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SPACES NOT DISTINGUISHING IDEAL CONVERGENCES OF REAL-VALUED FUNCTIONS

Abstract

We introduce an alternative definition of the concept of an ideal weak QN-space and compare it with the definition introduced by Bukovský, Das, and Šupina. We classify the properties of spaces expressing some kinds of indistinguishability for various pairs of ideal convergences and semi-convergences. We give combinatorial characterizations of the least cardinalities of spaces not having a particular property and show that they are invariant for classes of spaces that contain metric spaces and are closed under homeomorphisms. The counterexamples proving this are subsets of the Baire space ${}^{\omega}\omega$.

Introduction

The study of spaces not distinguishing a pair of convergences of sequences of continuous real-valued functions was initiated in [3] where the notions of a QN-space and a wQN-space were introduced. Over the last years this study has been extended for ideal convergences. Recently Kwela [11] has given a combinatorial characterization of the cardinal number non(IwQN-space) and obtained nontrivial estimations of this cardinal number for a wide class of ideals I on ω. This characterization is similar to the characterization of non(IQN-space) by Šupina [15]. However, there are several approaches how

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to understand "weak" (expressed by the letter "w") in the notion of an IwQN-space. According to Das and Chandra [6] a space X is an IwQN-space if for every sequence $\{f_n\}_n$ of continuous real-valued functions pointwise converging to 0 there exists an "increasing" sequence $\{n_k\}_k$ of natural numbers such that $\{f_{n_k}\}_k \xrightarrow{I\mathrm{QN}} 0$ on X. On the other hand, Bukovský, Das, and Šupina [2] say that a space X is an (I,J)wQN-space if for every sequence $\{f_n\}_n$ of continuous real-valued functions pointwise I-converging to 0 there exists "arbitrary" sequence $\{n_k\}_k$ of natural numbers such that $\{f_{n_k}\}_k \xrightarrow{J\mathrm{QN}} 0$ on X. The latter approach of choosing a subsequence is more handy because sometimes it is useful to consider ideals on countable sets different from ω, in which case it may not be obvious what "increasing" should mean. In this paper we consider another "weakening" of J-quasi-normal convergence meaning the existence of an ideal $K \leq_{\mathrm{K}} J$ (in the Katětov partial ordering of ideals) such that $\{f_n\}_n \xrightarrow{K\mathrm{QN}} 0$; i.e., instead of choosing a subsequence we choose a "simpler" ideal convergence. We shall call this condition $\leq_{\mathrm{K}} J$QN-convergence and applying it we get a weakening of the notion of an (I,J)QN-space that is stronger than (I,J)wQN-space in the sense of [2]. In fact, the $\leq_{\mathrm{K}} J$QN-convergence has weaker properties than one could expect from the notion of a convergence. Therefore we introduce the concept of a semi-convergence.

In Section 1, we collect several results showing dependence of the I-convergence and the IQN-convergence on transformations of the ideal I in connection to Katětov partial ordering of ideals. We introduce natural transformations between sequences of reals and functions in ${}^{\omega}\omega$ and present translations of the ideal convergences into the language of functions. All these technical results serve as a schema for arguments in proofs of further sections.

In Section 2 we consider two ideal convergences and two ideal semi-convergences (I-convergence, IQN-convergence, $\leq_{\mathrm{K}} I$-convergence, $\leq_{\mathrm{K}} I$QN-convergence) of sequences of functions. For each pair β and γ of these semi-convergences (with possibly distinct ideals) we consider two concepts, the notion of a (β,γ)-space and the notion of a w(β,γ)-space, expressing the property of a space X that for any sequence f of continuous real-valued functions on X, the β-convergence of f to 0 implies the γ-convergence of f to 0 and the γ-convergence of a subsequence of f to 0, respectively. Excluding trivial properties and identifying equivalent properties we obtain nine nontrivial cases and implications between them. We use slightly different notation for these concepts from the notations used in the cited papers. In particular, we use the terms $(I, J\mathrm{QN})$-space and w$(I, J\mathrm{QN})$-space instead of (I,J)QN-space and (I,J)wQN-space from [2] to indicate that parameters of the concept are convergences instead of the ideals on which the convergences depend. For example, the mentioned IwQN-space from [11] corresponds to w$(\mathrm{Fin}, I\mathrm{QN})$-

space where Fin stands for the standard pointwise convergence determined by the ideal Fin of finite sets.

In Section 3, for every of the nine nontrivial properties we characterize the existence of a space (a set of reals) not having this property (Theorem 3.4 and Theorem 3.9). We get combinatorial characterizations of minimal cardinalities of such spaces depending on ideals (Theorem 3.6 and Theorem 3.11). Five of these cardinals are the same and it remains an open question whether the corresponding properties are different.

Section 4 collects the results about heredity of the properties with respect to the Katětov partial ordering $\leq_{\mathrm{K}}$ of ideals. In particular, these results allow to replace non-tall ideals by Fin.

1 Preliminaries

By composition of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ we understand the relation $R \circ S = \{(x,z) : \exists y\ [(x,y) \in R \text{ and } (y,z) \in S]\} \subseteq X \times Z$. In this sense the composition of functions $f : X \to Y$ and $g : Y \to Z$ (as relations) is a function $f \circ g : X \to Z$ and $(f \circ g)(x) = g(f(x))$ for $x \in X$. Note that many authors prefer the opposite order of functions in the composition.

A class $\mathcal{X}$ of topological spaces is said to be reasonable if $\mathcal{X}$ contains all separable metric spaces and $\mathcal{X}$ is closed under homeomorphisms. For a property P of topological spaces let $\mathrm{non}_{\mathcal{X}}(P)$ denote the minimal cardinality of a space in $\mathcal{X}$ not having the property P. We write $\mathrm{non}_{\mathcal{X}}(P) = \infty$ if every space in $\mathcal{X}$ has P. Naturally, we suppose that $\kappa < \infty$ for every cardinal κ. Then the inequality $\mathrm{non}_{\mathcal{X}}(P_1) \leq \mathrm{non}_{\mathcal{X}}(P_2)$ says also this: If every space in $\mathcal{X}$ has P_1, then every space in $\mathcal{X}$ has P_2. Employing ∞ symbol is useful mainly in case when the properties have parameters. The cardinal $\mathrm{add}_{\mathcal{X}}(P)$ denotes the minimal cardinal κ such that there is a space X in $\mathcal{X}$ not having the property P which can be expressed as the union $X = \bigcup_{\xi<\kappa} X_\xi$ where every subspace X_ξ has the property P. In this paper we omit the subscript $\mathcal{X}$ if the values are the same for all reasonable classes $\mathcal{X}$. Then either $\mathrm{non}(P) \leq \mathfrak{c}$ or $\mathrm{non}(P) = \infty$.

The pseudo-intersection number $\mathfrak{p}$ (see e.g. [7]) is the minimal cardinality of a set $\mathcal{F} \subseteq [\omega]^\omega$ with the strong finite intersection property which has no infinite pseudo-intersection; i.e., every finite subfamily of $\mathcal{F}$ has an infinite intersection and there is no $a \in [\omega]^\omega$ such that $a \setminus b \in [\omega]^{<\omega}$ for all $b \in \mathcal{F}$. The bounding number $\mathfrak{b}$ is the least cardinality of a subset of ${}^\omega\omega$ without an upper bound with respect to the eventual domination of functions: $\varphi \leq^* \psi$ if $(\forall^\infty n \in \omega)\ \varphi(n) \leq \psi(n)$.

By an ideal on a set S (usually $S = \omega$) we understand a proper subfamily $I \subsetneq \mathcal{P}(S)$ containing $[S]^{<\omega}$ as a subset such that for every $a, b \subseteq S$, $a, b \in I$ implies $a \cup b \in I$, and $b \subseteq a$ and $a \in I$ together imply $b \in I$. Then

$$I^+ = \mathcal{P}(S) \setminus I \quad \text{and} \quad I^* = \{S \setminus a : a \in I\}$$

are the family of I-positive sets and the dual filter to the ideal I, respectively. An ideal I on S is tall if $(\forall a \in [S]^\omega)\ [a]^\omega \cap I \neq \emptyset$. If I and J are ideals on S and $a \subseteq S$, then by $I \vee J$ and $I \vee \langle a \rangle$ we denote the smallest ideals on S containing $I \cup J$ and $I \cup \{a\}$, respectively, if such ideals exist. If $a \in I^+$, then $I{\restriction}a = \{a \cap b : b \in I\}$ is an ideal on a. Let $\mathrm{Fin} = [\omega]^{<\omega}$.

A sequence of reals $\xi = \langle \xi_n : n \in \omega \rangle \in {}^\omega\mathbb{R}$ is said to be I-convergent to $x \in \mathbb{R}$, we write $\xi \xrightarrow{I} x$, if the set $\{n \in \omega : |\xi_n - x| \geq \varepsilon\} \in I$ for all $\varepsilon > 0$. Instead of $\xi \xrightarrow{\mathrm{Fin}} x$ we can write simply $\xi \to x$. A sequence of real-valued functions $f = \langle f_n : n \in \omega \rangle$ on a set X is said to converge quasi-normally to a function g, we write $\langle f_n : n \in \omega \rangle \xrightarrow{\mathrm{QN}} g$ or $f \xrightarrow{\mathrm{QN}} g$, if there is a sequence of non-negative reals $\langle \varepsilon_n : n \in \omega \rangle$ converging to 0 such that for every $x \in X$ for all but finitely many $n \in \omega$, $|f_n(x) - g(x)| \leq \varepsilon_n$ (see [1]). This convergence is also known as the equal convergence (see [5]).

The set ${}^X\mathbb{R}$ of all real-valued functions on a set X is an additive group. Therefore to introduce a notion of convergence for sequences of functions usually it is enough to describe those sequences whose limit is the constant zero function. We consider the following ideal generalizations of pointwise and quasi-normal convergences of sequences of real-valued functions:

Definition 1.1. Let I, J, and $K \subseteq J$ be ideals on ω, $X \neq \emptyset$, and let $f \in {}^\omega({}^X\mathbb{R})$. We define the following convergences of the sequence f:

(i) $f \xrightarrow{I} 0$ if $(\forall x \in X)(\forall \varepsilon > 0)\ \{n \in \omega : |f_n(x)| \geq \varepsilon\} \in I$.

(ii) $f \xrightarrow{J\mathrm{QN}} 0$ if there exists an $\varepsilon \in {}^\omega[0, \infty)$ such that $\varepsilon \xrightarrow{J} 0$ and $(\forall x \in X)$ $\{n \in \omega : |f_n(x)| \geq \varepsilon_n\} \in J$.

(iii) $f \xrightarrow{JK\mathrm{QN}} 0$ if there exists an $\varepsilon \in {}^\omega[0, \infty)$ such that $\varepsilon \xrightarrow{K} 0$ and $(\forall x \in X)$ $\{n \in \omega : |f_n(x)| \geq \varepsilon_n\} \in J$.

