathcal{G}(\mathcal{F}) \in \mathcal{K}_\mathcal{G}$, hence $\{CL(\mathbb{R}) \cap \mathcal{P}(X) \colon X \subseteq \mathbb{R}\} \subseteq \mathcal{K}_\mathcal{G}$. $\square$

A similar argument would prove the same result for the family $\mathcal{G} = (-\infty, g)$. Nevertheless, it will also follow from Corollary 4.6 below.

We will characterize those families $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ for which $\mathcal{K}_\mathcal{G} = \{CL(\mathbb{R}) \cap \mathcal{P}(X) \colon X \subseteq \mathbb{R}\}$. Like in the previous section, we characterize both inclusions separately. For $x \in \mathbb{R}$ denote $\mathcal{A}_x = CL(\mathbb{R}) \cap \mathcal{P}(\mathbb{R} \setminus \{x\}) = \{E \in CL(\mathbb{R}) \colon x \notin E\}$.

Theorem 4.2. *Let $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ be nonempty. The following conditions are equivalent.*

(1) $\{CL(\mathbb{R}) \cap \mathcal{P}(X) \colon X \subseteq \mathbb{R}\} \subseteq \mathcal{K}_\mathcal{G}$.
(2) $\{\mathcal{A}_x \colon x \in \mathbb{R}\} \subseteq \mathcal{K}_\mathcal{G}$.
(3) *For every $x \in \mathbb{R}$ there exists $f \in C(\mathbb{R}, \mathbb{R})$ such that $E_\mathcal{G}(\{f\}) = \mathcal{A}_x$.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3). For every $x \in \mathbb{R}$, we have $\{x\} \notin \mathcal{A}_x = E_\mathcal{G}(F_\mathcal{G}(\mathcal{A}_x))$, hence there exists $f \in F_\mathcal{G}(\mathcal{A}_x)$ such that $f \restriction \{x\} \notin \mathcal{G} \restriction \{x\}$. We have $\{f\} \subseteq F_\mathcal{G}(\mathcal{A}_x)$, hence $E_\mathcal{G}(\{f\}) \supseteq E_\mathcal{G}(F_\mathcal{G}(\mathcal{A}_x)) = \mathcal{A}_x$. Conversely, if $E \in CL(\mathbb{R}) \setminus \mathcal{A}_x$ then $x \in E$ and hence $f \restriction E \notin \mathcal{G} \restriction E$. It follows that $E \notin E_\mathcal{G}(\{f\})$, and we obtain $E_\mathcal{G}(\{f\}) \subseteq \mathcal{A}_x$.

(3) $\Rightarrow$ (1). Let $X \subseteq \mathbb{R}$, $\mathcal{E} = CL(\mathbb{R}) \cap \mathcal{P}(X)$, and $\mathcal{F} = F_\mathcal{G}(\mathcal{E})$. We will show that $E_\mathcal{G}(\mathcal{F}) = \mathcal{E}$. If not, then there exists $E \in E_\mathcal{G}(\mathcal{F})$ such that $E \nsubseteq X$. Let $x \in E \setminus X$. By (3) there exists $f \in C(\mathbb{R}, \mathbb{R})$ such that $E_\mathcal{G}(\{f\}) = \mathcal{A}_x$. Since $\mathcal{E} \subseteq \mathcal{A}_x$, we have $F_\mathcal{G}(\mathcal{E}) \supseteq F_\mathcal{G}(\mathcal{A}_x)$, hence $f \in F_\mathcal{G}(\mathcal{E}) = \mathcal{F}$. It follows that $E \in E_\mathcal{G}(\{f\})$, and we come to a contradiction. $\square$

Note that the family $\{\mathcal{A}_x \colon x \in \mathbb{R}\}$ in condition (2) of Theorem 4.2 cannot be replaced by a smaller one. More precisely, for every $z \in \mathbb{R}$ there exists $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ such that $\mathcal{A}_x \in \mathcal{K}_\mathcal{G}$ for all $x \neq z$ but $\mathcal{A}_z \notin \mathcal{K}_\mathcal{G}$. Indeed, let $\mathcal{G} = \{f \in C(\mathbb{R}, \mathbb{R}) \colon (\forall x \neq z)\, f(x) > 0\}$. For every $y \in \mathbb{R}$, let $f_y(x) = |x - y|$. If $y \neq z$ then $E_\mathcal{G}(\{f_y\}) = \mathcal{A}_y$, hence $\mathcal{A}_y \in \mathcal{K}_\mathcal{G}$. Since $F_\mathcal{G}(\mathcal{A}_z) = \{f \in C(\mathbb{R}, \mathbb{R}) \colon (\forall x \neq z)\, f(x) > 0\} = \mathcal{G}$, we obtain that $E_\mathcal{G}(F_\mathcal{G}(\mathcal{A}_z)) = CL(\mathbb{R})$, hence $\mathcal{A}_z \notin \mathcal{K}_\mathcal{G}$.

To characterize all families $\mathcal{G}$ such that $\mathcal{K}_\mathcal{G} \subseteq \{CL(\mathbb{R}) \cap \mathcal{P}(X) \colon X \subseteq \mathbb{R}\}$ we need the following notion. We say that a set $H \subseteq \mathbb{R}^2$ is *functionally connected* if for any two points $(x_1, y_1), (x_2, y_2) \in H$ such that $x_1 < x_2$, there exists a continuous function $h\colon [x_1, x_2] \to \mathbb{R}$ such that $h(x_1) = y_1$, $h(x_2) = y_2$, and the graph of h is included in H. If H is a functionally connected set then $\pi_1[H]$, the projection of H to the first coordinate, is connected. If $\pi_1[H]$ has at most one point then H is functionally connected. If $\pi_1[H]$ has more than one point and H is functionally connected then H must be pathwise connected. A connected set need not to be functionally connected, a simple example is the unit circle $\{(x, y)\colon x^2 + y^2 = 1\}$.

Lemma 4.3. *Let $a < b$ and let $H \subseteq \mathbb{R}^2$ be a functionally connected set such that $[a, b] \subseteq \pi_1[H]$. Let $h\colon [a, b] \to \mathbb{R}$ be a continuous function such that $h \subseteq H$, and let $u, v \in \mathbb{R}$ be such that points $(a, u), (b, v) \in H$. Then for every open interval J such that $u, v \in J$ and $\operatorname{rng}(h) \subseteq J$, there exists a continuous function $g\colon [a, b] \to J$ such that $g \subseteq H$, $g(a) = u$, and $g(b) = v$.*

Proof. Let a, b, H, h, u, v, and J be as above. There exists a continuous function $f\colon [a, b] \to \mathbb{R}$ such that $f \subseteq H$, $f(a) = u$, and $f(b) = v$. Let $a_2, b_2 \in (a, b)$ be such that $a_2 < b_2$ and $f(x) \in J$ for every $x \in [a, a_2] \cup [b_2, b]$.

