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Factorization of Observables

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Abstract Categorical approach to probability leads to better understanding of basic notions and constructions in generalized (fuzzy, operational, quantum) probability, where observables—dual notions to generalized random variables (statistical maps)—play a major role. First, to avoid inconsistencies, we introduce three categories $\mathbb{L}$, $\mathbb{S}$, and $\mathbb{P}$, the objects and morphisms of which correspond to basic notions of fuzzy probability theory and operational probability theory, and describe their relationships. To illustrate the advantages of categorical approach, we show that two categorical constructions involving observables (related to the representation of generalized random variables via products, or smearing of sharp observables, respectively) can be described as factorizing a morphism into composition of two morphisms having desired properties. We close with a remark concerning products.

Keywords Generalized probability · Categorical approach · Observable · Statistical map · Smearing · Effect algebra

1 Introduction

A fuzzification of the classical (Boolean) random events initiated by L. A. Zadeh [28], i.e., the transition from σ-fields of sets to measurable fuzzy sets, underwent a considerable evolution [2, 21, 24, 25]. In a broader context, fuzzy random events can be viewed as a particular case of effect algebras, D-posets, and other quantum structures [7]. Categorical methods help to understand the transition from classical probability theory to fuzzy

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probability theory, lead to a simplification and offer a more consistent terminology and notation [10–19].

In [20], R. J. Greechie and D. J. Foulis have proposed to study "lifting problems for morphisms and measures". Roughly, if A, B, C are mathematical structures, $\phi\colon A \longrightarrow B$ and $\iota\colon A \longrightarrow C$ are mappings, the problem is to find a mapping $\phi^*\colon C \longrightarrow B$ that makes the underlying diagram commutative in the sense that $\phi = \phi^* \circ \iota$. If ϕ^* can be found, it is said to be obtained by **lifting ϕ through ι to ϕ^***, see Fig. 1a.

Along these lines, we are interested in situations where (in a given category) we have a morphism $h\colon A \longrightarrow B$ and we are looking for two morphisms $g\colon A \longrightarrow C$ and $f : C \longrightarrow B$ such that h factorizes into the composition $h = f \circ g$, see Fig. 1b. Additionally (see Construction I in Section 2), we may require, for example, that g is injective and f is conservative (standard).

The classical probability theory starts with a probability space $(\Omega, \mathbf{A}, p)$, where Ω is the set (universe) of outcomes (elementary events) of a random experiment, $\mathbf{A}$ is a σ-field of random events (subsets of Ω), p is a probability measure on $\mathbf{A}$, and $p(A)$, $A \in \mathbf{A}$, measures how big (relatively to the sure event) A is. In a minimal categorical upgrading of the classical probability theory [18, 19], each probability space $(\Omega, \mathbf{A}, p)$ is replaced by its fuzzification $\left(\Omega, \mathcal{M}(\mathbf{A}), \int(\cdot)\mathrm{d}p\right)$, where $\mathcal{M}(\mathbf{A})$ is the **full Łukasiewicz tribe** of all measurable functions into [0,1] (fuzzy random events) and $\int(\cdot)\mathrm{d}p$ is the probability integral with respect to p. A fundamental role is played by observables, for example, probability measures and probability integrals are observables into a special object $\mathcal{M}(\mathbf{T})$, where $\mathbf{T}$ is the trivial σ-field $\{\emptyset, \Omega\}$, $card(\Omega) = 1$ (cf. [10]). Clearly, $\mathcal{M}(\mathbf{T})$ and [0,1] can be identified.

Recall that $\mathcal{M}(\mathbf{A})$ is a subalgebra of $\mathcal{M}(\mathbf{T})^{\Omega}$, equipped with the usual partial (pointwise) order, Łukasiewicz operations defined pointwise: $(u \oplus v)(\omega) = u(\omega) \oplus v(\omega) = \min\{1, u(\omega) + v(\omega)\}$, $(u \odot v)(\omega) = u(\omega) \odot v(\omega) = \max\{0, u(\omega) + v(\omega) - 1\}$, $u^*(\omega) = 1 - u(\omega)$, $\omega \in \Omega$, and pointwise sequential convergence. In what follows, we identify $A \in \mathbf{A}$ and its indicator function $\chi_A \in \mathcal{M}(\mathbf{A})$.

In fuzzy probability theory, as developed by S. Gudder in [21, 22], $\mathcal{M}(\mathbf{A})$ is studied as an effect algebra, denoted by $\mathcal{E}(\Omega, \mathbf{A})$, and effect σ-homomorphisms play a key role. The notation and terminology in [22] is not strictly consistent (albeit standard). For example, $M_1^+(\Omega, \mathbf{A})$ denotes the set of all probability measures on $\mathbf{A}$, but the same symbol also denotes **states**, i.e., the probability integrals $\int(\cdot)\mathrm{d}p$ on $\mathcal{E}(\Omega, \mathbf{A})$. Similarly, in theorems and their proofs a probability measure p is treated as $\int(\cdot)\mathrm{d}p$ (the notion of a probability measure on p. 879, resp. the notion of a distribution, p. 880). Likewise, an observable (effect σ-homomorphism) is a map on $\mathbf{B}$ into $\mathcal{E}(\Omega, \mathbf{A})$, but often it is understood as its unique sequentially continuous extension over $\mathcal{E}(\Xi, \mathbf{B})$. Further, a statistical map is defined as a