The I-convergence and the JQN-convergence are the same as in [2, 6]. The JKQN-convergence was introduced in [8] under the name "the (J, K)-equal convergence" while the "J-equal convergence" was used before with the meaning of (J, K)-equal convergence either with $K = \mathrm{Fin}$ or with $K = J$.

Obviously, JQN-convergence is stronger than J-convergence. By [8, Proposition 4.4], JKQN-convergence is stronger than J-convergence (and JQN-convergence) if and only if $K \subseteq J$. We usually assume $K \subseteq J$.

For an ideal I on ω and a function $\varphi \in {}^{\omega}\omega$ we define

$$\begin{aligned}
\varphi^{\rightarrow}(I) &= \{a \subseteq \omega : \varphi^{-1}(a) \in I\}, \\
\varphi^{\leftarrow}(I) &= \{a \subseteq \omega : \varphi(a) \in I\}, \\
F(I) &= \{\alpha \in {}^{\omega}\omega : (\forall n \in \omega)\ \alpha^{-1}(\{n\}) \in I\}.
\end{aligned}$$

Note that $\varphi^{\rightarrow}(I)$ is an ideal on ω if and only if $\varphi \in F(I)$; and $\varphi^{\leftarrow}(I)$ is an ideal on ω if and only if $\operatorname{rng}(\varphi) \in I^{+}$. If $\operatorname{rng}(\varphi) \in I^{*}$, then $\varphi^{\rightarrow}(\varphi^{\leftarrow}(I)) = I$. For any ideals I and J, $I \subseteq \varphi^{\rightarrow}(J)$ if and only if $\varphi^{\leftarrow}(I) \subseteq J$.

We assume that $\mathcal{P}(\omega)$ is endowed with the compact Polish topology homeomorphic to the Cantor topology on ${}^{\omega}2$ via characteristic functions. Observe that $F(I)$ is a dense subset of the Baire space ${}^{\omega}\omega$, $F(\mathrm{Fin}) \subseteq F(I)$, and $\varphi \in F(\varphi^{\leftarrow}(I))$ whenever $\varphi^{\leftarrow}(I)$ is an ideal on ω. In most cases $F(I)$ is not a nice subset of the Baire space, but if an ideal I is an F_{σ} subset of $\mathcal{P}(\omega)$, then $F(I)$ is $F_{\sigma\delta}$. To see this assume that $I = \bigcup_{m\in\omega} I_m$ where every set I_m is a closed subset of $\mathcal{P}(\omega)$. Then $F(I) = \bigcap_{n\in\omega} \bigcup_{m\in\omega} F_{n,m}$ where $F_{n,m} = \{\alpha \in {}^{\omega}\omega : (\exists a \in I_m)(\forall k \in \omega \setminus a)\ \alpha(k) \neq n\}$. Every set $F_{n,m}$ is closed because it is a projection of a closed set along the compact set I_m.

Let us recall Rudin-Keisler ($\leq_{\mathrm{RK}}$), Rudin-Blass ($\leq_{\mathrm{RB}}$), Katětov ($\leq_{\mathrm{K}}$), and Katětov-Blass ($\leq_{\mathrm{KB}}$) partial quasi-orderings of ideals I and J on ω:

$$\begin{aligned}
I \leq_{\mathrm{RK}} J &\Leftrightarrow (\exists \varphi \in F(J))\ I = \varphi^{\rightarrow}(J), \\
I \leq_{\mathrm{RB}} J &\Leftrightarrow (\exists \varphi \in F(\mathrm{Fin}))\ I = \varphi^{\rightarrow}(J), \\
I \leq_{\mathrm{K}} J &\Leftrightarrow (\exists \varphi \in F(J))\ I \subseteq \varphi^{\rightarrow}(J), \\
I \leq_{\mathrm{KB}} J &\Leftrightarrow (\exists \varphi \in F(\mathrm{Fin}))\ I \subseteq \varphi^{\rightarrow}(J)
\end{aligned}$$

(see e.g., [10]). Note that we use the quantifiers $\exists \varphi \in F(J)$ instead of $\exists \varphi \in {}^{\omega}\omega$ which appear in original definitions of these quasi-orderings. This replacement of the quantifiers does not change the meaning of the defined notions because $\varphi \in F(J)$ if and only if $\mathrm{Fin} \subseteq \varphi^{\rightarrow}(J)$.

In the rest of this section we present three technical lemmas which collect a special kind of arguments with the aim to improve understanding of main ideas of the proofs in the following sections.

For a sequence of functions $f \in {}^{\omega}({}^{X}\mathbb{R})$ we define sequences of functions $\varphi \circ f \in {}^{\omega}({}^{X}\mathbb{R})$ and $\varphi * f \in {}^{\omega}({}^{X}\mathbb{R})$ by mixing the terms $f_k(x)$, $k \in \omega$, for a given $x \in X$ using $\varphi \in {}^{\omega}\omega$ in the following two different ways:

$$(\varphi \circ f)_k(x) = f_{\varphi(k)}(x), \quad k \in \omega,$$

$$(\varphi * f)_n(x) = \begin{cases} \sup\{\min\{|f_k(x)|, 1\} : k \in \varphi^{-1}(\{n\})\}, & \text{if } n \in \mathrm{rng}(\varphi), \\ 0, & \text{otherwise.} \end{cases}$$

Note that $(\varphi * (\varphi \circ f))_n(x) = \min\{|f_n(x)|, 1\}$ for all $n \in \mathrm{rng}(\varphi)$ and $x \in X$.

Lemma 1.2. *Let I be an ideal on ω and let $\varphi \in F(I)$ and $f \in {}^{\omega}({}^{X}\mathbb{R})$.*

(a) *$f \xrightarrow{\varphi^{\to}(I)} 0$ if and only if $\varphi \circ f \xrightarrow{I} 0$.*

(b) *If $f \xrightarrow{\varphi^{\to}(I)\mathrm{QN}} 0$, then $\varphi \circ f \xrightarrow{I\mathrm{QN}} 0$.*

(c) *If $\varphi * f \xrightarrow{\varphi^{\to}(I)} 0$, then $f \xrightarrow{I} 0$.*

(d) *If $\varphi * f \xrightarrow{\varphi^{\to}(I)\mathrm{QN}} 0$, then $f \xrightarrow{I\mathrm{QN}} 0$.*

Proof. (a) For every $\varepsilon > 0$ and $x \in X$, $\{k \in \omega : |f_{\varphi(k)}(x)| \geq \varepsilon\} = \varphi^{-1}(\{n \in \omega : |f_n(x)| \geq \varepsilon\})$. Therefore $\{n \in \omega : |f_n(x)| \geq \varepsilon\} \in \varphi^{\to}(I)$ if and only if $\{k \in \omega : |f_{\varphi(k)}(x)| \geq \varepsilon\} \in I$.

(b) Let $\varepsilon \in {}^{\omega}[0,\infty)$ be such that $\varepsilon \xrightarrow{\varphi^{\to}(I)} 0$ and $\{n \in \omega : |f_n(x)| \geq \varepsilon_n\} \in \varphi^{\to}(I)$ for all $x \in X$. Then $\varphi \circ f \xrightarrow{I\mathrm{QN}} 0$ because for every $x \in X$, $\{k \in \omega : |f_{\varphi(k)}(x)| \geq \varepsilon_{\varphi(k)}\} = \varphi^{-1}(\{n \in \omega : |f_n(x)| \geq \varepsilon_n\}) \in I$ and, by (a), $\varphi \circ \varepsilon \xrightarrow{I} 0$.

(c) Denote $b_{x,\varepsilon} = \{n \in \omega : (\varphi * f)_n(x) \geq \varepsilon\}$. For every $\varepsilon \in (0,1)$ and $x \in X$, if $b_{x,\varepsilon} \in \varphi^{\to}(I)$, then $\{k \in \omega : |f_k(x)| \geq \varepsilon\} \subseteq \varphi^{-1}(b_{x,\varepsilon}) \in I$.

(d) Let $\varepsilon \in {}^{\omega}[0,\infty)$ be such that $\varepsilon \xrightarrow{\varphi^{\to}(I)} 0$ and $b_x = \{n \in \omega : (\varphi * f)_n(x) \geq \varepsilon_n\} \in \varphi^{\to}(I)$ for all $x \in X$. Denote $a = \{n \in \omega : \varepsilon_n \geq 1\}$; $a \in \varphi^{\to}(I)$. By (a), $\varphi \circ \varepsilon \xrightarrow{I} 0$. Now, $f \xrightarrow{I\mathrm{QN}} 0$ because for every $x \in X$, $\{k \in \omega : |f_k(x)| \geq \varepsilon_{\varphi(k)}\} \subseteq \{k \in \omega : \varepsilon_{\varphi(k)} \geq 1 \text{ or } \min\{|f_k(x)|, 1\} \geq \varepsilon_{\varphi(k)}\} \subseteq \varphi^{-1}(a \cup b_x) \in I$. □

Lemma 1.3. *Let I be an ideal on ω, let $\varphi \in {}^{\omega}\omega$ be such that $\mathrm{rng}(\varphi) \in I^+$, and let $f \in {}^{\omega}({}^{X}\mathbb{R})$.*

(a) *$f \xrightarrow{\varphi^{\leftarrow}(I)} 0$ if and only if $\varphi * f \xrightarrow{I} 0$.*

(b) *If $f \xrightarrow{\varphi^{\leftarrow}(I)\mathrm{QN}} 0$, then $\varphi * f \xrightarrow{I\mathrm{QN}} 0$.*

If $\mathrm{rng}(\varphi) \in I^*$, then:

(c) *$\varphi \circ f \xrightarrow{\varphi^{\leftarrow}(I)} 0$ if and only if $f \xrightarrow{I} 0$.*

(d) *If $\varphi \circ f \xrightarrow{\varphi^{\leftarrow}(I)\mathrm{QN}} 0$, then $f \xrightarrow{I\mathrm{QN}} 0$.*

PROOF. (a) For every $x \in X$ and $\varepsilon \in (0,1)$, $\{n \in \omega : (\varphi * f)_n(x) \geq 2\varepsilon\} \subseteq \varphi(\{k \in \omega : |f_k(x)| \geq \varepsilon\}) \subseteq \{n \in \omega : (\varphi * f)_n(x) \geq \varepsilon\}$.

(b) Let $\varepsilon \in {}^\omega[0,\infty)$ be such that $\varepsilon \xrightarrow{\varphi^{\leftarrow}(I)} 0$ and $b_x = \{k \in \omega : |f_k(x)| \geq \varepsilon_k\} \in \varphi^{\leftarrow}(I)$ for all $x \in X$. By (a), $\langle(\varphi * \varepsilon)_n + 2^{-n}\rangle_{n\in\omega} \xrightarrow{I} 0$. Now, $\varphi * f \xrightarrow{\mathrm{IQN}} 0$ because for every $x \in X$, $\{n \in \omega : (\varphi * f)_n(x) \geq (\varphi * \varepsilon)_n + 2^{-n}\} \subseteq \varphi(b_x) \in I$.