Let us first prove that there exists some $a_1 \in [a, a_2]$ and a continuous function $f_1\colon [a, a_1] \to J$ such that $f_1 \subseteq H$, $f_1(a) = f(a)$, and $f_1(a_1) = h(a_1)$. This is clear if $f(x) = h(x)$ for some $x \in [a, a_2]$. If this is not the case then the values $f(a) - h(a)$ and $f(a_2) - h(a_2)$ must have the same signs. Without a loss of generality, assume that $f(a) > h(a)$ and $f(a_2) > h(a_2)$. Let $f'\colon [a, a_2] \to \mathbb{R}$ be a continuous function such that $f' \subseteq H$, $f'(a) = f(a)$, and $f'(a_2) = h(a_2)$. Let $a_0 = \max\{x \in [a, a_2]\colon f'(x) = f(x)\}$ and $a_1 = \min\{x \in [a_0, a_2]\colon f'(x) = h(x)\}$. It follows that $f'(x) \in J$ for every $x \in [a_0, a_1]$, and we can define $f_1(x) = f(x)$ for $x \in [a, a_0]$ and $f_1(x) = f'(x)$ for $x \in [a_0, a_1]$. Then f_1 is as required.

Similarly, there exist $b_1 \in [b_2, b]$ and a continuous function $f_2\colon [b_1, b] \to J$ such that $f_2 \subseteq H$, $f_2(b_1) = h(b_1)$, and $f_2(b) = f(b)$. Let $g(x) = f_1(x)$ for $x \in [a, a_1]$, $g(x) = h(x)$ for $x \in [a_1, b_1]$, and $g(x) = f_2(x)$ for $x \in [b_1, b]$. Then g has the required properties. □

Lemma 4.4. *Let $H \subseteq \mathbb{R}^2$ be a functionally connected set such that $\pi_1[H] = \mathbb{R}$. Then for every $f \in C(\mathbb{R}, \mathbb{R})$ and $E \in CL(\mathbb{R})$ such that $f \restriction E \subseteq H$, there exists $g \in C(\mathbb{R}, \mathbb{R})$ such that $g \restriction E = f \restriction E$ and $g \subseteq H$.*

Proof. Let H, f and E be as assumed. Let us note that for each point $(x, y) \in H$ there exists $g \in C(\mathbb{R}, \mathbb{R})$ such that $g(x) = y$ and $g \subseteq H$.

For every closed interval I adjacent to E there exists a continuous function $g_I\colon \mathbb{R} \to \mathbb{R}$ such that $g_I \subseteq H$ and g_I coincides with f at the endpoints of I. Let $g(x) = f(x)$ for $x \in E$ and $g(x) = g_I(x)$ for $x \in I$ if I is a closed interval adjacent to E. We obtain a function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g \restriction E = f \restriction E$ and $g \subseteq H$.

It remains to show that g_I can be chosen so that g is continuous. This is clear if there are only finitely many such intervals, so we will assume the opposite. Let $\{I_n : n \in \omega\}$ be one-to-one enumeration of all closed intervals adjacent to E. We define continuous functions $g_n = g_{I_n}$ as follows.

If I_n is unbounded then let $g_n\colon I_n \to \mathbb{R}$ be arbitrary continuous function such that $g_n \subseteq H$ and g_n coincides with f at the only endpoint of I_n.

Assume that $I_n = [a_n, b_n]$. For every continuous function $h\colon I_n \to \mathbb{R}$ denote $\operatorname{osc}(h)$ its oscillation, that is, $\operatorname{osc}(h) = \max\{h(x) - h(y) : x, y \in I_n\}$. Let $o_n = \inf\{\operatorname{osc}(h) \colon h \in \mathcal{H}_n\}$, where $\mathcal{H}_n$ is the family of all functions $h \in C(I_n, \mathbb{R})$ such that $h \subseteq H$ and $h \restriction \{a_n, b_n\} = f \restriction \{a_n, b_n\}$. Choose $g_n \in \mathcal{H}_n$ such that $\operatorname{osc}(g_n) \leq o_n + 2^{-n}$.

As above, let $g\colon \mathbb{R} \to \mathbb{R}$ be a map such that $g(x) = f(x)$ for all $x \in E$ and $g(x) = g_n(x)$ for all $x \in I_n$, $n \in \omega$. We prove that g is continuous at every point $z \in \mathbb{R}$. Let us take a convergent sequence $z_k \to z$. We will assume that this sequence is increasing, as it suffices to consider only one-sided limits, and for decreasing sequences the proof is the same. We may further assume that $z_k \in \mathbb{R} \setminus E$ for all k since we have $g(x) = f(x)$ for $x \in E$ and f is continuous at z. For every k, let n_k be such that $z_k \in I_{n_k}$. If there exist m, l such that $n_k = m$ for all $k > l$, then $g(z_k) = g_m(z_k)$ for all $k > l$, hence $z \in \operatorname{cl} I_m$ and $g(z_k) \to g(z)$. So we may assume that $n_k \to \infty$ and $z \in E$.

To prove that $g(z_k) \to g(z)$, it will suffice to show that $\operatorname{osc}(g_{n_k}) \to 0$. Fix some $h \in C(\mathbb{R}, \mathbb{R})$ such that $h \subseteq H$ and $h(z) = g(z)$. By Lemma 4.3, for every n we have $o_n \leq \operatorname{diam}(f[I_n] \cup h[I_n])$. Since both f and h is continuous at z, we have $\operatorname{diam}(f[I_n] \cup h[I_n]) \to 0$, hence $\operatorname{osc}(g_{n_k}) \leq o_{n_k} + 2^{-n_k} \to 0$. $\square$

Recall that a family $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ is said to be complete if $f \in \mathcal{G}$ for every $f \in C(\mathbb{R}, \mathbb{R})$ such that $f \subseteq \bigcup \mathcal{G}$. A complete family $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ is connected if and only if $\bigcup \mathcal{G}$ is functionally connected.