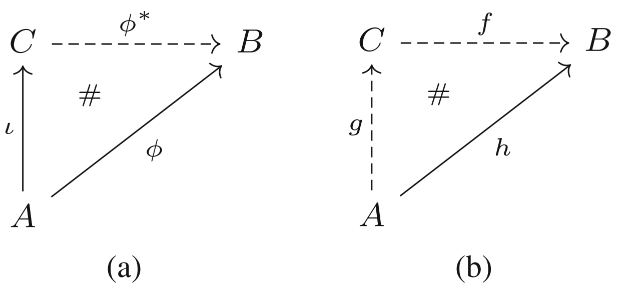

Fig. 1 Lifting (**a**), factorization (**b**)

special map on $M_1^+(\Omega, \mathbf{A})$ into $M_1^+(\Xi, \mathbf{B})$, but often treated as a map on states on $\mathcal{E}(\Omega, \mathbf{A})$ into states on $\mathcal{E}(\Xi, \mathbf{B})$.

In operational probability theory, as developed by S. Bugajski in [2, 3], the fundamental role is played by statistical maps (generalized random variables) mapping $M_1^+(\Omega, \mathbf{A})$ into $M_1^+(\Xi, \mathbf{B})$.

It is known that D-posets and effect algebras are isomorphic structures [4, 7, 8, 26, 27]. It is also known that effect σ-homomorphisms can be viewed as **sequentially continuous** ***D*-homomorphisms**, i.e., maps preserving partial order, the top and the bottom elements (constant functions 0_Ω and 1_Ω in case of $\mathcal{M}(\mathbf{A})$), and the partial operation of difference ($u \ominus v$ is defined iff $v \leq u$). Note that the sequential continuity of probability measures and integrals follows from the Lebesgue dominated convergence theorem. An interested reader can find details about effect algebras, D-posets, and their relationship in [4, 7, 26]. To avoid inconsistencies, in the next section we introduce three categories $\mathbb{L}$, $\mathbb{S}$, and $\mathbb{P}$, the objects and morphisms of which correspond to basic notions of fuzzy probability theory and operational probability theory. We describe their relationships. In the present paper we utilize only very basic notions of the category theory (objects are structured sets and morphisms are structure preserving maps). Additional information can be found, for example, in [1, 5].

2 Three Categories

Definition 1 Let $(\Omega, \mathbf{A})$ and $(\Xi, \mathbf{B})$ be measurable spaces. A sequentially continuous D-homomorphism $h \colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ is said to be an **observable**. Moreover, if $h(B) \in \mathbf{A}$ for all $B \in \mathbf{B}$, then h is said to be **conservative**.

Denote $\mathbb{L}$ the category of full Łukasiewicz tribes as objects (i.e., objects are of the form $\mathcal{M}(\mathbf{A})$ for some σ-field of sets $\mathbf{A}$) and observables as morphisms.

As stated in the introduction, observables into $\mathcal{M}(\mathbf{T}) = [0, 1]$ are exactly probability integrals; they are called **states**. Denote $\mathcal{S}(\mathcal{M}(\mathbf{A}))$ the set of all states on $\mathcal{M}(\mathbf{A})$.

Let $h \colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ be an observable. The compositions $s \circ h$, where s is a state on $\mathcal{M}(\mathbf{A})$, define a map $\overline{T_h}$ on the set $\mathcal{S}(\mathcal{M}(\mathbf{A}))$ into the set $\mathcal{S}(\mathcal{M}(\mathbf{B}))$, i.e., $\overline{T_h}(s) = s \circ h$. The "overlined" symbol $\overline{T_h}$ is used to distinguish such maps from statistical maps.

Definition 2 Let $h \colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ be an observable. Then $\overline{T_h} \colon \mathcal{S}(\mathcal{M}(\mathbf{A})) \longrightarrow \mathcal{S}(\mathcal{M}(\mathbf{B}))$ is said to be a **state map**.

Denote $\mathbb{S}$ the category of states on full Łukasiewicz tribes (i.e., sets of the form $\mathcal{S}(\mathcal{M}(\mathbf{A}))$) as objects and state maps as morphisms.

There are equivalent ways how to define a statistical map (cf. [2]), e.g., via a Markov kernel.

Recall that a map K on $\Omega \times \mathbf{B}$ into [0,1] is said to be a **Markov kernel** (also probability kernel) if, for each $B \in \mathbf{B}$, $K(\cdot, B)$ is $\mathbf{A}$-measurable and, for each $\omega \in \Omega$, $K(\omega, \cdot)$ is a probability measure on $\mathbf{B}$. Observe that K yields a map $\overline{K} \colon \mathbf{B} \longrightarrow \mathcal{M}(\mathbf{A})$, $\overline{K}(B) = K(\cdot, B)$. Since each $K(\omega, \cdot)$ is a probability measure on $\mathbf{B}$, $\overline{K}$ is a sequentially continuous D-homomorphism on $\mathbf{B}$ into $\mathcal{M}(\mathbf{A})$. Conversely, to each sequentially continuous D-homomorphism k on $\mathbf{B}$ into $\mathcal{M}(\mathbf{A})$ there corresponds a unique Markov kernel K such that $\overline{K} = k$. In our notation, if $h \colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ is an observable, then the restriction $h|\mathbf{B}$ of h to $\mathbf{B}$ is a sequentially continuous D-homomorphism and, for each $\omega \in \Omega$, $((h|\mathbf{B})(\cdot))(\omega)$ is a probability measure on $\mathbf{B}$ (as a composition of $h|\mathbf{B}$ and the probability point-measure

δ_ω), hence a Markov kernel. Since h (as a sequentially continuous D-homomorphism) is the unique extension of $h|\mathbf{B}$ and since each Markov kernel corresponds to a unique sequentially continuous D-homomorphism on $\mathbf{B}$ into $\mathcal{M}(\mathbf{A})$ there is a one-to-one correspondence between observables and Markov kernels (cf. [22]). Even though two Markov kernels cannot be composed, hence cannot be explicitly used in categorical constructions, they can be advantageously used in some calculations.