(c)–(d) Let $f'_n(x) = \min\{|f_n(x)|, 1\}$. Then $\{n \in \omega : \varphi * (\varphi \circ f)_n \neq f'_n\} \subseteq \omega \setminus \mathrm{rng}(\varphi) \in I$ and consequently, $\varphi * (\varphi \circ f) \xrightarrow{I} 0 \Leftrightarrow f' \xrightarrow{I} 0 \Leftrightarrow f \xrightarrow{I} 0$ and $\varphi * (\varphi \circ f) \xrightarrow{I\mathrm{QN}} 0 \Leftrightarrow f' \xrightarrow{I\mathrm{QN}} 0 \Leftrightarrow f \xrightarrow{I\mathrm{QN}} 0$. Therefore it is enough to substitute $\varphi \circ f$ for f into (a) and (b) and then replace $\varphi * (\varphi \circ f)$ by f. $\square$

We will need the following two reductions between the functions $\alpha \in {}^\omega\omega$ and the sequences of reals $\xi = \langle \xi_k : k \in \omega\rangle \in {}^\omega\mathbb{R}$, namely

$$\sigma : {}^\omega\omega \to {}^\omega(0,1] \quad \text{and} \quad \tau : {}^\omega\mathbb{R} \to {}^\omega\omega$$

defined by

$$\sigma_k(\alpha) = \sigma(\alpha)(k) = 2^{-\alpha(k)},$$
$$\tau(\xi)(k) = \min\{n \in \omega : |\xi_k| + 2^{-k} > 2^{-n}\}.$$

The functions $\sigma_k : {}^\omega\omega \to \mathbb{R}$, $k \in \omega$, are continuous.

If ξ and η are sequences of reals (or natural numbers), then we denote

$$\|\xi \leq \eta\| = \{k \in \omega : \xi(k) \leq \eta(k)\} = \{k \in \omega : \xi_k \leq \eta_k\}.$$

In the same way we define $\|\xi < \eta\|$ and $\|\xi = \eta\|$.

Lemma 1.4. *Let I be an ideal on ω and let $\alpha, \varphi \in {}^\omega\omega$ and $\xi \in {}^\omega\mathbb{R}$.*

(1) *$\sigma(\alpha) \xrightarrow{I} 0$ if and only if $\alpha \in F(I)$.*

(2) *$\tau(\xi) \in F(I)$ if and only if $\xi \xrightarrow{I} 0$.*

(3) *$\{k \in \omega : |\xi_k| \geq \sigma_k(\alpha)\})\| \subseteq \|\tau(\xi) \leq \alpha\|$.*

(4) *$\{k \in \omega : \sigma_k(\alpha) \geq |\xi_k|\} \supseteq \|\alpha < \tau(\xi)\|$.*

(5) *$\{k \in \omega : |\xi_{\varphi(k)}| \geq \sigma_k(\alpha)\} \subseteq \|\varphi \circ \tau(\xi) \leq \alpha\|$.*

(6) *$\{k \in \omega : \sigma_{\varphi(k)}(\alpha) \geq |\xi_k|\} \supseteq \|\varphi \circ \alpha < \tau(\xi)\|$.*

PROOF. (1) and (2) hold by definitions; (3) and (4) are special cases of (5) and (6).

(5) $\{k \in \omega : |\xi_{\varphi(k)}| \geq \sigma_k(\alpha)\} \subseteq \{k \in \omega : |\xi_{\varphi(k)}| + 2^{-\varphi(k)} > 2^{-\alpha(k)}\} = \{k \in \omega : \tau(\xi)(\varphi(k)) \leq \alpha(k)\} = \|\varphi \circ \tau(\xi) \leq \alpha\|$.

(6) $\{k \in \omega : \sigma_{\varphi(k)}(\alpha) \geq |\xi_k|\} \supseteq \{k \in \omega : 2^{-\alpha(\varphi(k))} \geq |\xi_k| + 2^{-k}\} = \{k \in \omega : \alpha(\varphi(k)) < \tau(\xi)(k)\} = \|\varphi \circ \alpha < \tau(\xi)\|$. $\square$

2 (β, γ)-space and $\mathrm{w}(\beta, \gamma)$-space

It is only in this section that the symbols β and γ denote convergences and semi-convergences of sequences of functions: By a semi-convergence on ${}^X\mathbb{R}$ we mean an asymmetric binary relation β between sequences of functions $f = \langle f_n : n \in \omega \rangle \in {}^\omega({}^X\mathbb{R})$ and functions $\varphi \in {}^X\mathbb{R}$, we write $f \xrightarrow{\beta} \varphi$, satisfying the following conditions (S_0)–(S_2):

(S_0) If $f_n = 0$ for all $n \in \omega$, then $f \xrightarrow{\beta} 0$.

(S_1) If $f \xrightarrow{\beta} \varphi$ and $\psi \in {}^X\mathbb{R}$, then $\langle f_n + \psi : n \in \omega \rangle \xrightarrow{\beta} \varphi + \psi$.

(S_2) If $f \xrightarrow{\beta} 0$ and $|g_n| \le |f_n|$ for all $n \in \omega$, then $g \xrightarrow{\beta} 0$.

We say that the semi-convergence β is nice, if (S_3) holds:

(S_3) If $f \xrightarrow{\beta} 0$ and $c \in {}^\omega\mathbb{R}$ is such that $c \xrightarrow{\mathrm{Fin}} 0$, then $\langle f_n + c_n : n \in \omega \rangle \xrightarrow{\beta} 0$.

Due to (S_1), a semi-convergence β is determined by the relation $f \xrightarrow{\beta} 0$.

Every convergence we consider is a nice semi-convergence. Natural requirements for a convergence are, for example, uniqueness of limits, additivity, scalar multiplication, and so on. We shall use the phrases "β-convergence" and "β-converges to" also in the case that β is a semi-convergence and not a convergence because the prefix 'semi-' is clearly expressed by the prefix 'β-'.

Let $C(X)$ be the family of continuous real-valued functions on X. Following [2] but in accordance with the denotation in the paper [4] we define:

Definition 2.1. Let β and γ be arbitrary semi-convergences.

(i) X is a $\beta\gamma$-space, if for every $f \in {}^\omega C(X)$, $f \xrightarrow{\beta} 0$ implies $f \xrightarrow{\gamma} 0$.

(ii) X is a $\mathrm{w}\beta\gamma$-space (the letter 'w' means 'weak'), if for every $f \in {}^\omega C(X)$ such that $f \xrightarrow{\beta} 0$ there is $\varphi \in {}^\omega\omega$ such that $\varphi \circ f \xrightarrow{\gamma} 0$.

We will write a (β, γ)-space and a $\mathrm{w}(\beta, \gamma)$-space because β and γ may be represented by strings of several letters.

The prefixes (β, γ) and $\mathrm{w}(\beta, \gamma)$ are considered to be properties of spaces. Therefore, for example, the formula $(\beta, \gamma) \Rightarrow (\beta', \gamma')$ means "every (β, γ)-space is a (β', γ')-space". Sometimes, for the sake of brevity, we will say "a space X is (β, γ)" meaning that "a space X is a (β, γ)-space".

The investigation of such properties has a quite long history. For example, a QN-space and a wQN-space in [3] are a $(\mathrm{Fin}, \mathrm{FinQN})$-space and a $\mathrm{w}(\mathrm{Fin}, \mathrm{FinQN})$-space, respectively, where Fin and FinQN stand for I-convergence and IQN-convergence, respectively, with $I = \mathrm{Fin}$. An $(I, J\mathrm{QN})$-space and a $\mathrm{w}(I, J\mathrm{QN})$-space have the same meaning as an (I, J)QN-space and an (I, J)wQN-space in [2].

Some of the known definitions of 'weak $(I, J\mathrm{QN})$' (see [6] and [11]) require to choose a strictly increasing subsequence of f. The definition of $\mathrm{w}(\beta, \gamma)$ (like the definition of (I, J)wQN in [2]) has no such restriction. This allows to extend the definition to convergences of sequences with arbitrary countable set of indices. In fact, the set of indices for γ may differ from the set of indices for β in the property $\mathrm{w}(\beta, \gamma)$ (not in the property (β, γ)).

A property $\mathrm{w}(\beta, \gamma)$ equals to the property $(\beta, \mathrm{w}\gamma)$ where $\mathrm{w}\gamma$ is the semi-convergence defined by $f \xrightarrow{\mathrm{w}\gamma} 0 \Leftrightarrow (\exists \varphi \in {}^\omega\omega)\ \varphi \circ f \xrightarrow{\gamma} 0$ (in most cases $\mathrm{w}\gamma$ is not nice). Ideal convergences can be weakened to semi-convergences in several ways (e.g., by choosing a restriction of an ideal, a subideal, a super-ideal, or in some sense a simpler ideal, etc., instead of choosing a subsequence). The following semi-convergence $\gamma(\leq_{\mathrm{K}} I)$ seems to be the most suitable weakening of $\gamma(I)$-convergence for $\gamma(I) = I$ and $\gamma(I) = I\mathrm{QN}$ different from $\mathrm{w}\gamma(I)$:

Definition 2.2. Let $\gamma(I)$ be a semi-convergence depending on an ideal I. For $f \in {}^\omega({}^X\mathbb{R})$ we define $f \xrightarrow{\gamma(\leq_{\mathrm{K}} I)} 0 \Leftrightarrow (\exists J \leq_{\mathrm{K}} I)\ f \xrightarrow{\gamma(J)} 0$.

Lemma 2.3. *If $\gamma(I)$ is a nice semi-convergence for all ideals I, then $\gamma(\leq_{\mathrm{K}} I)$ is nice, too. In particular, $\leq_{\mathrm{K}} I$ and $\leq_{\mathrm{K}} I\mathrm{QN}$ are nice semi-convergences.* $\square$

Moreover,

$$f \xrightarrow{\leq_{\mathrm{K}} I} 0 \Leftrightarrow (\exists J \leq_{\mathrm{RK}} I)\ f \xrightarrow{J} 0, \qquad f \xrightarrow{\leq_{\mathrm{K}} I\mathrm{QN}} 0 \Leftrightarrow (\exists J \leq_{\mathrm{RK}} I)\ f \xrightarrow{J\mathrm{QN}} 0,$$

$$\begin{array}{ccc} I\text{-convergence} & \Rightarrow & \leq_{\mathrm{K}} I\text{-convergence} \\ \Uparrow & & \Uparrow \\ I\mathrm{QN}\text{-convergence} & \Rightarrow & \leq_{\mathrm{K}} I\mathrm{QN}\text{-convergence}. \end{array}$$

The property of an $(I, \leq_{\mathrm{K}} J\mathrm{QN})$-space can be considered to be an alternative definition of a 'weak $(I, J\mathrm{QN})$-space'. It is between $(I, J\mathrm{QN})$ and $\mathrm{w}(I, J\mathrm{QN})$ (see Lemma 2.9).