Theorem 4.5. *Let $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ be nonempty. The following conditions are equivalent.*

(1) $\mathcal{K}_{\mathcal{G}} \subseteq \{CL(\mathbb{R}) \cap \mathcal{P}(X) \colon X \subseteq \mathbb{R}\}$.
(2) *$\mathcal{G}$ is a complete and connected family.*

Proof. (1) $\Rightarrow$ (2). Denote $H = \bigcup \mathcal{G}$. Clearly, $\mathcal{G} \subseteq \{g \in C(\mathbb{R}, \mathbb{R}) \colon g \subseteq H\}$. To prove the opposite inclusion, let $g \in C(\mathbb{R}, \mathbb{R})$ be such that $g \subseteq H$. For every $x \in \mathbb{R}$ we have $g(x) \in \mathcal{G}[x]$, hence $\{x\} \in E_{\mathcal{G}}(\{g\})$. It follows that $\mathbb{R} \in E_{\mathcal{G}}(\{g\})$, hence $g \in \mathcal{G}$. Thus, $\mathcal{G} = \{g \in C(\mathbb{R}, \mathbb{R}) \colon g \subseteq H\}$, hence $\mathcal{G}$ is complete.

It remains to show that H is functionally connected. Let $(x_1, y_1), (x_2, y_2) \in H$ and $x_1 < x_2$. Let $f \in C(\mathbb{R}, \mathbb{R})$ be such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since $\{x_1\}, \{x_2\} \in E_{\mathcal{G}}(\{f\})$ and $E_{\mathcal{G}}(\{f\}) \in \mathcal{K}_{\mathcal{G}}$, it follows that $\{x_1, x_2\} \in E_{\mathcal{G}}(\{f\})$. Hence, there exists $g \in \mathcal{G}$ such that $g(x_1) = y_1$ and $g(x_2) = y_2$.

(2) $\Rightarrow$ (1). Let $\mathcal{E} \in \mathcal{K}_{\mathcal{G}}$, that is, there exists $\mathcal{F} \subseteq C(\mathbb{R}, \mathbb{R})$ such that $\mathcal{E} = E_{\mathcal{G}}(\mathcal{F})$. Let $X = \bigcup \mathcal{E}$. We will show that $\mathcal{E} = CL(\mathbb{R}) \cap \mathcal{P}(X)$.

Let us take $E \in CL(\mathbb{R}) \cap \mathcal{P}(X)$ and an arbitrary $f \in \mathcal{F}$. For every $x \in E$ we have $\{x\} \in \mathcal{E}$, hence $f(x) \in \mathcal{G}[x]$. It follows that $f \restriction E \subseteq \bigcup \mathcal{G}$. Since $\mathcal{G}$ is connected, by Lemma 4.4 there exists $g \in C(\mathbb{R}, \mathbb{R})$ such that $f \restriction E = g \restriction E$ and $g \subseteq \bigcup \mathcal{G}$. Since $\mathcal{G}$ is complete, we have $g \in \mathcal{G}$. This shows that $E \in E_{\mathcal{G}}(\mathcal{F})$, so $CL(\mathbb{R}) \cap \mathcal{P}(X) \subseteq \mathcal{E}$. The opposite inclusion is clear. $\square$

From Theorems 4.2 and 4.5 we obtain the following characterization.

Corollary 4.6. *Let $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ be nonempty. The following conditions are equivalent.*

(1) $\mathcal{K}_{\mathcal{G}} = \{CL(\mathbb{R}) \cap \mathcal{P}(X) \colon X \subseteq \mathbb{R}\}$.
(2) *$\mathcal{G}$ is a complete and connected family, and for every $x \in \mathbb{R}$ there exists a function $f \in C(\mathbb{R},\mathbb{R})$ such that $f \setminus \bigcup \mathcal{G} = f \restriction \{x\}$.*

5. Results for family $(-\infty, g) \cup (g, \infty)$

For $f, g \in C(\mathbb{R},\mathbb{R})$, if $f(x) \neq g(x)$ for every x then either $f < g$ or $f > g$. Hence, $\{f \in C(\mathbb{R},\mathbb{R}) \colon (\forall x \in \mathbb{R})\, f(x) \neq g(x)\} = (-\infty, g) \cup (g, \infty)$.

Recall that a family $\mathcal{X}$ of subsets of a topological space is said to be separated if for every distinct $X, Y \in \mathcal{X}$ there exist disjoint open sets U, V such that $X \subseteq U$, $Y \subseteq V$ and $(\forall Z \in \mathcal{X})(Z \subseteq U$ or $Z \subseteq V)$.

Theorem 5.1. *Let $g \in C(\mathbb{R},\mathbb{R})$, $\mathcal{G} = (-\infty, g) \cup (g, \infty)$. Then*

$$\mathcal{K}_{\mathcal{G}} = \left\{ CL(\mathbb{R}) \cap \bigcup_{X \in \mathcal{X}} \mathcal{P}(X) \colon \mathcal{X} \subseteq \mathcal{P}(\mathbb{R}) \text{ is separated} \right\}.$$

Proof. Let $\mathcal{E} \in \mathcal{K}_{\mathcal{G}}$, that is, $\mathcal{E} = E_{\mathcal{G}}(\mathcal{F})$ for some $\mathcal{F} \subseteq C(\mathbb{R},\mathbb{R})$. For $x \in \bigcup \mathcal{E}$, denote $\mathcal{E}_x = \{E \in \mathcal{E} : x \in E\}$, and let $\mathcal{X} = \left\{ \bigcup \mathcal{E}_x : x \in \bigcup \mathcal{E} \right\}$. We will show that $\mathcal{X}$ is separated and $\mathcal{E} = CL(\mathbb{R}) \cap \bigcup_{X \in \mathcal{X}} \mathcal{P}(X)$. Let us note that for every $x, y \in \mathbb{R}$, $\{x, y\} \in \mathcal{E}$ if and only if for every $f \in \mathcal{F}$,

$$(f(x) > g(x) \text{ and } f(y) > g(y)) \text{ or } (f(x) < g(x) \text{ and } f(y) < g(y)).$$

Hence, the relation $\sim$, defined by $x \sim y \Leftrightarrow \{x, y\} \in \mathcal{E}$, is an equivalence relation on $\bigcup \mathcal{E}$, and $\mathcal{X}$ is the corresponding partition of $\bigcup \mathcal{E}$ into equivalence classes.