In [9] the following definition has been given.

Definition 3 Let $(\Omega, \mathbf{A})$, $(\Xi, \mathbf{B})$ be measurable spaces. Let T be a map of the set $M_1^+(\Omega, \mathbf{A})$ of all probability measures on $\mathbf{A}$ into the set $M_1^+(\Xi, \mathbf{B})$ of all probability measures on $\mathbf{B}$ such that, for each $B \in \mathbf{B}$, the assignment $\omega \mapsto (T(\delta_\omega))(B)$ yields a measurable map of Ω into $[0, 1]$ and

$$(T(p))(B) = \int_\Omega (T(\delta_\omega))(B)\mathrm{d}p \tag{BG}$$

for all $p \in M_1^+(\Omega, \mathbf{A})$ and all $B \in \mathbf{B}$. Then T is said to be a **statistical map**.

Denote $\mathbb{P}$ the category having sets of the form $M_1^+(\Omega, \mathbf{A})$ as objects and statistical maps as morphisms.

Remark 1 Some authors define observable as an effect σ-homomorphism x on the σ-field $\mathbf{B}$ into a suitable effect algebra subjected to some additional assumptions ([6, 23]). As pointed out in [22], each effect σ-homomorphism $x\colon \mathbf{B} \longrightarrow \mathcal{M}(\mathbf{A})$ can be uniquely extended to an effect σ-homomorphism $\tilde{x}\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$. This way $\tilde{x}$ becomes a morphism (the domain and the range of $\tilde{x}$ are objects of the same category) and related constructions become categorical.

Remark 2 There is a one-to-one correspondence between Markov kernels and statistical maps (cf. Theorem 2.3 in [2]). As pointed out in [22], the assumption that a statistical map is affine (Definition 2.1 in [2]) is redundant.

Remark 3 Condition (BG) yields a map $\overline{T}$ on $\mathcal{S}(\mathcal{M}(\mathbf{A}))$ into $\mathcal{S}(\mathcal{M}(\mathbf{B}))$. Indeed, since $\int \chi_B \mathrm{d}(T(p)) = (T(p))(B)$, $B \in \mathbf{B}$, $p \in M_1^+(\Omega, \mathbf{A})$, the system $\left\{(T(p))(B) = \int (T(\delta_\omega))(B)\mathrm{d}p;\ B \in \mathbf{B}\right\}$ uniquely determines the integral $\int(\cdot)\mathrm{d}(T(p))$ on $\mathcal{M}(\mathbf{B})$: the $\overline{T}$-image of the integral $\int(\cdot)\mathrm{d}p$ on $\mathcal{M}(\mathbf{A})$. It follows from the relationships between observables, Markov kernels and statistical maps that $\overline{T}$ is a state map. Observe that the same system also defines the statistical map T.

In category theory, an equivalence of categories is a relation between two categories that establishes that these categories are "essentially the same". If a category is equivalent to the opposite (or dual) of another category then one speaks of a duality of categories, and says that the two categories are **dually equivalent**.

An equivalence of categories consists of a functor between the involved categories, which is required to have an "inverse" functor. However, in contrast to the situation common for isomorphisms in an algebraic setting, the composition of the functor and its "inverse" is not necessarily the identity mapping. Instead it is sufficient that each object be naturally isomorphic to its image under this composition. Thus one may describe the functors as being "inverse up to isomorphism". The concept of isomorphism of categories, where a strict form of inverse functor is required, is of much less practical use than the equivalence concept.

In [19] the following theorem has been proved.

Theorem 1 *The categories* $\mathbb{L}$ *and* $\mathbb{S}$ *are dually equivalent.*

The relationship between $\mathbb{P}$ and $\mathbb{S}$ is different.

Theorem 2 *The categories* $\mathbb{P}$ *and* $\mathbb{S}$ *are isomorphic.*

Proof We have to construct two covariant functors $F\colon \mathbb{P} \longrightarrow \mathbb{S}$ and $G\colon \mathbb{S} \longrightarrow \mathbb{P}$ such that $G \circ F = Id_{\mathbb{P}}$ and $F \circ G = Id_{\mathbb{S}}$.

For each measurable space $(\Omega, \mathbf{A})$ let us define $F\left(M_1^+(\Omega, \mathbf{A})\right) = \mathcal{S}(\mathcal{M}(\mathbf{A}))$ and $G(\mathcal{S}(\mathcal{M}(\mathbf{A}))) = M_1^+(\Omega, \mathbf{A})$. Clearly, $(G \circ F)\left((M_1^+(\Omega, \mathbf{A})\right) = M_1^+(\Omega, \mathbf{A})$ and $(F \circ G)(\mathcal{S}(\mathcal{M}(\mathbf{A}))) = \mathcal{S}(\mathcal{M}(\mathbf{A}))$.