The next lemma describes some cases in which the sequence φ of indices in Definition 2.1 (ii) tends to infinity according to an ideal convergence:

Lemma 2.4 ([2]). *Let β be nice, J be an ideal on ω, $f \in {}^\omega C(X)$, and $f \xrightarrow{\beta} 0$.*

(1) *Let the γ-convergence imply the J-convergence (e.g., $\gamma = J$, $\gamma = J\mathrm{QN}$). If X is $\mathrm{w}(\beta, \gamma)$, then there is $\varphi \in F(J)$ such that $\varphi \circ f \xrightarrow{\gamma} 0$.*

(2) *Let the $\gamma(I)$-convergence imply the I-convergence for all ideals I (e.g., $\gamma(I) = I$, $\gamma(I) = I\mathrm{QN}$). If X is $\mathrm{w}(\beta, \gamma(\leq_{\mathrm{K}} J))$, then there are an ideal $I \leq_{\mathrm{K}} J$ and $\varphi \in F(I)$ such that $\varphi \circ f \xrightarrow{\gamma(I)} 0$.*

Proof. Let $h_n(x) = |f_n(x)| + 2^{-n}$. If $\varphi \circ h \xrightarrow{I} 0$, then $\varphi \in F(I)$. □

The following two lemmas exclude trivial properties.

Lemma 2.5. *For ideals I and J on ω the following conditions are equivalent.*

(1) *$I \subseteq J$.*

(2) *Every space is an (I, J)-space.*

(3) *Every space is an $(I\mathrm{QN}, J\mathrm{QN})$-space.*

(4) *Every space is an $(I\mathrm{QN}, J)$-space.*

(5) *There is a nonempty (I, J)-space.*

(6) *There is a nonempty $(I\mathrm{QN}, J\mathrm{QN})$-space.*

(7) *There is a nonempty $(I\mathrm{QN}, J)$-space.*

Proof. Obviously, (1) ⇒ (2) ⇒ (5) ⇒ (7), (1) ⇒ (3) ⇒ (6) ⇒ (7), and (1) ⇒ (4) ⇒ (7).

(7) ⇒ (1) Let $X \neq \emptyset$. Given $a \in I$ assign a sequence $f^a \in {}^{\omega}C(X)$ defined by $f^a_n(x) = 1$, if $n \in a$, and $f^a_n(x) = 0$, if $n \in \omega \setminus a$. For every $a \in I$, $f^a \xrightarrow{I\mathrm{QN}} 0$ and, if $f^a \xrightarrow{J} 0$, then $a \in J$. □

Lemma 2.6. *For any ideals I and J on ω there is no nonempty space which is either $(\leq_{\mathrm{K}} I, J\mathrm{QN})$, or $(\leq_{\mathrm{K}} I, J)$, or $(\leq_{\mathrm{K}} I\mathrm{QN}, J\mathrm{QN})$, or $(\leq_{\mathrm{K}} I\mathrm{QN}, J)$.*

Proof. It is enough to prove that there is no nonempty $(\leq_{\mathrm{K}} I\mathrm{QN}, J)$-space because the other properties imply this property. Fix $a \in [\omega]^{\omega} \setminus J^*$ and let $\varphi \in {}^{\omega}\omega$ be a one-to-one function with $\mathrm{rng}(\varphi) = a$ and $I' = \varphi^{\rightarrow}(\mathrm{Fin}) = \mathrm{Fin} \vee \langle \omega \setminus a \rangle$. Then $I' \leq_{\mathrm{K}} I$ because $I' \subseteq \varphi^{\rightarrow}(I)$. Therefore, if X is a $(\leq_{\mathrm{K}} I\mathrm{QN}, J)$-space, then X is an $(I'\mathrm{QN}, J)$-space. By Lemma 2.5 (7), $X = \emptyset$ because $I' \nsubseteq J$ (in fact, $\omega \setminus a \in I' \setminus J$). □

Lemma 2.7. *If β is nice, then $(\beta, \leq_{\mathrm{K}} \mathrm{Fin}\mathrm{QN}) \Leftrightarrow \mathrm{w}(\beta, \mathrm{Fin}\mathrm{QN})$.*

Proof. $J \leq_K \mathrm{Fin}$ if and only if there is $a \in [\omega]^\omega$ such that $J \subseteq \mathrm{Fin} \vee \langle \omega \setminus a\rangle$ (if $J \subseteq \psi^{\rightarrow}(\mathrm{Fin})$, then $\psi \in F(\mathrm{Fin})$ and let $a = \mathrm{rng}(\psi)$). Assume that $f \xrightarrow{\beta} 0$.

($\Rightarrow$) By applying ($\beta, \leq_K \mathrm{FinQN}$) we find $J \subseteq \mathrm{Fin} \vee \langle \omega \setminus a\rangle$ such that $f \xrightarrow{J\mathrm{QN}} 0$. Let φ be a one-to-one enumeration of a. Then $f \circ \varphi \xrightarrow{\mathrm{FinQN}} 0$ because $f{\restriction}a$ is $[a]^{<\omega}$QN-converging to 0 and $f \circ \varphi$ is an enumeration of $f{\restriction}a$.

($\Leftarrow$) By Lemma 2.4 (1), by applying $\mathrm{w}(\beta, \mathrm{FinQN})$ we can find $\varphi \in F(\mathrm{Fin})$ such that $f \circ \varphi \xrightarrow{\mathrm{FinQN}} 0$. Then every subsequence of $f \circ \varphi$ FinQN-converges to 0 and hence we can choose φ strictly increasing. Denote $a = \mathrm{rng}(\varphi)$ and $J = \mathrm{Fin} \vee \langle \omega \setminus a\rangle$. Then $J \leq_K \mathrm{Fin}$ and $f \xrightarrow{J\mathrm{QN}} 0$. This verifies ($\beta, \leq_K \mathrm{FinQN}$). □

A similar result for J-pointwise convergence holds for all ideals J:

Lemma 2.8. *Let I and J be ideals on ω, β be nice, and γ arbitrary.*

(a) $\mathrm{w}(\beta, J) \Leftrightarrow \mathrm{w}(\beta, \leq_K J) \Leftrightarrow (\beta, \leq_K J)$.

(b) $\mathrm{w}(\beta, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(\beta, \leq_K J\mathrm{QN})$.

(c) $\mathrm{w}(I, \gamma) \Leftrightarrow \mathrm{w}(\leq_K I, \gamma)$.

(d) $\mathrm{w}(I\mathrm{QN}, \gamma) \Leftrightarrow \mathrm{w}(\leq_K I\mathrm{QN}, \gamma)$.

Proof. The following implications are trivial: (a) $\mathrm{w}(\beta, J) \Rightarrow \mathrm{w}(\beta, \leq_K J)$ and $(\beta, \leq_K J) \Rightarrow \mathrm{w}(\beta, \leq_K J)$; (b) $\mathrm{w}(\beta, J\mathrm{QN}) \Rightarrow \mathrm{w}(\beta, \leq_K J\mathrm{QN})$; (c) $\mathrm{w}(\leq_K I, \gamma) \Rightarrow \mathrm{w}(I, \gamma)$; (d) $\mathrm{w}(\leq_K I\mathrm{QN}, \gamma) \Rightarrow \mathrm{w}(I\mathrm{QN}, \gamma)$. We prove the inverse implications. Let X be arbitrary space and let $f \in {}^\omega C(X)$.

(a) $\mathrm{w}(\beta, \leq_K J) \Rightarrow \mathrm{w}(\beta, J)$ and $\mathrm{w}(\beta, \leq_K J) \Rightarrow (\beta, \leq_K J)$. Assume that by applying $\mathrm{w}(\beta, \leq_K J)$ and due to Lemma 2.4 we have found $K \leq_K J$ and $\varphi \in F(K)$ such that $\varphi \circ f \xrightarrow{K} 0$. Let $\psi \in {}^\omega\omega$ be such that $K \subseteq \psi^{\rightarrow}(J)$. Then $\varphi \circ f \xrightarrow{\psi^{\rightarrow}(J)} 0$ and by Lemma 1.2 (a) (both directions), $(\psi \circ \varphi) \circ f \xrightarrow{J} 0$ and $f \xrightarrow{(\psi\circ\varphi)^{\rightarrow}(J)} 0$, where $(\psi \circ \varphi)^{\rightarrow}(J) \leq_K J$.

(b) $\mathrm{w}(\beta, \leq_K J\mathrm{QN}) \Rightarrow \mathrm{w}(\beta, J\mathrm{QN})$. Assume that by applying $\mathrm{w}(\beta, \leq_K J\mathrm{QN})$ we have $K \leq_K J$ and $\varphi \in {}^\omega\omega$ such that $\varphi \circ f \xrightarrow{K\mathrm{QN}} 0$. Let $\psi \in F(J)$ be such that $K \subseteq \psi^{\rightarrow}(J)$. Then $\varphi \circ f \xrightarrow{\psi^{\rightarrow}(J)\mathrm{QN}} 0$. By Lemma 1.2 (b), $(\psi \circ \varphi) \circ f \xrightarrow{J\mathrm{QN}} 0$.

(c)–(d) $\mathrm{w}(I, \gamma) \Rightarrow \mathrm{w}(\leq_K I, \gamma)$ and $\mathrm{w}(I\mathrm{QN}, \gamma) \Rightarrow \mathrm{w}(\leq_K I\mathrm{QN}, \gamma)$. Assume that $f \xrightarrow{\leq_K I} 0$ (or $f \xrightarrow{\leq_K I\mathrm{QN}} 0$). Hence for some $\varphi \in F(I)$, $f \xrightarrow{\varphi^{\rightarrow}(I)} 0$ (or $f \xrightarrow{\varphi^{\rightarrow}(I)\mathrm{QN}} 0$). Then by Lemma 1.2 (a) (or Lemma 1.2 (b)), $\varphi \circ f \xrightarrow{I} 0$ (or $\varphi \circ f \xrightarrow{I\mathrm{QN}} 0$). Now, applying $\mathrm{w}(I, \gamma)$ (or $\mathrm{w}(I\mathrm{QN}, \gamma)$) we find $\psi \in {}^\omega\omega$ such that $(\psi \circ \varphi) \circ f \xrightarrow{\gamma} 0$. □

There are together 2×16 properties of the form (β, γ) and $\mathrm{w}(\beta, \gamma)$ for $\beta \in \{I\mathrm{QN}, I, \le_{\mathrm{K}} I\mathrm{QN}, \le_{\mathrm{K}} I\}$ and $\gamma \in \{J\mathrm{QN}, J, \le_{\mathrm{K}} J\mathrm{QN}, \le_{\mathrm{K}} J\}$:

$-$ $(I\mathrm{QN}, J\mathrm{QN})$	$-$ $(I\mathrm{QN}, J)$	2 $(I\mathrm{QN}, \le_{\mathrm{K}} J\mathrm{QN})$	8 $(I\mathrm{QN}, \le_{\mathrm{K}} J)$
1 $(I, J\mathrm{QN})$	$-$ (I, J)	3 $(I, \le_{\mathrm{K}} J\mathrm{QN})$	9 $(I, \le_{\mathrm{K}} J)$
$-$ $(\le_{\mathrm{K}} I\mathrm{QN}, J\mathrm{QN})$	$-$ $(\le_{\mathrm{K}} I\mathrm{QN}, J)$	4 $(\le_{\mathrm{K}} I\mathrm{QN}, \le_{\mathrm{K}} J\mathrm{QN})$	8 $(\le_{\mathrm{K}} I\mathrm{QN}, \le_{\mathrm{K}} J)$
$-$ $(\le_{\mathrm{K}} I, J\mathrm{QN})$	$-$ $(\le_{\mathrm{K}} I, J)$	5 $(\le_{\mathrm{K}} I, \le_{\mathrm{K}} J\mathrm{QN})$	9 $(\le_{\mathrm{K}} I, \le_{\mathrm{K}} J)$
6 $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN})$	8 $\mathrm{w}(I\mathrm{QN}, J)$	6 $\mathrm{w}(I\mathrm{QN}, \le_{\mathrm{K}} J\mathrm{QN})$	8 $\mathrm{w}(I\mathrm{QN}, \le_{\mathrm{K}} J)$
7 $\mathrm{w}(I, J\mathrm{QN})$	9 $\mathrm{w}(I, J)$	7 $\mathrm{w}(I, \le_{\mathrm{K}} J\mathrm{QN})$	9 $\mathrm{w}(I, \le_{\mathrm{K}} J)$
6 $\mathrm{w}(\le_{\mathrm{K}} I\mathrm{QN}, J\mathrm{QN})$	8 $\mathrm{w}(\le_{\mathrm{K}} I\mathrm{QN}, J)$	6 $\mathrm{w}(\le_{\mathrm{K}} I\mathrm{QN}, \le_{\mathrm{K}} J\mathrm{QN})$	8 $\mathrm{w}(\le_{\mathrm{K}} I\mathrm{QN}, \le_{\mathrm{K}} J)$
7 $\mathrm{w}(\le_{\mathrm{K}} I, J\mathrm{QN})$	9 $\mathrm{w}(\le_{\mathrm{K}} I, J)$	7 $\mathrm{w}(\le_{\mathrm{K}} I, \le_{\mathrm{K}} J\mathrm{QN})$	9 $\mathrm{w}(\le_{\mathrm{K}} I, \le_{\mathrm{K}} J)$

The properties $(I\mathrm{QN}, J\mathrm{QN})$, $(I\mathrm{QN}, J)$, (I, J) are trivial by Lemma 2.5 and the properties $(\le_{\mathrm{K}} I\mathrm{QN}, J\mathrm{QN})$, $(\le_{\mathrm{K}} I\mathrm{QN}, J)$, $(\le_{\mathrm{K}} I, J\mathrm{QN})$, $(\le_{\mathrm{K}} I, J)$ are trivial by Lemma 2.6. This is indicated by the minus signs $-$. Due to Lemma 2.8 the remaining 25 properties in the table are divided into the following 9 equivalence classes. The numbers 1–9 indicate the equivalence classes and the circled numbers indicate the representatives that we will use from now on.

1. $(I, J\mathrm{QN})$.

2. $(I\mathrm{QN}, \le_{\mathrm{K}} J\mathrm{QN})$.

3. $(I, \le_{\mathrm{K}} J\mathrm{QN})$

4. $(\le_{\mathrm{K}} I\mathrm{QN}, \le_{\mathrm{K}} J\mathrm{QN})$.

5. $(\le_{\mathrm{K}} I, \le_{\mathrm{K}} J\mathrm{QN})$.

6. $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(I\mathrm{QN}, \le_{\mathrm{K}} J\mathrm{QN})$
   $\Leftrightarrow \mathrm{w}(\le_K I\mathrm{QN}, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(\le_K I\mathrm{QN}, \le_{\mathrm{K}} J\mathrm{QN})$. Lemma 2.8 (b), (d).

7. $\mathrm{w}(I, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(I, \le_{\mathrm{K}} J\mathrm{QN}) \Leftrightarrow \mathrm{w}(\le_K I, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(\le_K I, \le_{\mathrm{K}} J\mathrm{QN})$.
   Lemma 2.8 (b), (c).

8. $\mathrm{w}(I\mathrm{QN}, J) \Leftrightarrow \mathrm{w}(I\mathrm{QN}, \le_{\mathrm{K}} J) \Leftrightarrow (I\mathrm{QN}, \le_{\mathrm{K}} J)$
   $\Leftrightarrow \mathrm{w}(\le_K I\mathrm{QN}, J) \Leftrightarrow \mathrm{w}(\le_K I\mathrm{QN}, \le_{\mathrm{K}} J) \Leftrightarrow (\le_K I\mathrm{QN}, \le_{\mathrm{K}} J)$.
   Lemma 2.8 (a), (d).

9. $\mathrm{w}(I, J) \Leftrightarrow \mathrm{w}(I, \le_{\mathrm{K}} J) \Leftrightarrow (I, \le_{\mathrm{K}} J)$
   $\Leftrightarrow \mathrm{w}(\le_K I, J) \Leftrightarrow \mathrm{w}(\le_K I, \le_{\mathrm{K}} J) \Leftrightarrow (\le_K I, \le_{\mathrm{K}} J)$. Lemma 2.8 (a), (c).

As a consequence we get the following implications between the properties:

Lemma 2.9. *Let I, J, and $K \subseteq J$ be any ideals on ω. Then*

$$\begin{array}{ccccccc}
(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN}) & \Rightarrow & (I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN}) & \Rightarrow & \mathrm{w}(I\mathrm{QN}, J\mathrm{QN}) & \Rightarrow & \mathrm{w}(I\mathrm{QN}, J) \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN}) & \Rightarrow & (I, \leq_{\mathrm{K}} J\mathrm{QN}) & \Rightarrow & \mathrm{w}(I, J\mathrm{QN}) & \Rightarrow & \mathrm{w}(I, J) \\
 & & \Uparrow & & & & \\
 & & (I, J\mathrm{QN}) & & \Uparrow & & \\
 & & \Uparrow & & & & \\
 & & (I, JK\mathrm{QN}) & \Rightarrow & \mathrm{w}(I, JK\mathrm{QN}) & &
\end{array}$$

Proof. By 6., $(I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN}) \Rightarrow \mathrm{w}(I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN}) \Leftrightarrow \mathrm{w}(I\mathrm{QN}, J\mathrm{QN})$ and, by 7., $(I, \leq_{\mathrm{K}} J\mathrm{QN}) \Rightarrow \mathrm{w}(I, \leq_{\mathrm{K}} J\mathrm{QN}) \Leftrightarrow \mathrm{w}(I, J\mathrm{QN})$. The other implications are obvious. $\square$

In next section we find combinatorial characterizations of cardinal invariants $\mathrm{non}(P)$ for all properties P in Lemma 2.9. Not all these properties can be distinguished through this invariant since it is the same for 5 of them, see Theorem 3.11. We do not know whether all these properties are provably or consistently different at least for some pairs of ideals. By next lemma, for every property within Lemma 2.9 weaker than $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})$ there exists a nonempty space with this property. For the remaining 3 properties the existence of a nonempty space requires $I \subseteq J$.

Lemma 2.10. *For arbitrary ideals I and J on ω the following conditions hold:*

(1) $(\forall \varphi \in F(I))(\exists \psi \in F(\mathrm{Fin}))\ \psi \circ \varphi \in F(\mathrm{Fin})$ *and* $\varphi \in F(\psi^{\to}(\mathrm{Fin}))$.

(2) *There is a nonempty $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})$-space.*

Proof. (1) Let $\varphi \in F(I)$. For every $n \in \omega$ define $\psi(n) = \min(\varphi^{-1}(\{k\}))$ where k is the nth member of $\mathrm{rng}(\varphi)$, and hence $(\psi \circ \varphi)(n) = \varphi(\psi(n)) = k$ if and only if k is the nth member of $\mathrm{rng}(\varphi)$. Therefore the functions $\psi \circ \varphi$ and ψ are one-to-one. For every $k \in \omega$, $\varphi^{-1}(\{k\}) \in \psi^{\to}(\mathrm{Fin})$ because $\psi^{-1}(\varphi^{-1}(\{k\})) = (\psi \circ \varphi)^{-1}(\{k\}) \in [\omega]^{\leq 1}$.

(2) We prove that the singleton space $X = 1$ is $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})$. Let $f \in {}^{\omega}\mathbb{R} \simeq {}^{\omega}C(X)$ be arbitrary such that $f \xrightarrow{I'} 0$ for some $I' \leq_{\mathrm{K}} I$. Then, by Lemma 1.4 (2), $\tau(f) \in F(I')$, by (1) there is $\psi \in F(\mathrm{Fin})$ such that $\tau(f) \in F(\psi^{\to}(\mathrm{Fin}))$, and by Lemma 1.4 (2), $f \xrightarrow{\psi^{\to}(\mathrm{Fin})} 0$. Now, $f \xrightarrow{\psi^{\to}(\mathrm{Fin})\mathrm{QN}} 0$ because $|X| = 1$ and $\psi^{\to}(\mathrm{Fin}) \leq_{\mathrm{K}} J$. $\square$

Lemma 2.11 ([2]). *There is a nonempty $(I, J\mathrm{QN})$-space if and only if $I \subseteq J$.*

Proof. A singleton is an $(I, J\mathrm{QN})$-space, if $I \subseteq J$. Conversely, an $(I, J\mathrm{QN})$-space is an (I, J)-space and, by Lemma 2.5, it exists only if $I \subseteq J$. $\square$

A base of an ideal I is a set $B \subseteq I$ such that $(\forall x \in I)(\exists b \in B)\ x \subseteq b$.

Theorem 2.12. *Let J, K, and $I \subseteq J \cap K$ be ideals on ω and $\mathcal{X}$ be a class of spaces. Then:*

(a) $\mathrm{non}_{\mathcal{X}}((I, JK\mathrm{QN})$*-space*$) \geq \omega_1$.

(b) *If I has a countable base, then* $\mathrm{non}_{\mathcal{X}}((I, JK\mathrm{QN})$*-space*$) \geq \mathfrak{b}$.