Let $X = \bigcup \mathcal{E}_x$ and $Y = \bigcup \mathcal{E}_y$ be distinct elements of $\mathcal{X}$. Then $\{x, y\} \notin \mathcal{E}$, hence there exists $f \in \mathcal{F}$ such that $(f(x) - g(x))(f(y) - g(y)) \leq 0$. We have $\{x\}, \{y\} \in \mathcal{E}$, so $(f(x) - g(x))(f(y) - g(y)) \neq 0$. Without a loss of generality we may assume that $f(x) < g(x)$ and $f(y) > g(y)$. Let $U = \{u \in \mathbb{R} \colon f(u) < g(u)\}$ and $V = \{v \in \mathbb{R} : f(v) > g(v)\}$. Then U, V are disjoint open sets such that $X \subseteq U$, $Y \subseteq V$. Also, for every $z \in \bigcup \mathcal{E}$ we have $f(z) \neq g(z)$, hence $z \in U$ or $z \in V$. Clearly, $z \in U$ implies $\mathcal{E}_z \subseteq U$, and similarly $z \in V$ implies $\mathcal{E}_z \subseteq V$, hence the family $\mathcal{X}$ is separated.

We show that $\mathcal{E} = CL(\mathbb{R}) \cap \bigcup_{X \in \mathcal{X}} \mathcal{P}(X)$. The inclusion from left to right follows from the definition of $\mathcal{X}$. Conversely, if $E \in CL(\mathbb{R}) \cap \mathcal{P}(X)$ for some $X \in \mathcal{X}$ then we have $x \sim y$ for all $x, y \in E$, hence for every $f \in \mathcal{F}$ we have either $f <_E g$ or $g <_E f$, where $f <_E g$ is a shorthand for $(\forall x \in E)\, f(x) < g(x)$. It follows that $f \restriction E \in \mathcal{G} \restriction E$, thus $E \in E_{\mathcal{G}}(\mathcal{F}) = \mathcal{E}$.

We have proved that $\mathcal{K}_{\mathcal{G}} \subseteq \left\{ CL(\mathbb{R}) \cap \bigcup_{X \in \mathcal{X}} \colon \mathcal{X} \subseteq \mathcal{P}(\mathbb{R}) \text{ is separated} \right\}$. For the reverse inclusion, let us take $\mathcal{E} = CL(\mathbb{R}) \cap \bigcup_{X \in \mathcal{X}} \mathcal{P}(X)$ for some separated family $\mathcal{X} \subseteq \mathcal{P}(\mathbb{R})$, and let $\mathcal{F} = F_{\mathcal{G}}(\mathcal{E})$. Then $f \in \mathcal{F}$ if and only if $f \in C(\mathbb{R},\mathbb{R})$ and $f \restriction E \in \mathcal{G} \restriction E$ for every $E \in \mathcal{E}$. Hence, $\mathcal{F} = \{f \in C(\mathbb{R},\mathbb{R}) \colon (\forall X \in \mathcal{X})(f <_X g \text{ or } g <_X f)\}$.

Let us further show that $E_{\mathcal{G}}(\mathcal{F}) = \mathcal{E}$. Assume that $E \in CL(\mathbb{R})$ and $E \notin \mathcal{E}$. Then either $E \not\subseteq \bigcup \mathcal{X}$ or there exist distinct sets $X, Y \in \mathcal{X}$ such that E intersects both of them. In the first case we take $z \in E \setminus \bigcup \mathcal{X}$ and $f \in C(\mathbb{R}, \mathbb{R})$ such that $f(z) = g(z)$ and $f(x) > g(x)$ for all $x \neq z$. Then $f \in \mathcal{F}$ but $f(z) \notin \mathcal{G}[z]$, hence $E \notin E_{\mathcal{G}}(\mathcal{F})$. In the second case let U, V be disjoint open sets such that $X \subseteq U$, $Y \subseteq V$ and $(\forall Z \in \mathcal{X})(Z \subseteq U$ or $Z \subseteq V)$. There exists $f \in C(\mathbb{R}, \mathbb{R})$ such that $U = \{x \in \mathbb{R} \colon f(x) > g(x)\}$ and $V = \{x \in \mathbb{R} \colon f(x) < g(x)\}$. We have $f \in \mathcal{F}$ but $f \restriction E \notin \mathcal{G} \restriction E$, hence again $E \notin E_{\mathcal{G}}(\mathcal{F})$. It follows that $E_{\mathcal{G}}(\mathcal{F}) \subseteq \mathcal{E}$, hence the equality holds true, and thus $\mathcal{E} \in \mathcal{K}_{\mathcal{G}}$. Therefore, $\big\{CL(\mathbb{R}) \cap \bigcup_{X \in \mathcal{X}} \mathcal{P}(X) \colon \mathcal{X} \subseteq \mathcal{P}(\mathbb{R})$ is separated$\big\} \subseteq \mathcal{K}_{\mathcal{G}}$. □

If a family $\mathcal{X} \subseteq \mathcal{P}(\mathbb{R})$ is separated then for every distinct sets $X, Y \in \mathcal{X}$ one can find regular open sets U, V such that $X \subseteq U$, $Y \subseteq V$ and $(\forall Z \in \mathcal{X})(Z \subseteq U$ or $Z \subseteq V)$. Indeed, if U, V are disjoint open sets then their regularizations $U' = \operatorname{int}(\operatorname{cl} U)$, $V' = \operatorname{int}(\operatorname{cl} V)$ satisfy $U \subseteq U'$, $V \subseteq V'$ and are disjoint as well. We may also take $\mathbb{R} \setminus \operatorname{cl} U'$ instead of V'.

Let us recall that for $x \in \mathbb{R}$ we have denoted $\mathcal{A}_x = CL(\mathbb{R}) \cap \mathcal{P}(\mathbb{R} \setminus \{x\})$. For every open set $U \subseteq \mathbb{R}$ we also denote $\mathcal{B}_U = CL(\mathbb{R}) \cap (\mathcal{P}(U) \cup \mathcal{P}(\mathbb{R} \setminus \operatorname{cl} U))$.

Theorem 5.2. *Let $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ be nonempty. The following conditions are equivalent.*

(1) $\big\{CL(\mathbb{R}) \cap \bigcup_{X \in \mathcal{X}} \mathcal{P}(X) \colon \mathcal{X} \subseteq \mathcal{P}(\mathbb{R})$ *is separated*$\big\} \subseteq \mathcal{K}_{\mathcal{G}}$.
(2) $\{\mathcal{A}_x \colon x \in \mathbb{R}\} \cup \{\mathcal{B}_U \colon U \subseteq \mathbb{R}$ *is regular open*$\} \subseteq \mathcal{K}_{\mathcal{G}}$.
(3) *For any $x \in \mathbb{R}$ there exists $f \in C(\mathbb{R}, \mathbb{R})$ such that $E_{\mathcal{G}}(\{f\}) = \mathcal{A}_x$, and for any $x, y \in \mathbb{R}$ and any regular open set $U \subseteq \mathbb{R}$ such that $x \in U$ and $y \notin \operatorname{cl} U$ there exists $f \in F_{\mathcal{G}}(\mathcal{B}_U)$ such that $f \restriction \{x, y\} \notin \mathcal{G} \restriction \{x, y\}$.*

Proof. (1) ⇒ (2) is obvious.