Let $(\Omega, \mathbf{A})$, $(\Xi, \mathbf{B})$ be measurable spaces. Let $T\colon M_1^+(\Omega, \mathbf{A}) \longrightarrow M_1^+(\Xi, \mathbf{B})$ be a statistical map and let $\overline{T}\colon \mathcal{S}(\mathcal{M}(\mathbf{A})) \longrightarrow \mathcal{S}(\mathcal{M}(\mathbf{B}))$ be the corresponding state map. Define $F(T) = \overline{T}$ and $G(\overline{T}) = T$. Clearly, $G(F(T)) = T$ and $F(G(\overline{T}) = \overline{T}$.

We have to prove that F and G are covariant functors, namely, that they preserve identity maps and composition of morphisms.

For each measurable space $(\Omega, \mathbf{A})$ let us define $int_{\mathbf{A}}\colon M_1^+(\Omega, \mathbf{A}) \longrightarrow \mathcal{S}(\mathcal{M}(\mathbf{A}))$ so that $int_{\mathbf{A}}(p) = \int(\cdot)\mathrm{d}p$, $p \in M_1^+(\Omega, \mathbf{A})$. Dually, define $meas_{\mathbf{A}}\colon \mathcal{S}(\mathcal{M}(\mathbf{A})) \longrightarrow M_1^+(\Omega, \mathbf{A})$ so that $meas_{\mathbf{A}}\left(\int(\cdot)\mathrm{d}p\right) = p$, $p \in M_1^+(\Omega, \mathbf{A})$, see Fig. 2a.

Then (by Remark 3) the following hold: for each $p \in M_1^+(\Omega, \mathbf{A})$ we have $\int(\cdot)\mathrm{d}(T(p)) = \overline{T}(int_{\mathbf{A}}(p)) = int_{\mathbf{B}}(T(p))$, and for each $\int(\cdot)\mathrm{d}p \in \mathcal{S}(\mathcal{M}(\mathbf{A}))$ we have $T(p) = T\left(meas_{\mathbf{A}}\left(\int(\cdot)\mathrm{d}p\right)\right) = meas_{\mathbf{B}}\left(\overline{T}\left(\int(\cdot)\mathrm{d}p\right)\right)$, see Fig. 2b, where $a \longmapsto b$ means that the map in question maps a to b. Consequently, the diagram in Fig. 2b is commutative.

It follows directly from the two schemes that F and G preserve identity maps. Finally, let $S\colon M_1^+(\Xi, \mathbf{B}) \longrightarrow M_1^+(\Lambda, \mathbf{C})$ be some other statistical map and let $\overline{S}\colon \mathcal{S}(\mathcal{M}(\mathbf{B})) \longrightarrow \mathcal{S}(\mathcal{M}(\mathbf{C}))$ be the corresponding state map. Then, repeating each scheme twice, it follows that F and G preserve compositions of morphisms. □

Let us remark that the results in [22] are stated and proved using (implicitly) the fact that $\mathbb{P}$ and $\mathbb{S}$ are isomorphic.

3 Construction I

Let $T\colon M_1^+(\Omega, \mathbf{A}) \longrightarrow M_1^+(\Xi, \mathbf{B})$ be a statistical map. If T maps each point-probability measure δ_ω, $\omega \in \Omega$, to a **degenerated** (i.e., $\{0, 1\}$-valued) probability measure, then T is

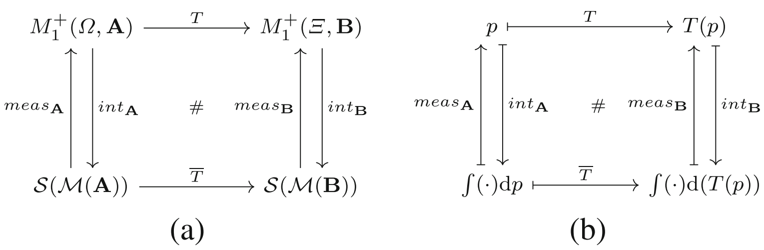

Fig. 2 The categories $\mathbb{P}$ and $\mathbb{S}$ are isomorphic

said to be **standard** (classical, conservative). In [3, p. 353], S. Bugajski has proved a kind of "hidden variables" theorem (representation of nonstandard maps by a standard ones). It is based on the construction of product of statistical maps. It is stated that "a similar result has been obtained recently by S. Gudder".

Let $T_i\colon M_1^+(\Omega, \mathbf{A}) \longrightarrow M_1^+(\Xi_i, \mathbf{B}_i)$, $i = 1, 2$, be statistical maps and let $(\Xi_1 \times \Xi_2, \mathbf{B}_1 \times \mathbf{B}_2)$ be the usual product of the involved target measurable spaces. Then

$$((T_1 \otimes T_2)(p))\,(B_1 \times B_2) = \int_\Omega (T_1(\delta_\omega))\,(B_1) \cdot (T_2(\delta_\omega))\,(B_2)\,\mathrm{d}p,$$

$B_1 \in \mathbf{B}_1$, $B_2 \in \mathbf{B}_2$, $p \in M_1^+(\Omega, \mathbf{A})$, defines a statistical map $T_1 \otimes T_2\colon M_1^+(\Omega, \mathbf{A}) \longrightarrow M_1^+(\Xi_1 \times \Xi_2, \mathbf{B}_1 \times \mathbf{B}_2)$; it is called the **product** of T_1 and T_2 [2, 3, 22].