Proof. Let X be a space, countable in case (a), and of cardinality $< \mathfrak{b}$ in case (b). Let $f \in {}^{\omega}C(X)$ be such that $f \xrightarrow{I} 0$. We prove that $f \xrightarrow{I\mathrm{QN}} 0$. For every $x \in X$ and $k \in \omega$ let $b_{x,k} = \{n \in \omega : f_n(x) \geq 2^{-k}\} \in I$. There is an increasing sequence of sets $\{a_n : n \in \omega\} \subseteq I$ such that $\bigcup_{n\in\omega} a_n = \omega$ and for every $x \in X$ and $k \in \omega$ there is $n \in \omega$ such that $b_{x,k} \subseteq a_n$ (case (a) $|X| \leq \omega$ and let $a_n = n \cup$ "a finite union of $b_{x,k}$'s"; case (b) let $\{a_n : n \in \omega\}$ be a base of I). For every $x \in X$ let $\varphi_x \in {}^{\omega}\omega$ be such that $(\forall k \in \omega)\ b_{x,k} \subseteq a_{\varphi_x(k)}$. Since $|X| < \mathfrak{b}$ there is an increasing function $\varphi \in {}^{\omega}\omega$ such that $(\forall x \in X)(\exists k_x \in \omega)$ $(\forall k \geq k_x)\ \varphi_x(k) \leq \varphi(k)$. Define $\varepsilon_n = 2^{-k}$, if $n \in a_{\varphi(k+1)} \setminus a_{\varphi(k)}$. Clearly, $\varepsilon \xrightarrow{I} 0$. For every $x \in X$ and $k \geq k_x$, $a_{\varphi(k)} \supseteq a_{\varphi_x(k)} \supseteq b_{x,k}$ and therefore $\{n \in \omega : f_n(x) \geq \varepsilon_n\} \subseteq a_{\varphi(k_x)} \cup \bigcup_{k\geq k_x}\{n \in a_{\varphi(k+1)} \setminus a_{\varphi(k)} : f_n(x) \geq 2^{-k}\} \subseteq a_{\varphi(k_x)} \cup \bigcup_{k\geq k_x}(b_{x,k} \setminus a_{\varphi(k)}) = a_{\varphi(k_x)} \in I$. $\square$

Hence, if $I \subseteq J$, then $\mathrm{non}((I, J\mathrm{QN})\text{-space}) \geq \omega_1$. In fact, $\mathrm{non}(P) \geq \omega_1$ also for all properties P within Lemma 2.9 that are weaker than $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})$ for any ideals I and J on countable sets (see Remark 3.7). By Theorem 3.6 and Theorem 3.11 the subscripts $\mathcal{X}$ can be omitted at $\mathrm{non}(P)$.

3 Cardinal invariants

An ideal J on ω is said to be a $P(I)$-ideal for an ideal I, if for every sequence of sets $\{a_n : n \in \omega\} \subseteq J$ there is $c \in J^*$ such that $a_n \cap c \in I$ for all $n \in \omega$. Sequences of sets in this definition can be equivalently replaced by partitions of ω and partitions $\{a_n : n \in \omega\} \subseteq J$ can be expressed by functions $\alpha \in F(J)$ defined by $\alpha(k) = n$ for $k \in a_n$; i.e., $a_n = \|\alpha = n\|$ for all $n \in \omega$. Therefore J is a $P(I)$-ideal if and only if $(\forall \alpha \in F(J))(\exists c \in J^*)(\forall n \in \omega)\ \|\alpha = n\| \cap c \in I$. We need similar properties.

Definition 3.1. Let I, J, K be ideals on ω.

J is a weak $P(I)$-ideal, if $(\forall \alpha \in F(J))(\exists c \in J^+)(\forall n \in \omega)\ \|\alpha = n\| \cap c \in I$.

J is a $W(I)$-ideal, if $(\forall\alpha \in F(J))(\forall\varphi \in F(J))(\exists c \in J^+)(\forall n \in \omega)$
$$\varphi(\|\alpha = n\| \cap c) \in I.$$

K is a weak $P(I,J)$-ideal, if $(\forall\alpha \in F(K))(\exists c \in J^+)(\forall n \in \omega)\ \|\alpha = n\| \cap c \in I$.

K is a $W(I,J)$-ideal, if $(\forall\alpha \in F(K))(\forall\varphi \in F(J))(\exists c \in J^+)(\forall n \in \omega)$
$$\varphi(\|\alpha = n\| \cap c) \in I.$$

Hence, J is a weak $P(I)$-ideal $\Leftrightarrow$ J is a weak $P(I,J)$-ideal; J is a $W(I)$-ideal $\Leftrightarrow$ J is a $W(I,J)$-ideal. A weak $P(\mathrm{Fin})$-ideal is called a weak P-ideal.

The property of a $P(I)$-ideal was introduced by Filipów and Staniszewski in [8] as a generalization of the property of a P-ideal; it can be found under a different name in [12]. The property of a weak $P(I)$-ideal was introduced by Šupina [15] as a generalization of a weak P-ideal; he collected several characterizations of a weak P-ideal. A weak $P(I,J)$-ideal, a $W(I)$-ideal, and a $W(I,J)$-ideal are new properties. However, the property of a weak $P(I,J)$-ideal is related to the property $W(I,J,K)$ of Staniszewski [14] where $W(J,K,I)$ $\Leftrightarrow$ for every partition $\{a_n : n \in \omega\} \subseteq K$ of ω there exists $c \in J^+$ such that $a_0 \cap c \in J$ and $a_{n+1} \cap c \in I$ for all $n \in \omega$. If $K \subseteq J$, then $W(J,K,I)$ holds if and only if K is a $P(I,J)$-ideal (consider $a_n = \|\alpha = n\|$ and note that $c \setminus a_0 \in J^+$ whenever $c \in J^+$ and $a_0 \in K \subseteq J$).

Clearly, Fin is a $W(\mathrm{Fin})$-ideal. Every $W(I,J)$-ideal is a weak $P(I,J)$-ideal and every $P(\mathrm{Fin},J)$-ideal is a $W(\mathrm{Fin},J)$-ideal. K is a $W(I,J)$-ideal if and only if K is a weak $P(\varphi^{\leftarrow}(I),J)$-ideal for every $\varphi \in F(J)$ such that $\mathrm{rng}(\varphi) \notin I$ (and $I \subseteq \varphi^{\rightarrow}(J)$; use $c = \varphi^{-1}(x)$ for $x \in I \setminus \varphi^{\rightarrow}(J)$).

Lemma 3.2. (a) *If $I \not\subseteq J$, then J is a weak $P(I)$-ideal and every ideal is a weak $P(I,J)$-ideal.* (b) *If $I \not\leq_{\mathrm{K}} J$, then J is a $W(I)$-ideal and every ideal is a $W(I,J)$-ideal.*

PROOF. (a) The definition of a weak $P(I,J)$-ideal is fulfilled by any $c \in I \setminus J$.

(b) If K is not a $W(I,J)$-ideal and $\alpha \in F(K)$ and $\varphi \in F(J)$ witness this, then $I \subseteq \varphi^{\rightarrow}(J)$: For $x \in I$, $\varphi^{-1}(x) \in J$ because for $c = \varphi^{-1}(x)$ and every $n \in \omega$, $\varphi(\|\alpha = n\| \cap c) \subseteq \varphi(c) = x \in I$. $\square$

Example 3.3. (1) Let an ideal I be "sufficiently" thinner than a maximal ideal. There are ideals $J_1 \supseteq J_0 \supseteq I$ such that J_0 is a weak $P(I)$-ideal that is not a $P(I)$-ideal and every ideal $J \supseteq J_1$ is not a weak $P(I)$-ideal.

(2a) For every ideal $I \neq \mathrm{Fin}$ there is an ideal J such that $I \not\subseteq J$ and I is not a $W(I,J)$-ideal; I is trivially a weak $P(I,J)$-ideal because $I \not\subseteq J$.

(2b) There is an ideal I that is not a $W(I)$-ideal; I is trivially a $P(I)$-ideal.

Proof. (1) Let $\{a_n : n \in \omega\} \subseteq I^+$ be a partition of ω, $J_0 = \{x \subseteq \omega : (\exists n \in \omega)\ x \setminus \bigcup_{k=1}^{n} a_k \in I\}$, and $J_1 = \{x \subseteq \omega : (\forall^\infty n \in \omega)\ x \cap a_k \in I\}$. J_0 is a weak $P(I)$-ideal because $a_0 \in J^+$ and $(\forall x \in J_0)\ a_0 \cap x \in I$. J_0 is not a $P(I)$-ideal because there is no $x \in J_0$ such that $a_n \setminus x \in I$ for all $n \geq 1$. An ideal $J \supseteq J_1$ is not a weak $P(I)$-ideal because $a_n \in J_1$ and if $x \subseteq \omega$ is such that $a_n \cap x \in I$ for all $n \in \omega$, then $x \in J_1 \subseteq J$.

(2a) Let $\{a_n : n \in \omega\} \subseteq I \cap [\omega]^\omega$ be a partition of ω, let $\{k_{n,m} : m \in \omega\}$ be the increasing enumeration of a_n, and let $b_m = \{k_{n,m} : n \in \omega\}$. Define $\alpha, \varphi \in {}^\omega\omega$ so that $\|\alpha = n\| = a_n$ and $\|\varphi = m\| = b_m$ for all $n, m \in \omega$; then $\alpha \in F(I)$. Let $J = \mathcal{J}(I) = \{x \subseteq \omega : (\forall n \in \omega)\ \varphi(\|\alpha = n\| \cap x) \in I\}$. Then J is an ideal on ω ($\omega \notin J$ because $\varphi(\|\alpha = n\|) = \omega \notin I$) and $\varphi \in F(J)$. $I \not\subseteq J$ because $a_n \in I \setminus J$. By definition of J it follows that I is not a $W(I, J)$-ideal.

(2b) Let I be the union of the increasing sequence of ideals $\{I_n : n \in \omega\}$ defined by induction: I_0 is the ideal generated by the partition $\{a_n : n \in \omega\}$ from (2a) and $I_{n+1} = \mathcal{J}(I_n) \vee I_0$; I_{n+1} is well defined (i.e., $\omega \notin I_{n+1}$) because $a_n \notin \mathcal{J}(I_n)$ for all $n \in \omega$. I is not a $W(I)$-ideal because $I \supseteq \mathcal{J}(I)$. $\square$

Next Theorem 3.4 (a) is similar to [9, Theorem 1.2]; Corollary 3.8 (a) is similar to [9, Theorem 5.1]; and Corollary 3.8 (b) is similar to [11, Proposition 2] because $\|\beta < \alpha\| = \bigcup_{n \in \omega}(\|\alpha = n\| \cap \bigcup_{k<n} \|\beta = k\|)$.

Theorem 3.4. *Let I, J, and $K \subseteq J$ be ideals on ω.*

(a) *The following conditions are equivalent:*

(1) *K is not a weak $P(I, J)$-ideal.*

(2) $(\exists \alpha \in F(K))(\forall \beta \in F(I))\ \|\beta \leq \alpha\| \in J$.

(3) $(\exists \alpha \in F(K))(\forall \beta \in F(I))\ \|\beta < \alpha\| \in J$.

(4) *Every space is an $(I, JK\mathrm{QN})$-space.*

(5) *$F(I)$ as a subset of the Baire space is an $(I, JK\mathrm{QN})$-space.*

(b) *The following conditions are equivalent:*

(1) *K is not a $W(I, J)$-ideal.*

(2) $(\exists \alpha \in F(K))(\exists \varphi \in F(J))(\forall \beta \in F(I))\ \|\varphi \circ \beta \leq \alpha\| \in J$.

(3) $(\exists \alpha \in F(K))(\exists \varphi \in F(J))(\forall \beta \in F(I))\ \|\varphi \circ \beta < \alpha\| \in J$.