(2) ⇒ (3). The first part of (3) follows from Theorem 4.2. For the second part, let U be a regular open set such that $x \in U$ and $y \notin \operatorname{cl} U$. By (2), we have $\mathcal{B}_U \in \mathcal{K}_{\mathcal{G}}$, hence $E_{\mathcal{G}}(F_{\mathcal{G}}(\mathcal{B}_U)) = \mathcal{B}_U$. Since $\{x, y\} \notin \mathcal{B}_U$, there exists $f \in F_{\mathcal{G}}(\mathcal{B}_U)$ such that $f \restriction \{x, y\} \notin \mathcal{G} \restriction \{x, y\}$.

(3) ⇒ (1). Let $\mathcal{X} \subseteq \mathcal{P}(\mathbb{R})$ be a separated family and let $\mathcal{E} = CL(\mathbb{R}) \cap \bigcup_{X \in \mathcal{X}} \mathcal{P}(X)$. Denote $\mathcal{F} = F_{\mathcal{G}}(\mathcal{E})$. We show that $\mathcal{E} = E_{\mathcal{G}}(\mathcal{F})$. Let us take $E \in CL(\mathbb{R})$, $E \notin \mathcal{E}$. Then either there exists $x \in E \setminus \bigcup \mathcal{X}$, or there exist $x, y \in E$ and distinct sets $X, Y \in \mathcal{X}$ such that $x \in X$ and $y \in Y$.

If $x \in E \setminus \bigcup \mathcal{X}$ then let $f \in F_{\mathcal{G}}(\mathcal{A}_x)$ be such that $f(x) \notin \mathcal{G}[x]$. Since $\mathcal{E} \subseteq \mathcal{A}_x$, we have $F_{\mathcal{G}}(\mathcal{A}_x) \subseteq F_{\mathcal{G}}(\mathcal{E})$, hence $f \in \mathcal{F}$. It follows that $\{x\} \notin E_{\mathcal{G}}(\mathcal{F})$, hence $E \notin E_{\mathcal{G}}(\mathcal{F})$. If $x \in X$, $y \in Y$ for some distinct $X, Y \in \mathcal{X}$ then there exists an open regular set U such that $X \subseteq U$, $Y \subseteq \mathbb{R} \setminus \operatorname{cl} U$ and each $Z \in \mathcal{X}$ is covered either by U or by $\mathbb{R} \setminus \operatorname{cl} U$. It follows that $\mathcal{E} \subseteq \mathcal{B}_U$. By (3), there exists $f \in F_{\mathcal{G}}(\mathcal{B}_U)$ such that $f \restriction \{x, y\} \notin \mathcal{G} \restriction \{x, y\}$. We have $f \in \mathcal{F}$, hence $\{x, y\} \notin E_{\mathcal{G}}(\mathcal{F})$, so $E \notin E_{\mathcal{G}}(\mathcal{F})$. In both cases it follows that $E_{\mathcal{G}}(\mathcal{F}) = \mathcal{E}$, hence $\mathcal{E} \in \mathcal{K}_{\mathcal{G}}$. □

Let us note that the family $\mathcal{H} = \{\mathcal{A}_x \colon x \in \mathbb{R}\} \cup \{\mathcal{B}_U \colon U$ is regular open$\}$ in the second condition of Theorem 5.2 is not minimal. Indeed, $\mathcal{B}_{\emptyset} = \mathcal{B}_{\mathbb{R}} = CL(\mathbb{R})$ is an element of $\mathcal{K}_{\mathcal{G}}$ for every $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$, hence $\mathcal{H} \setminus \{\mathcal{B}_{\emptyset}\} \subseteq \mathcal{K}_{\mathcal{G}} \Leftrightarrow \mathcal{H} \subseteq \mathcal{K}_{\mathcal{G}}$

holds for every $\mathcal{G}$. We do not know whether there exists a regular open set U and a family $\mathcal{G} \subseteq C(\mathbb{R},\mathbb{R})$ such that $\mathcal{H} \setminus \{\mathcal{B}_U\} \subseteq \mathcal{K}_\mathcal{G}$ and $\mathcal{B}_U \notin \mathcal{K}_\mathcal{G}$. We also do not know whether one can find a minimal family $\mathcal{M}$ of nonempty hereditary families of closed sets having the property that for every $\mathcal{G}$, if $\mathcal{M} \subseteq \mathcal{K}_\mathcal{G}$ then $CL(\mathbb{R}) \cap \bigcup_{X\in\mathcal{X}} \mathcal{P}(X) \in \mathcal{K}_\mathcal{G}$ for every separated family $\mathcal{X}$.

To characterize families $\mathcal{G}$ such that $CL(\mathbb{R}) \cap \bigcup_{X\in\mathcal{X}} \mathcal{P}(X) \in \mathcal{K}_\mathcal{G}$ holds for every separated family $\mathcal{X}$, we need few more notions. Given a fixed family $\mathcal{G} \subseteq C(\mathbb{R},\mathbb{R})$ and points $a = (a_1, a_2)$, $b = (b_1, b_2)$ in $\mathbb{R}^2$, let us write $a \sim b$ if there exists a function $f \in \mathcal{G}$ such that $f(a_1) = a_2$ and $f(b_1) = b_2$. Clearly, if $a \sim b$ and $a_1 = b_1$ then $a = b$. We say that family $\mathcal{G}$ is *transitive* if for any points $a, b, c \in \mathbb{R}^2$ having distinct first coordinates, if $a \sim b$ and $b \sim c$ then $a \sim c$. We say that family $\mathcal{G}$ is *sequential* if $a \sim b$ holds true whenever $a, b \in \bigcup \mathcal{G}$ and there exists a sequence of points $\{a_n : n \in \omega\}$ in $\bigcup \mathcal{G}$ such that $a_n \to a$, $a_n \sim b$, and the first coordinates of points a, b, and a_n, $n \in \omega$, are pairwise distinct.