Theorem 3 *Let $T\colon M_1^+(\Omega, \mathbf{A}) \longrightarrow M_1^+(\Xi, \mathbf{B})$ be a statistical map. Then there is an injective statistical map $U\colon M_1^+(\Omega, \mathbf{A}) \longrightarrow M_1^+(\Omega \times \Xi, \mathbf{A} \times \mathbf{B})$ and a standard statistical map $S\colon M_1^+(\Omega \times \Xi, \mathbf{A} \times \mathbf{B}) \longrightarrow M_1^+(\Xi, \mathbf{B})$ such that $T = S \circ U$.*

S. Bugajski has proposed the following interpretation: "The standard map S returns all probability distributions produced by the original map T on its outcome, so it could be considered as a faithful representative of T."

As it follows from Fig. 3, the theorem of S. Bugajski can be viewed in a broader context: U is the product $I \otimes T$ of the identity I and T, L_1 and L_2 are **lateral** (marginal) projections, i.e., for $q \in M_1^+(\Omega \times \Xi, \mathbf{A} \times \mathbf{B})$, $A \in \mathbf{A}$, and $B \in \mathbf{B}$, $(L_1(q))\,(A) = q(A \times \Xi)$, $(L_2(q))\,(B) = q(\Omega \times B)$, and $S = L_2$. So, the statistical map T is **factorized** into $S \circ U$ "via the **product space** $(\Omega \times \Xi, \mathbf{A} \times \mathbf{B})$" with U injective and S standard; alternatively, T is **lifted via the product of domain and range**.

It is shown in the last section that the notion of products of statistical maps "is not categorical". The original proof of Theorem 3 is based on nontrivial measure-theoretic calculations.

We prove an analogous factorization theorem for observables. However, the construction is now "categorical": each observable is factorized/lifted via the categorical product of domain and range.

Let $\mathcal{M}(\mathbf{A}) \subseteq [0,1]^\Omega$ and $\mathcal{M}(\mathbf{B}) \subseteq [0,1]^\Xi$ be objects of $\mathbb{L}$. Along with the natural projections (observables) pr_1 and pr_2, the product $\mathcal{M}(\mathbf{A}) \times \mathcal{M}(\mathbf{B})$ consists of all pairs (u, v), $u \in \mathcal{M}(\mathbf{A})$, $v \in \mathcal{M}(\mathbf{B})$, where the structure and convergence is defined coordinatewise; observe that if Σ is the disjoint union of Ω and Ξ (their coproduct in the category of sets and maps) then each (u, v) can be visualized as a function w on Σ, where u and v are "glued in a disjoint way" to form w $(pr_1(u, v) = u,\ pr_2(u, v) = v)$.

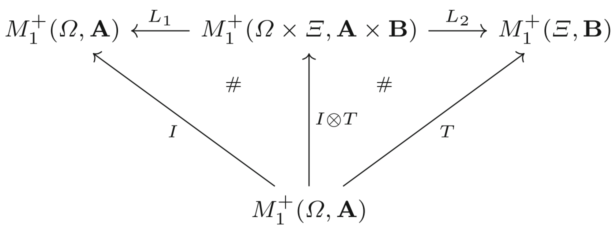

Fig. 3 Product of I and T

Theorem 4 *Let* $h\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ *be an observable. Then there is an injective observable* $g\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{B})\times\mathcal{M}(\mathbf{A})$ *and a conservative observable* $f\colon \mathcal{M}(\mathbf{B})\times\mathcal{M}(\mathbf{A}) \longrightarrow \mathcal{M}(\mathbf{A})$ *such that* $f \circ g = h$.

Proof Consider the commutative diagram shown in Fig. 4. Put $g = id \times h$ and $f = pr_2$, where $id \times h$ is the unique categorical arrow defined by $(id \times h)(u) = (u, h(u)), u \in \mathcal{M}(\mathbf{B})$. Clearly, g is injective and f is conservative. □

Since the categories $\mathbb{P}$ and $\mathbb{S}$ are isomorphic and the categories $\mathbb{S}$ and $\mathbb{L}$ are dual, we can form dual commutative diagrams and compare their properties.

Let $h\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ be an observable, let $T\colon M_1^+(\Omega, \mathbf{A}) \longrightarrow M_1^+(\Xi, \mathbf{B})$ be the dual statistical map (i.e., $T = T_h$), let $k\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{B} \times \mathbf{A})$ be the observable dual to the statistical map $L_2\colon M_1^+(\Xi \times \Omega, \mathbf{B} \times \mathbf{A}) \longrightarrow M_1^+(\Xi, \mathbf{B})$ (i.e., $T_k = L_2$), and let $l\colon \mathcal{M}(\mathbf{B} \times \mathbf{A}) \longrightarrow \mathcal{M}(\mathbf{A})$ be the observable dual to the statistical map $I \otimes T\colon M_1^+(\Xi \times \Omega, \mathbf{B} \times \mathbf{A}) \longrightarrow M_1^+(\Xi, \mathbf{B})$ (i.e., $T_l = I \otimes T$). Then the resulting dual diagram (see Fig. 5a) commutes, the observable k is injective, but the observable l fails to be conservative whenever T fails to be standard.