(4) *Every space is a $\mathrm{w}(I, JK\mathrm{QN})$-space.*

(5) *$F(I)$ as a subset of the Baire space is a $\mathrm{w}(I, JK\mathrm{QN})$-space.*

(c) *The following conditions are equivalent:*

(1) $(\exists J' \leq_{\mathrm{RK}} J$ *on* $\omega)$ J' *is not a weak* $P(I)$*-ideal.*

(2) $(\exists J' \leq_{\mathrm{RK}} J$ *on* $\omega)(\exists \alpha \in F(J'))(\forall \beta \in F(I))$ $\|\beta \leq \alpha\| \in J'$.

(3) $(\exists J' \leq_{\mathrm{RK}} J$ *on* $\omega)(\exists \alpha \in F(J'))(\forall \beta \in F(I))$ $\|\beta < \alpha\| \in J'$.

(4) *Every space is an* $(I, \leq_{\mathrm{K}} J\mathrm{QN})$*-space.*

(5) $F(I)$ *as a subset of the Baire space is an* $(I, \leq_{\mathrm{K}} J\mathrm{QN})$*-space.*

(d) *The following conditions are equivalent:*

(1) $(\forall I' \leq_{\mathrm{RK}} I$ *on* $\omega)(\exists J' \leq_{\mathrm{RK}} J$ *on* $\omega)$ J' *is not a weak* $P(I')$*-ideal.*

(2) $(\forall I' \leq_{\mathrm{RK}} I$ *on* $\omega)(\exists J' \leq_{\mathrm{RK}} J$ *on* $\omega)(\exists \alpha \in F(J'))(\forall \beta \in F(I'))$
$\|\beta \leq \alpha\| \in J'$.

(3) $(\forall I' \leq_{\mathrm{RK}} I$ *on* $\omega)(\exists J' \leq_{\mathrm{RK}} J$ *on* $\omega)(\exists \alpha \in F(J'))(\forall \beta \in F(I'))$
$\|\beta < \alpha\| \in J'$.

(4) *Every space is a* $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})$*-space.*

(5) $(\forall I' \leq_{\mathrm{K}} I$ *on* $\omega)$ $F(I') \subseteq {}^{\omega}\omega$ *is a* $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})$*-space.*

The quantifiers $(\forall I' \leq_{\mathrm{RK}} I)$ and $(\exists J' \leq_{\mathrm{RK}} J)$ in cases (c) and (d) of Theorem 3.4 can be equivalently replaced by $(\forall I' \leq_{\mathrm{K}} I)$ and $(\exists J' \leq_{\mathrm{K}} J)$, respectively, because $\leq_{\mathrm{K}} = \subseteq \circ \leq_{\mathrm{RK}}$ and the properties following these quantifiers are monotone with respect to the inclusion of ideals.

Proof. (a) (1) $\Rightarrow$ (2) If $\alpha \in F(K)$ witnesses that K is not weak $P(I, J)$, then for every $\beta \in F(I)$, $\|\beta \leq \alpha\| \in J$ because $\|\alpha = n\| \cap \|\beta \leq \alpha\| \subseteq \|\beta \leq n\| \in I$ for all $n \in \omega$.

(2) $\Rightarrow$ (1) Assume that $\alpha \in F(K)$ witnesses (2) and let $c \subseteq \omega$ be arbitrary such that $\|\alpha = n\| \cap c \in I$ for all $n \in \omega$. Define $\beta \in {}^{\omega}\omega$ by $\beta(k) = \alpha(k)$, if $k \in c$, and $\beta(k) = k$, otherwise. Then $\beta \in F(I)$ because $\|\beta = n\| \subseteq (\|\alpha = n\| \cap c) \cup \{n\} \in I$ for all $n \in \omega$ and $c \in J$ because $c \subseteq \|\beta \leq \alpha\| \in J$. Therefore K is not weak $P(I, J)$.

(2) $\Rightarrow$ (3) holds because $\|\beta < \alpha\| \subseteq \|\beta \leq \alpha\|$.

(3) $\Rightarrow$ (2) Let $\bar{\alpha}(n) = \max\{\alpha(n) - 1, 0\}$. If $\alpha \in F(K)$ and $\|\beta < \alpha\| \in J$, then $\bar{\alpha} \in F(K)$ and $\|\beta \leq \bar{\alpha}\| \subseteq \|\beta < \alpha\| \cup \|\alpha = 0\| \in J$ because $K \subseteq J$.

(4) $\Rightarrow$ (5) is trivial.

(2) $\Rightarrow$ (4) We apply Lemma 1.4 (1)–(3). Let X be arbitrary space, let $f \in {}^{\omega}C(X)$ be such that $f \xrightarrow{I} 0$, and let $\xi^x = \langle f_k(x) : k \in \omega \rangle$ for $x \in X$.

Then $\tau(\xi^x) \in F(I)$ because $\xi^x \xrightarrow{I} 0$. If $\alpha \in F(K)$ fulfills (2) for all β of the form $\tau(\xi^x)$, then $\sigma(\alpha) \xrightarrow{K} 0$ and for every $x \in X$, $\{k \in \omega : |\xi^x_k| \geq \sigma_k(\alpha)\} \subseteq \|\tau(\xi^x) \leq \alpha\| \in J$. Therefore $f \xrightarrow{JK\mathrm{QN}} 0$ and X is an $(I, JK\mathrm{QN})$-space.

(5) $\Rightarrow$ (3) We use Lemma 1.4 (1), (2), (4). Let $X = F(I)$. Define $f \in {}^\omega C(X)$ by $f_k(\beta) = \sigma_k(\beta)$. Since X is an $(I, JK\mathrm{QN})$-space and $f \xrightarrow{I} 0$, there is an $\varepsilon \in {}^\omega[0,\infty)$ with $\varepsilon \xrightarrow{K} 0$ witnessing the $JK\mathrm{QN}$-convergence of f. Then $\tau(\varepsilon) \in F(K)$ and for every $\beta \in X$, $\|\beta < \tau(\varepsilon)\| \subseteq \{k \in \omega : f_k(\beta) \geq \varepsilon_k\} \in J$.

(b) (1) $\Rightarrow$ (2) If $(\alpha, \varphi) \in F(K) \times F(J)$ witnesses that K is not $W(I,J)$, then for every $\beta \in F(I)$, $\|\varphi \circ \beta \leq \alpha\| = \bigcup_{n\in\omega}(\|\alpha = n\| \cap \varphi^{-1}(\|\beta \leq n\|)) \in J$ because $\varphi(\|\alpha = n\| \cap \varphi^{-1}(\|\beta \leq n\|)) \subseteq \|\beta \leq n\| \in I$ for all $n \in \omega$.

(2) $\Rightarrow$ (1) Let $\alpha \in F(K)$ and $\varphi \in F(J)$ witness (2) and let $c \subseteq \omega$ be arbitrary such that $\varphi(\|\alpha = n\| \cap c) \in I$ for all $n \in \omega$. Define $\beta \in {}^\omega\omega$ by $\beta(m) = \min\{n \in \omega : m \in \varphi(\|\alpha = n\| \cap c)\}$, if $m \in \varphi(c)$, and $\beta(m) = m$, otherwise. Then $\beta \in F(I)$ because $\|\beta = n\| \subseteq \varphi(\|\alpha = n\| \cap c) \cup \{n\} \in I$ for all $n \in \omega$. Then $c \in J$ because $c \subseteq \{k \in \omega : \beta(\varphi(k)) \leq \alpha(k)\} = \|\varphi \circ \beta \leq \alpha\| \in J$. Therefore K is not $W(I,J)$.

(2) $\Leftrightarrow$ (3) holds by same arguments like in (a) and (4) $\Rightarrow$ (5) is trivial.

(2) $\Rightarrow$ (4) We use Lemma 1.4 (1), (2), (5). Let X be arbitrary space, let $f \in {}^\omega C(X)$ be such that $f \xrightarrow{I} 0$, and let $\xi^x = \langle f_k(x) : k \in \omega\rangle$ for $x \in X$. Then $\tau(\xi^x) \in F(I)$ because $\xi^x \xrightarrow{I} 0$. If $\alpha \in F(K)$ fulfills (2) for all β of the form $\tau(\xi^x)$, then $\sigma(\alpha) \xrightarrow{K} 0$ and for every $x \in X$, $\{k \in \omega : |\xi^x_{\varphi(k)}| \geq \sigma_k(\alpha)\} \subseteq \|\varphi \circ \tau(\xi^x) \leq \alpha\| \in J$. Therefore $\varphi \circ f \xrightarrow{JK\mathrm{QN}} 0$ and X is an $\mathrm{w}(I, JK\mathrm{QN})$-space.

(5) $\Rightarrow$ (3) We use Lemma 1.4 (1), (2), (6). Let $X = F(I)$ and define $f \in {}^\omega C(X)$ by $f_k(\beta) = \sigma_k(\beta)$. Since X is an $\mathrm{w}(I, JK\mathrm{QN})$-space and $f \xrightarrow{I} 0$, there are $\varphi \in F(J)$ and $\varepsilon \in {}^\omega[0,\infty)$ K-converging to 0 witnessing the $JK\mathrm{QN}$-convergence of $\varphi \circ f$ to 0. Then $\tau(\varepsilon) \in F(K)$ and for every $\beta \in X$, $\|\varphi \circ \beta < \tau(\varepsilon)\| \subseteq \{k \in \omega : f_{\varphi(k)}(\beta) \geq \varepsilon_k\} \in J$.

(c)–(d) In both cases, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) easily follow from (a1) $\Leftrightarrow$ (a2) $\Leftrightarrow$ (a3) with $K = J = J'$ and (4) $\Rightarrow$ (5) is trivial. Therefore it is enough to prove (2) $\Rightarrow$ (4) and (5) $\Rightarrow$ (3). We prove it in case (d); (c) is the same.

(d2) $\Rightarrow$ (d4) We use Lemma 1.4 (1)–(3). Let X be arbitrary space and let $f \in {}^\omega C(X)$ be such that $f \xrightarrow{\leq_{\mathrm{K}} I} 0$; i.e., there is $I' \leq_{\mathrm{RK}} I$ such that $f \xrightarrow{I'} 0$. For every $x \in X$ let $\xi^x_k = f_k(x)$ for $x \in X$. Then $\tau(\xi^x) \in F(I')$ because $\xi^x \xrightarrow{I'} 0$. If $J' \leq_{\mathrm{RK}} J$ and $\alpha \in F(J')$ fulfill (d2) for all β of the form $\tau(\xi^x)$, then $\sigma(\alpha) \xrightarrow{J'} 0$ and for every $x \in X$, $\{k \in \omega : |f_k(x)| \geq \sigma_k(\alpha)\} \subseteq \|\tau(\xi^x) \leq \alpha\| \in J'$. Therefore $f \xrightarrow{J'\mathrm{QN}} 0$ and $f \xrightarrow{\leq_{\mathrm{K}} J\mathrm{QN}} 0$ and hence X is a $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})$-space.