Let $(I, <)$ be a linearly ordered set, and for every $i \in I$, let $\mathcal{G}_i \subseteq C(\mathbb{R},\mathbb{R})$ be nonempty. We say that indexed system $\{\mathcal{G}_i : i \in I\}$ is *sliced* if for every $i \in I$ there exist functions $g_i^-, g_i^+ \in C(\mathbb{R},\mathbb{R}^*)$ such that

$$\bigcup_{j<i} \mathcal{G}_j \subseteq (-\infty, g_i^-), \quad \mathcal{G}_i \subseteq (g_i^-, g_i^+) \quad \text{and} \quad \bigcup_{j>i} \mathcal{G}_j \subseteq (g_i^+, \infty).$$

Let us note that the assumption $\mathcal{G}_i \neq \emptyset$ implies that $g_i^- < g_i^+$, for every $i \in I$.

Lemma 5.3. *Let $\{\mathcal{G}_i : i \in I\}$ be a sliced system. Then for each $i \in I$ there exist functions $h_i^-, h_i^+ \in C(\mathbb{R},\mathbb{R}^*)$ such that $\mathcal{G}_i \subseteq (h_i^-, h_i^+)$ and $h_i^+ \leq h_j^-$ whenever $i < j$.*

Proof. For every i, let $g_i^-, g_i^+ \in C(\mathbb{R},\mathbb{R}^*)$ be such that $\bigcup_{j<i} \mathcal{G}_j \subseteq (-\infty, g_i^-)$, $\mathcal{G}_i \subseteq (g_i^-, g_i^+)$, and $\bigcup_{j>i} \mathcal{G}_j \subseteq (g_i^+, \infty)$. It is clear that if $\bigcup_{j<i} \mathcal{G}_j \neq \emptyset$ then $g_i^- \in C(\mathbb{R},\mathbb{R})$ and, similarly, if $\bigcup_{j>i} \mathcal{G}_j \neq \emptyset$ then $g_i^+ \in C(\mathbb{R},\mathbb{R})$. Since each interval $(g_i^-(0), g_i^+(0))$ contains a rational number, I is at most countable.

For simplicity let us assume that I is infinite. In the finite case the proof will be the same. Let $\{i(n) : n < \omega\}$ be a one-to-one enumeration of I. By induction let us define

$$\begin{aligned} h_{i(n)}^-(x) &= \max\left(\left\{h_{i(m)}^+(x) : m < n \text{ and } i(m) < i(n)\right\} \cup \left\{g_{i(n)}^-(x)\right\}\right), \\ h_{i(n)}^+(x) &= \min\left(\left\{h_{i(m)}^-(x) : m < n \text{ and } i(m) > i(n)\right\} \cup \left\{g_{i(n)}^+(x)\right\}\right). \end{aligned}$$

It is clear that $h_i^-, h_i^+ \in C(\mathbb{R},\mathbb{R}^*)$, and $\mathcal{G}_i \subseteq (h_i^-, h_i^+)$. Moreover, if $m < n$ then either $i(m) < i(n)$ and then $h_{i(m)}^+ \leq h_{i(n)}^-$ by the definition of $h_{i(n)}^-$, or $i(m) > i(n)$ and then $h_{i(n)}^+ \leq h_{i(m)}^-$ by the definition of $h_{i(n)}^+$. Hence, $h_i^+ \leq h_j^-$ for any $i < j$. $\square$

Lemma 5.4. *Let $\mathcal{G} \subseteq C(\mathbb{R},\mathbb{R})$ be a complete, transitive and sequential family. Then there exists a sliced system $\{\mathcal{G}_i : i \in I\}$ such that $\mathcal{G} = \bigcup_{i\in I} \mathcal{G}_i$ and each $\mathcal{G}_i$ is complete and connected.*

Proof. We may assume that $\mathcal{G}$ is nonempty. Denote $H = \bigcup \mathcal{G}$. For $a, b \in \mathbb{R}^2$, let us write $a \approx b$ if there exists c such that $a \sim c \sim b$. We prove that $\approx$ is an equivalence relation on H. The symmetry and the reflexivity of $\approx$ is clear. For the transitivity it suffices to prove that $a \sim b \sim c \sim d$ implies $a \approx d$. Let $a = (a_1, a_2)$, $b = (b_1, b_2)$, $c = (c_1, c_2)$, $d = (d_1, d_2)$. We may assume that $a \neq b \neq c \neq d$, hence $a_1 \neq b_1 \neq c_1 \neq d_1$. If $a_1 \neq c_1$ then by transitivity of $\mathcal{G}$ we have $a \sim c$, and we are done. A similar argument works if $b_1 \neq d_1$, so we may assume that $a_1 = c_1$ and $b_1 = d_1$. Without a loss of generality, let $a_1 < b_1$. Let $f \in \mathcal{G}$ be such that $f(b_1) = b_2$, and let $b' = (b'_1, b'_2)$ be such that $b'_1 > b_1$ and $b'_2 = f(b'_1)$. Then we have $a \sim b'$ and $b' \sim c$. Since $b'_1 \neq d_1$, it follows that $b' \sim d$, and thus $a \approx d$.

Let $\{H_i : i \in I\}$ be the partition of H corresponding to the equivalence $\approx$, and for every $i \in I$ let $\mathcal{G}_i = \{f \in \mathcal{G} : f \subseteq H_i\}$. Let $f \in \mathcal{G}$ be arbitrary. For all points $a, b \in f$ we have $a \approx b$, hence $f \subseteq H_i$ for some i. It follows that $\mathcal{G} = \bigcup\{\mathcal{G}_i : i \in I\}$. By the definition of $\approx$ and the completeness of $\mathcal{G}$, each $\mathcal{G}_i$ is connected and complete, and we have $\bigcup \mathcal{G}_i = H_i$. Clearly, if $\mathcal{G}_i \neq \mathcal{G}_j$ then $(\forall f \in \mathcal{G}_i)(\forall g \in \mathcal{G}_j)\, f < g$ or $(\forall f \in \mathcal{G}_i)(\forall g \in \mathcal{G}_j)\, f > g$. Thus there exists a linear order on I such that $i < j$ if and only if $f < g$ for all $f \in \mathcal{G}_i$ and $g \in \mathcal{G}_j$.