Similarly, let $T_h\colon M_1^+(\Omega, \mathbf{A}) \longrightarrow M_1^+(\Xi, \mathbf{B})$ be the statistical map dual to the observable $h\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$, let $T_{pr_2}\colon M_1^+(\Omega, \mathbf{A}) \longrightarrow M_1^+(\Sigma, \mathcal{M}(\mathbf{B}) \times \mathcal{M}(\mathbf{A}))$ be the statistical map dual to $pr_2\colon \mathcal{M}(\mathbf{B}) \times \mathcal{M}(\mathbf{A}) \longrightarrow \mathcal{M}(\mathbf{A})$, and let $T_{id\times h}\colon M_1^+(\Sigma, \mathcal{M}(\mathbf{B}) \times \mathcal{M}(\mathbf{A})) \longrightarrow M_1^+(\Xi, \mathbf{B})$ be the statistical map dual to $id \times h\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{B}) \times \mathcal{M}(\mathbf{A})$. Then the resulting diagram (see Fig. 5) commutes, the statistical map T_{pr_2} is injective, but the statistical map $T_{id\times h}$ fails to be standard whenever h fails to be conservative.

Finally, observe that the factorizations established in Theorem 2.1 and Theorem 2.2 are not unique. Indeed, the statistical map $U = I \otimes I \otimes T\colon M_1^+(\Omega, \mathbf{A}) \longrightarrow M_1^+(\Omega\times\Omega\times\Xi, \mathbf{A}\times\mathbf{A}\times\mathbf{B})$ is injective, the statistical map $S = L_3\colon M_1^+(\Omega\times\Omega\times\Xi, \mathbf{A}\times\mathbf{A}\times\mathbf{B}) \longrightarrow M_1^+(\Xi, \mathbf{B})$ is standard, and $T = S \circ U$. Analogously, $id \times id \times h\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{B})\times\mathcal{M}(\mathbf{B})\times\mathcal{M}(\mathbf{A})$ is injective, $pr_3\colon \mathcal{M}(\mathbf{B})\times\mathcal{M}(\mathbf{B})\times\mathcal{M}(\mathbf{A}) \longrightarrow \mathcal{M}(\mathbf{A})$ is conservative, and $h = (id \times id \times h) \circ pr_3$.

4 Construction II

In this section we show that if the ranges of observables are full Łukasiewicz tribes, i.e., objects of the category $\mathbb{L}$, then the smearing, as constructed in [6, 23], reduces to a rather trivial commuting diagram. This leads to the following question. How "big and useful" is

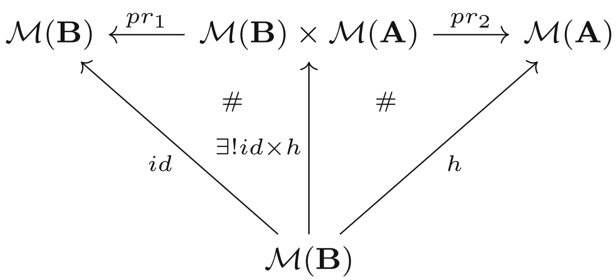

Fig. 4 Factorization of an observable

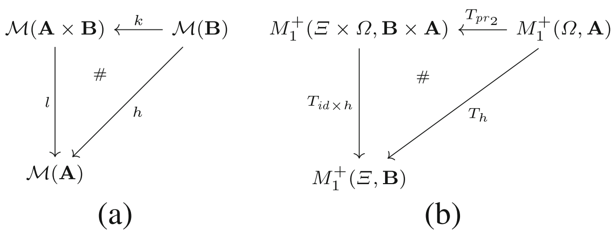

Fig. 5 Dual commutative diagrams

the class of all effect algebras, satisfying additional conditions related to the generalized Loomis-Sikorski theorem used in constructions of smearing, and not belonging to $\mathbb{L}$?

First, we cite the original definitions and theorems from [6] and [23] and then we present a modified construction of smearing (restricted to $\mathbb{L}$).

In [6], an observable on a quantum structure is any σ-homomorphism of quantum structures from the Borel σ-algebra of the real line into a monotone σ-complete effect algebra with the RDP (Riesz decomposition property).

In [23], an observable is defined as follows. Let $(\Omega, \mathbf{A})$ be a measurable space. An $(\Omega, \mathbf{A})$-observable on a σ-orthocomplete MV-effect algebra E is a mapping $\xi : \mathbf{A} \longrightarrow E$ such that

(i) $\xi(\Omega) = 1$;
(ii) $\xi\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} \xi(A_i)$, whenever $(A_i)_{i=1}^{\infty}$ is a sequence of mutually disjoint elements of $\mathbf{A}$.

An observable is sharp if its range consists of sharp elements.

In [6], the following smearing theorem has been proved.

Theorem 4.1, [6] Let M be a monotone σ-complete effect algebra with RDP having at least one σ-additive state, and let $(\Omega, \mathcal{T}, h)$ be the canonical representation of M such that every $f \in \mathcal{T}$ is $\mathcal{B}_0(\mathcal{T})$-measurable. There is a sharp observable ξ from $\mathcal{B}_0(\mathcal{T})$ into M such that given an observable x on M, $m \in \mathcal{S}_\sigma(M)$, and $E \in \mathcal{B}(\mathbb{R})$,

$$m\,(x(E)) = \int_\Omega f_E(\omega)\,\mathrm{d}m \circ \xi(\omega), \qquad (*)$$

where f_E is an arbitrary function from $\mathcal{T}$ such that $h(f_E) = x(E)$.