For every i that is not a maximal element of I, let us define $h_1(x) = \inf\{\sup A_{x,\varepsilon} : \varepsilon > 0\}$ and $h_2(x) = \sup\{\inf B_{x,\varepsilon} : \varepsilon > 0\}$, where $A_{x,\varepsilon} = H_i \cap \big((x-\varepsilon, x+\varepsilon) \times \mathbb{R}\big)$ and $B_{x,\varepsilon} = \bigcup_{j>i} H_j \cap \big((x-\varepsilon, x+\varepsilon) \times \mathbb{R}\big)$. Then h_1 is upper semi-continuous, h_2 is lower semi-continuous, and we have $f \leq h_1 \leq h_2 \leq g$ for all $f \in \mathcal{G}_i$ and $g \in \bigcup_{j>i} \mathcal{G}_j$. For $x \in \mathbb{R}$ denote $a = (x, h_1(x))$ and assume that $a \in H$. Then there exists a sequence $\{a_n : n \in \omega\}$ in H_i converging to a and such that each a_n's first coordinate is distinct from x. Let $b \in H_i$ be such that for every n, first coordinates of a, b, and a_n are distinct. We have $a_n \sim b$ for every n. Since $\mathcal{G}$ is sequential, it follows that $a \sim b$, hence $a \in H_i$. Similarly, if $b = (x, h_2(x))$ and $b \in H$ then there exists $k \in I$ such that $k = \min\{j \in I : j > i\}$, and $b \in H_k$. It follows that if $h_1(x) = h_2(x) = y$ then $(x, y) \notin H$.

By a theorem of Michael (see [3], Exercise 1.7.15 (d)), there exists a continuous function $h^+ \colon \mathbb{R} \to \mathbb{R}$ such that $h_1 \leq h^+ \leq h_2$ and for every $x \in \mathbb{R}$, if $h_1(x) < h_2(x)$ then $h_1(x) < h^+(x) < h_2(x)$. It follows that $f < h^+ < g$ for any $f \in \mathcal{G}_i$ and $g \in \bigcup_{j>i} \mathcal{G}_j$. A similar argument shows that if $\bigcup_{j<i} \mathcal{G}_j \neq \emptyset$ then there exists $h^- \in C(\mathbb{R}, \mathbb{R})$ such that $f < h^- < g$ for any $f \in \bigcup_{j<i} \mathcal{G}_j$ and $g \in \mathcal{G}_i$. Hence, $\{\mathcal{G}_i : i \in I\}$ is a sliced system. $\square$

Theorem 5.5. *Let $\mathcal{G} \subseteq C(\mathbb{R}, \mathbb{R})$ be nonempty. Then the following conditions are equivalent.*

(1) $\mathcal{K}_{\mathcal{G}} \subseteq \big\{CL(\mathbb{R}) \cap \bigcup_{X \in \mathcal{X}} \mathcal{P}(X) \colon \mathcal{X}$ *is separated*$\big\}$.

(2) *There exists a sliced system $\{\mathcal{G}_i : i \in I\}$ of complete connected families such that $\mathcal{G} = \bigcup_{i \in I} \mathcal{G}_i$.*

Proof. (1) $\Rightarrow$ (2). Denote $H = \bigcup \mathcal{G}$. Let $g \in C(\mathbb{R}, \mathbb{R})$ and $g \subseteq H$. Since $E_{\mathcal{G}}(\{g\}) \in \mathcal{K}_{\mathcal{G}}$, there exists a separated family $\mathcal{X} \subseteq \mathcal{P}(\mathbb{R})$ such that $E_{\mathcal{G}}(\{g\}) = CL(\mathbb{R}) \cap \bigcup_{X \in \mathcal{X}} \mathcal{P}(X)$. For every $x \in \mathbb{R}$ we have $g(x) \in \mathcal{G}[x]$, hence $\{x\} \in$

$E_{\mathcal{G}}(\{g\})$, and thus $\bigcup \mathcal{X} = \mathbb{R}$. For any disjoint open sets $U, V \subseteq \mathbb{R}$, if $U \cup V = \mathbb{R}$ then either $U = \mathbb{R}$ or $V = \mathbb{R}$. It follows that $\mathcal{X} = \{\mathbb{R}\}$, hence $\mathbb{R} \in E_{\mathcal{G}}(\{g\})$ and thus $g \in \mathcal{G}$. Hence, $\mathcal{G} = \{g \in C(\mathbb{R}, \mathbb{R})\colon g \subseteq H\}$, so $\mathcal{G}$ is complete.

Let us show that $\mathcal{G}$ is transitive. Let $a = (a_1, a_2)$, $b = (b_1, b_2)$, $c = (c_1, c_2)$ be such that a_1, b_1, c_1 are distinct, and let $a \sim b \sim c$. Since for any $(x, y) \in H$ there exists $g \in \mathcal{G}$ such that $g(x) = y$, we can find $g, h \in \mathcal{G}$ such that $g(a_1) = a_2$, $g(b_1) = h(b_1) = b_2$, and $h(c_1) = c_2$. Let $f \in C(\mathbb{R}, \mathbb{R})$ be any function such that $f(a_1) = a_2$, $f(b_1) = b_2$, and $f(c_1) = c_2$. Then $\{a_1, b_1\} \in E_{\mathcal{G}}(\{f\})$, $\{b_1, c_1\} \in E_{\mathcal{G}}(\{f\})$, and thus also $\{a_1, c_1\} \in E_{\mathcal{G}}(\{f\})$. It follows that there exists $f' \in \mathcal{G}$ such that $f'(a_1) = a_2$ and $f'(c_1) = (c_2)$, hence $a \sim c$.

To show that $\mathcal{G}$ is also sequential, assume that $a, b \in H$ and there exists a sequence $\{a_n : n \in \omega\}$ in H such that $a_n \to a$, $a_n \sim b$, and first coordinates of points a, b, and a_n, $n \in \omega$, are pairwise distinct. We have to prove that $a \sim b$. There exists a function $f \in C(\mathbb{R}, \mathbb{R})$ such that $f(x) = y$, $f(u) = v$, and $f(x_n) = y_n$ for all n, where $(x, y) = a$, $(u, v) = b$, and $(x_n, y_n) = a_n$. Since $E_{\mathcal{G}}(\{f\}) \in \mathcal{K}_{\mathcal{G}}$, by (1) there exists a separated family $\mathcal{X} \subseteq \mathcal{P}(\mathbb{R})$ such that $E_{\mathcal{G}}(\{f\}) = CL(\mathbb{R}) \cap \bigcup_{X \in \mathcal{X}} \mathcal{P}(X)$. We have $\{x\} \in E_{\mathcal{G}}(\{f\})$ and $\{x_n, u\} \in E_{\mathcal{G}}(\{f\})$, for every n.