Without going into details, $\mathcal{S}_\sigma(M)$ denotes the set of all σ-additive states on M and the assumption that M is a monotone σ-complete effect algebra with RDP having at least one σ-additive state guarantees the existence of canonical representation $(\Omega, \mathcal{T}, h)$ of M (generalized Loomis-Sikorski theorem) such that every $f \in \mathcal{T}$ is $\mathcal{B}_0(\mathcal{T})$-measurable. $\mathcal{T}$ is a suitable system of measurable functions into [0,1]. Condition $(*)$ defines the smearing of sharp observable ξ to get x. The smearing is a map on a suitable set of states on $\mathcal{T}$ into states on $\mathcal{B}(\mathbb{R})$, the σ-field of Borel sets on the real line.

The smearing theorem in [23] has the following form.

Theorem 3.4, [23] Every observable on a σ-lattice effect algebra is defined by a smearing of a sharp observable.

Again, without going into details, the generalized Loomis-Sikorski theorem is needed to construct the sharp observable in question. Observe that in the first smearing theorem the domain of observable is the σ-field of real Borel sets, while in the second smearing theorem the domain is an arbitrary σ-field of sets.

In case if observables are morphisms of the category $\mathbb{L}$, i.e., sequentially continuous D-homomorphisms the domain and range of which are full Łukasiewicz tribes, then the two smearing theorems reduce to a trivial commutative diagram in Fig. 6, where the identity map $id\colon \mathcal{M}(\mathbf{A}) \longrightarrow \mathcal{M}(\mathbf{A})$ is the desired strict (conservative) observable; each observable $h\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ can be factorized as $h = id \circ h$, and the smearing is simply the corresponding statistical map T_h.

This explains the question asked at the beginning of the present section.

5 Remarks on Products

In this section we point out that the notion of products of statistical maps in [2, 3, 22] is not categorical.

First, we recall the notion of a degenerated statistical map. A classical degenerated random variable is defined as follows. Fix $r \in R$ and define $f\colon \Omega \longrightarrow R$ by putting $f(\omega) = r$ for all $\omega \in \Omega$. Then $D_f(p) = \delta_r$ for all $p \in M_1^+(\Omega, \mathbf{A})$, where the distribution map D_f is defined by $D_f(p) = p \circ f^{\leftarrow}$, and the preimage map $f^{\leftarrow}$ sends $B \in \mathbf{B}$ to $\{\omega \in \Omega;\ f(\omega) \in B\}$.

A degenerated state map is defined analogously: for a fixed $q \in M_1^+(\Omega, \mathbf{B})$ the degenerated state map sends each probability integral on $\mathbf{A}$ to $\int(\cdot)\mathrm{d}q$. Finally, we need an observable $h\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ such that the corresponding $\overline{T_h}$ maps each probability integral $\int(\cdot)\mathrm{d}p$, $p \in M_1^+(\Omega, \mathbf{A})$, on $\mathcal{M}(\mathbf{A})$ to the probability integral $\int(\cdot)\mathrm{d}q$ on $\mathcal{M}(\mathbf{B})$. It suffices to define h as follows: for $u \in \mathcal{M}(\mathbf{B})$ let $h(u)$ be the constant function v_q on Ω the value of which is $\int u\mathrm{d}q$. Then, for all $p \in M_1^+(\Omega, \mathbf{A})$, we have $\int h(u)\mathrm{d}p = \int v_q\mathrm{d}p = \int u\mathrm{d}q$, and hence $\overline{T_h}\left(\int(\cdot)\mathrm{d}p\right) = \int(\cdot)\mathrm{d}q$.

The corresponding **degenerated statistical map** T_h maps each $p \in M_1^+(\Omega, \mathbf{A})$ into $q \in M_1^+(\Omega, \mathbf{B})$. Observe that each distribution map D_f is a statistical map.

Let $T_i\colon M_1^+(\Omega, \mathbf{A}) \longrightarrow M_1^+(\Xi_i, \mathbf{B}_i)$, $i = 1, 2$, be statistical maps and let $(\Xi_1 \times \Xi_2, \mathbf{B}_1 \times \mathbf{B}_2)$ be the usual product of the involved target measurable spaces. Recall that the product

$$T_1 \otimes T_2\colon M_1^+(\Omega, \mathbf{A}) \longrightarrow M_1^+(\Xi_1 \times \Xi_2, \mathbf{B}_1 \times \mathbf{B}_2)$$

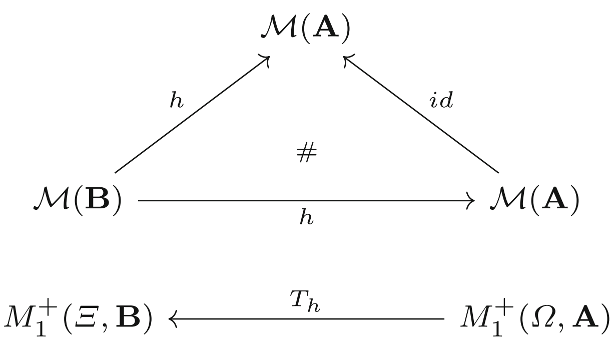

Fig. 6 Trivial smearing

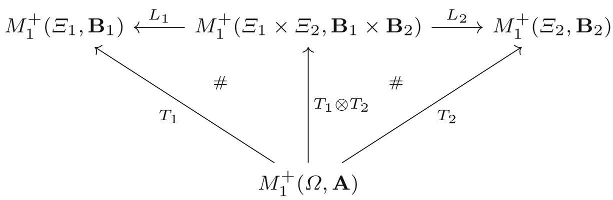

Fig. 7 The product of two statistical maps fails to be categorical

is defined by

$$((T_1 \otimes T_2)(p))\,(B_1 \times B_2) = \int_\Omega (T_1(\delta_\omega))\,(B_1) \cdot (T_2(\delta_\omega))\,(B_2)\,\mathrm{d}p,$$