It suffices to show that $\{x, u\} \in E_{\mathcal{G}}(\{f\})$. If this is not the case then there exist distinct sets $X, Y \in \mathcal{X}$ such that $x \in X$ and $u \in Y$. Since $\mathcal{X}$ is separated, there exist disjoint open sets U, V such that $X \subseteq U$, $Y \subseteq V$, and $(\forall Z \in \mathcal{X})(Z \subseteq U \text{ or } Z \subseteq V)$. Since $x_n \to x$, there exists n such that $x_n \in U$. But then $x_n \notin Y$, and this is in contradiction with $\{x_n, u\} \in E_{\mathcal{G}}(\{f\})$.

We have proved that $\mathcal{G}$ is complete, transitive and sequential. Then condition (2) follows from Lemma 5.4.

(2) $\Rightarrow$ (1). Let $\{\mathcal{G}_i \colon i \in I\}$ be a sliced system of complete connected families such that $\mathcal{G} = \bigcup_{i \in I} \mathcal{G}_i$. By Lemma 5.3, for every i there exist $h_i^-, h_i^+ \in C(\mathbb{R}, \mathbb{R}^*)$ such that $\mathcal{G}_i \subseteq (h_i^-, h_i^+)$ and $h_i^+ \leq h_j^-$ whenever $i < j$.

Let $f \in C(\mathbb{R}, \mathbb{R})$ be arbitrary. For every i, denote $X_i = \{x \in \mathbb{R} : f(x) \in \mathcal{G}[x] \text{ and } h_i^-(x) < f(x) < h_i^+(x)\}$, and let $\mathcal{X} = \{X_i : i \in I\}$. For every $i \in I$, let us take $U_i = \{x \in \mathbb{R} \colon h_i^-(x) < f(x) < h_i^+(x)\}$ and $V_i = \{x \in \mathbb{R} \colon f(x) < h_i^-(x) \text{ or } h_i^+(x) < f(x)\}$. Then U_i, V_i are disjoint open sets such that $X_i \subseteq U_i$ and $X_j \subseteq V_i$ for every $j \neq i$. It follows that $\mathcal{X}$ is a separated family.

We will prove that $E_{\mathcal{G}}(\{f\}) = CL(\mathbb{R}) \cap \bigcup_{X \in \mathcal{X}} \mathcal{P}(X)$. If $E \in E_{\mathcal{G}}(\{f\})$ then there exists $g \in \mathcal{G}$ such that $f \restriction E = g \restriction E$. We have $g \in \mathcal{G}_i$ for some $i \in I$, and it easy to see that $E \subseteq X_i$. We can conclude that $E_{\mathcal{G}}(\{f\}) \subseteq CL(\mathbb{R}) \cap \bigcup_{X \in X} \mathcal{P}(X)$.

To prove the opposite inclusion, assume that $E \notin E_{\mathcal{G}}(\{f\})$, hence $f \restriction E \neq g \restriction E$ for every $g \in \mathcal{G}$. If there exists $x \in E$ such that $f(x) \notin \mathcal{G}[x]$ then $x \notin \bigcup \mathcal{X}$, and hence $E \notin CL(\mathbb{R}) \cap \bigcup_{X \in \mathcal{X}} \mathcal{P}(X)$. Assume further that $E \subseteq \{x \in \mathbb{R} \colon f(x) \in \mathcal{G}[x]\}$. If there exists $i \in I$ such that $f \restriction E \subseteq \bigcup \mathcal{G}_i$, then by Lemma 4.4 there exists $g \in \mathcal{G}_i$ such that $f \restriction E = g \restriction E$, which is impossible. Hence, there exist $i \neq j$ and $x, y \in E$ such that $(x, f(x)) \in \bigcup \mathcal{G}_i$ and $(y, f(y)) \in \bigcup \mathcal{G}_j$. It follows that $x \in X_i$ and $y \in X_j$, hence $E \notin CL(\mathbb{R}) \cap \bigcup_{X \in \mathcal{X}} \mathcal{P}(X)$.

We have proved that for every $f \in C(\mathbb{R}, \mathbb{R})$ there exists a separated family $\mathcal{X}_f \subseteq \mathcal{P}(\mathbb{R})$ such that $E_{\mathcal{G}}(\{f\}) = CL(\mathbb{R}) \cap \bigcup_{X \in \mathcal{X}_f} \mathcal{P}(X)$. For arbitrary $\mathcal{F} \subseteq$

$C(\mathbb{R},\mathbb{R})$, we have $E_{\mathcal{G}}(\mathcal{F}) = \bigcap_{f\in\mathcal{F}} E_{\mathcal{G}}(\{f\})$. Let us take

$$\mathcal{X} = \left\{ \bigcap_{f\in\mathcal{F}} X_f \colon \langle X_f \colon f\in\mathcal{F}\rangle \in \prod_{f\in\mathcal{F}} \mathcal{X}_f \right\}.$$

We show that $\mathcal{X}$ is separated. Let $\langle X_f \colon f\in\mathcal{F}\rangle, \langle Y_f \colon f\in\mathcal{F}\rangle \in \prod_{f\in\mathcal{F}} X_f$ be such that $\bigcap_{f\in\mathcal{F}} X_f \neq \bigcap_{f\in\mathcal{F}} Y_f$. Then there exists $h\in\mathcal{F}$ such that $X_h \neq Y_h$. Since $\mathcal{X}_h$ is separated, there exist disjoint open sets $U, V \subseteq \mathbb{R}$ such that $X_h \subseteq U$, $Y_h \subseteq V$, and $(\forall Z\in\mathcal{X}_h)(Z\subseteq U$ or $Z\subseteq V)$. It follows that $\bigcap_{f\in\mathcal{F}} X_f \subseteq U$ and $\bigcap_{f\in\mathcal{F}} Y_f \subseteq V$. For every $\langle Z_f \colon f\in\mathcal{F}\rangle \in \prod_{f\in\mathcal{F}} \mathcal{X}_f$ we have $Z_h\subseteq U$ or $Z_h \subseteq V$, hence also $\bigcap_{f\in\mathcal{F}} Z_f \subseteq U$ or $\bigcap_{f\in\mathcal{F}} Z_f \subseteq V$.

For $E\in CL(\mathbb{R})$, we have $E\in E_{\mathcal{G}}(\mathcal{F})$ if and only if $(\forall f\in\mathcal{F})(\exists X\in\mathcal{X}_f)\, E\subseteq X$ if and only if $(\exists X\in\mathcal{X})\, E\subseteq X$. Hence, $E_{\mathcal{G}}(\mathcal{F}) = CL(\mathbb{R}) \cap \bigcup_{X\in\mathcal{X}} \mathcal{P}(X)$ and condition (1) follows. $\square$

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