for $B_1 \in \mathbf{B}_1$, $B_2 \in \mathbf{B}_2$, $p \in M_1^+(\Omega, \mathbf{A})$. Denote

$$L_i \colon M_1^+(\Xi_1 \times \Xi_2, \mathbf{B}_1 \times \mathbf{B}_2) \longrightarrow M_1^+(\Xi_i, \mathbf{B}_i), \quad i = 1, 2,$$

the usual lateral (marginal) projections (for each $r \in M_1^+(\Xi_1 \times \Xi_2, \mathbf{B}_1 \times \mathbf{B}_2)$, i.e., $L_1(r)$ is the unique measure $p \in M_1^+(\Xi_1, \mathbf{B}_1)$ such that $p(B) = r(B \times \Xi_2)$, $B \in \mathbf{B}_1$, analogously for L_2).

It follows from the definition of the product $T_1 \otimes T_2$ (see also condition (BG)), that $(T_1 \otimes T_2)(p)$, $p \in M_1^+(\Omega, \mathbf{A})$, is determined by values at δ_ω, $\omega \in \Omega$. Since (2.2 p. 349 in [3]) $(T_1 \otimes T_2)(\delta_\omega) = T_1(\delta_\omega) \times T_2(\delta_\omega)$ is the product measure on $\mathbf{B}_1 \times \mathbf{B}_2$, we have $L_i\,((T_1 \otimes T_2)(\delta_\omega)) = T_i(\delta_\omega)$, $\omega \in \Omega$, $i = 1, 2$. Consequently, $L_i \circ (T_1 \otimes T_2)(p) = T_i(p)$, $p \in M_1^+(\Omega, \mathbf{A})$, $i = 1, 2$, see Fig. 7.

Now, contrariwise, assume that $T_1 \otimes T_2$ is the categorical product of T_1 and T_2, i.e., if $T \colon M_1^+(\Omega, \mathbf{A}) \longrightarrow M_1^+(\Xi_1 \times \Xi_2, \mathbf{B}_1 \times \mathbf{B}_2)$ is a statistical map such that $L_i \circ T = T_i$, $i = 1, 2$, then $T = T_1 \otimes T_2$.

Fix $p_i \in M_1^+(\Xi_i, \mathbf{B}_i)$, $i = 1, 2$. Let $T_i \colon M_1^+(\Omega, \mathbf{A}) \longrightarrow M_1^+(\Xi_i, \mathbf{B}_i)$, $i = 1, 2$, be the degenerated statistical map sending each $p \in M_1^+(\Omega, \mathbf{A})$ into p_i. Then $((T_1 \otimes T_2)(p))\,(B_1 \times B_2) = p_1(B_1) \cdot p_2(B_2)$, $B_1 \in \mathbf{B}_1$, $B_2 \in \mathbf{B}_2$, and $(T_1 \otimes T_2)(p) = p_1 \times p_2$.

Let $\Xi_1 = \{a, b\}$, $\Xi_2 = \{c, d\}$, let $(\Xi_1, \mathbf{B}_1)$, $(\Xi_2, \mathbf{B}_2)$ be the corresponding discrete measurable spaces, and let $p_i \in M_1^+(\Xi_i, \mathbf{B}_i)$, $i = 1, 2$, be the uniform measure (assigning each singleton measure 1/2). Let $p \in M_1^+(\Xi_1 \times \Xi_2, \mathbf{B}_1 \times \mathbf{B}_2)$ be the uniform measure (assigning each singleton measure 1/4. Define $q \in M_1^+(\Xi_1 \times \Xi_2, \mathbf{B}_1 \times \mathbf{B}_2)$ as follows: $q(a, c) = q(b, d) = 1/8$, $q(a, d) = q(b, c) = 3/8$. Let $(\Omega, \mathbf{A})$ be a nontrivial measurable space. Let $T_i \colon M_1^+(\Omega, \mathbf{A}) \longrightarrow M_1^+(\Xi_i, \mathbf{B}_i)$, $i = 1, 2$, be the degenerated statistical map sending each $p \in M_1^+(\Omega, \mathbf{A})$ into p_i. Clearly $L_i \circ (T_1 \otimes T_2) = T_i$, $i = 1, 2$. Finally, let $T \colon M_1^+(\Omega, \mathbf{A}) \longrightarrow M_1^+(\Xi_1 \times \Xi_2, \mathbf{B}_1 \times \mathbf{B}_2)$ be the degenerated statistical map sending each $p \in M_1^+(\Omega, \mathbf{A})$ into q. Clearly, $L_i \circ T = T_i$, $i = 1, 2$, and $T \neq T_1 \otimes T_2$. This is a contradiction.

We believe that even modest application of arrows and commuting diagrams leads to simpler and more intuitive ways how to describe rather complex constructions in generalized probability theory and its applications to quantum structures.

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