(d5) $\Rightarrow$ (d3) We use Lemma 1.4 (1), (2), (4). Let $I' \leq_{\mathrm{RK}} I$ be an ideal on ω, let $X = F(I')$ and let $f \in {}^{\omega}C(X)$ be defined by $f_k(\beta) = \sigma_k(\beta)$. Since X is an $(I', \leq_{\mathrm{K}} J\mathrm{QN})$-space and $f \xrightarrow{I'} 0$, there are $J' \leq_{\mathrm{RK}} J$ and $\varepsilon \in {}^{\omega}[0, \infty)$ J'-converging to 0 witnessing the $J'\mathrm{QN}$-convergence of f. Then $\tau(\varepsilon) \in F(J')$ and for every $\beta \in X$, $\|\beta < \tau(\varepsilon)\| \subseteq \{k \in \omega : f_k(\beta) \geq \varepsilon_k\} \in J'$. $\square$

Corollary 3.5. *For any ideals I and J on ω,*

$$I \not\leq_{\mathrm{K}} J \Rightarrow J \textit{ is a } W(I)\textit{-ideal} \Rightarrow (\forall J' \leq_{\mathrm{RK}} J)\ J' \textit{ is weak } P(I)\textit{-ideal}$$
$$\Rightarrow J \textit{ is a weak } P(I)\textit{-ideal.}$$

PROOF. The first implication follows by Lemma 3.2. The other implications are consequences of characterizations of properties of ideals in Theorem 3.4 for $K = J$ because by Lemma 2.9, $(I, J\mathrm{QN}) \Rightarrow (I, \leq_{\mathrm{K}} J\mathrm{QN}) \Rightarrow \mathrm{w}(I, J\mathrm{QN})$. $\square$

Theorem 3.6. *Let I, J, and $K \subseteq J$ be ideals on ω. For reasonable classes of spaces the following holds:*

(a) *If K is a weak $P(I, J)$-ideal, then*

$$\mathrm{non}((I, JK\mathrm{QN})\textit{-space}) = \min\{|X| : X \subseteq F(I) \textit{ and }$$
$$(\forall \alpha \in F(K))(\exists \beta \in X)\ \|\beta < \alpha\| \notin J\} \leq \mathfrak{c},$$

otherwise, every space is an $(I, JK\mathrm{QN})$-space.

(b) *If K is a $W(I, J)$-ideal, then*

$$\mathrm{non}(\mathrm{w}(I, JK\mathrm{QN})\textit{-space}) = \min\{|X| : X \subseteq F(I) \textit{ and }$$
$$(\forall \alpha \in F(K))(\forall \varphi \in F(J))(\exists \beta \in X)\ \|\varphi \circ \beta < \alpha\| \notin J\} \leq \mathfrak{c},$$

otherwise, every space is a $\mathrm{w}(I, JK\mathrm{QN})$-space.

(c) *If every ideal $\leq_{\mathrm{K}}$-below J is a weak $P(I)$-ideal, then*

$$\mathrm{non}((I, \leq_{\mathrm{K}} J\mathrm{QN})\textit{-space}) = \min\{|X| : X \subseteq F(I) \textit{ and } (\forall J' \leq_{\mathrm{RK}} J \textit{ on } \omega)$$
$$(\forall \alpha \in F(J'))(\exists \beta \in X)\ \|\beta < \alpha\| \notin J'\} \leq \mathfrak{c},$$

otherwise, every space is an $(I, \leq_{\mathrm{K}} JK\mathrm{QN})$-space.

(d) *If there is an ideal $I' \leq_{\mathrm{K}} I$ on ω such that every ideal $J' \leq_{\mathrm{K}} J$ on ω is a weak $P(I')$-ideal, then*

$$\mathrm{non}((\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})\textit{-space}) = \min\{\mathrm{non}((I', \leq_{\mathrm{K}} J\mathrm{QN})\textit{-space}) : I' \leq_{\mathrm{RK}} I\} \leq \mathfrak{c},$$

otherwise, every space is a $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} JK\mathrm{QN})$-space.

Proof. (a)–(c) The equivalences (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) in assertions (a)–(c) of Theorem 3.4 hold by same proofs when reduced to spaces and subsets of $F(I)$ of a given cardinality. For example, in case (a), for every (finite or infinite) cardinal number κ the following conditions are equivalent:

(2) $(\forall X \in [F(I)]^{\leq\kappa})(\exists \alpha \in F(K))(\forall \beta \in X)\ \|\beta \leq \alpha\| \in J$.

(3) $(\forall X \in [F(I)]^{\leq\kappa})(\exists \alpha \in F(K))(\forall \beta \in X)\ \|\beta < \alpha\| \in J$.

(4) Every space of cardinality $\leq \kappa$ is an $(I, JK\mathrm{QN})$-space.

(5) $(\forall X \in [F(I)]^{\leq\kappa})$ X is an $(I, JK\mathrm{QN})$-space.

(d) Denote $\kappa = \min\{\mathrm{non}((I', \leq_{\mathrm{K}} J\mathrm{QN})\text{-space}) : I' \leq_{\mathrm{RK}} I\}$. For every ideal $I' \leq_{\mathrm{RK}} I$, a $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})$-space is an $(I', \leq_{\mathrm{K}} J\mathrm{QN})$-space and hence $\mathrm{non}((\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})\text{-space}) \leq \kappa$. Let X be arbitrary space with $|X| < \kappa$ and let $f \in {}^{\omega}C(X)$ be such that $f \xrightarrow{\leq_{\mathrm{K}} I} 0$; i.e., there is $I' \leq_{\mathrm{RK}} I$ such that $f \xrightarrow{I'} 0$. Then $f \xrightarrow{\leq_{\mathrm{K}} J\mathrm{QN}} 0$ because $|X| < \mathrm{non}((I', \leq_{\mathrm{K}} J\mathrm{QN})\text{-space})$. Therefore $\mathrm{non}((\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})\text{-space}) \geq \kappa$. □

Remark 3.7. By [13, Theorem 4.3], $\mathrm{non}((I, \leq_{\mathrm{K}} J\mathrm{QN})\text{-space}) \geq \omega_1$ for any ideals I and J. Then by Theorem 3.6 (d), $\mathrm{non}((\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})\text{-space}) \geq \omega_1$ and hence $\mathrm{non}(P) \geq \omega_1$ for all properties P within Lemma 2.9 that are weaker than $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})$.

For $K = J$ the assertions (a) and (b) of Theorem 3.6 take this form:

Corollary 3.8. *Let I and J be ideals on ω. For all reasonable classes of spaces the following holds:*

(a) *If J is a weak $P(I)$-ideal, then*

$$\mathrm{non}((I, J\mathrm{QN})\textit{-space}) = \min\{|X| : X \subseteq F(I) \textit{ and } (\forall \alpha \in F(J))(\exists \beta \in X)\ \|\beta < \alpha\| \notin J\} \leq \mathfrak{c},$$

otherwise, every space is an $(I, J\mathrm{QN})$-space.

(b) *If J is a $W(I)$-ideal, then*

$$\mathrm{non}(\mathrm{w}(I, J\mathrm{QN})\textit{-space}) = \min\{|X| : X \subseteq F(I) \textit{ and } (\forall \alpha \in F(K))(\forall \varphi \in F(J))(\exists \beta \in X)\ \|\varphi \circ \beta < \alpha\| \notin J\} \leq \mathfrak{c},$$

otherwise, every space is a $\mathrm{w}(I, J\mathrm{QN})$-space. □

In next theorem an ideal I on ω is considered to be a subset of the Cantor space $\mathcal{P}(\omega)$ and $C(I)$ is the family of continuous real-valued functions on I.

Theorem 3.9. *For ideals I and J on ω the following conditions are equivalent:*

(1) $I \leq_{\mathrm{K}} J$.

(2) *Every space is* $\mathrm{w}(I, J)$.

(3) *Every space is* $(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$.

(4) *Every space is* $(I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$.

(5) *Every space is* $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN})$.

(6) *Every space is* $\mathrm{w}(I\mathrm{QN}, J)$.

($\bar{2}$) *I as a subset of the Cantor space is* $\mathrm{w}(I, J)$.

($\bar{3}$) *I as a subset of the Cantor space is* $(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$.

($\bar{4}$) *I as a subset of the Cantor space is* $(I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$.

($\bar{5}$) *I as a subset of the Cantor space is* $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN})$.

($\bar{6}$) *I as a subset of the Cantor space is* $\mathrm{w}(I\mathrm{QN}, J)$.

PROOF. The implications (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6), ($\bar{3}$) $\Rightarrow$ ($\bar{4}$) $\Rightarrow$ ($\bar{5}$) $\Rightarrow$ ($\bar{6}$), (2) $\Rightarrow$ (6), ($\bar{2}$) $\Rightarrow$ ($\bar{6}$) hold by Lemma 2.9 and the implications (2) $\Rightarrow$ ($\bar{2}$), (3) $\Rightarrow$ ($\bar{3}$), (6) $\Rightarrow$ ($\bar{6}$) are trivial. We prove (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3), ($\bar{6}$) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) Assume that $I \leq_{\mathrm{K}} J$, i.e., there $\varphi \in {}^{\omega}\omega$ such that $I \subseteq \varphi^{\rightarrow}(J)$. Let X be arbitrary space and let $f \in {}^{\omega}({}^{X}\mathbb{R})$. If $f \xrightarrow{I} 0$, then $f \xrightarrow{\varphi^{\rightarrow}(J)} 0$ and by Lemma 1.2 (a), $\varphi \circ f \xrightarrow{J} 0$. Therefore X is a $\mathrm{w}(I, J)$-space.

(1) $\Rightarrow$ (3) Let $I \leq_{\mathrm{K}} J$. If $I' \leq_{\mathrm{K}} I$, then $I' \leq_{\mathrm{K}} J$ and every space is an $(I'\mathrm{QN}, I'\mathrm{QN})$-space. Therefore every space is a $(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$-space.

($\bar{6}$) $\Rightarrow$ (1) Let $f \in {}^{\omega}C(I)$ be defined by

$$f_n(x) = \begin{cases} 1, & \text{if } n \in x, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{for } x \in I.$$

Then $f \xrightarrow{I\mathrm{QN}} 0$ because $\{n \in \omega : |f_n(x)| \geq 2^{-n}\} = x \in I$ for all $x \in I$. Let $\varphi \in {}^{\omega}\omega$ be such that $\varphi \circ f \xrightarrow{J} 0$. Then for every $x \in I$, $\varphi^{-1}(x) = \{k \in \omega : f_{\varphi(k)}(x) \geq 1\} \in J$. Therefore $I \subseteq \varphi^{\rightarrow}(J)$ and hence $I \leq_{\mathrm{K}} J$. $\square$

For ideals I and J on ω we define

$$\begin{aligned}
\mathfrak{k}_{I,J} &= \min\{|X| : X \subseteq I \text{ and } (\forall \varphi \in F(J))\ X \setminus \varphi^{\rightarrow}(J) \neq \emptyset\}, \quad \text{if } I \not\leq_{\mathrm{K}} J,\\
\mathfrak{k}^*_{I,J} &= \min\{|X| : X \subseteq I \text{ and } (\forall \varphi \in F(\mathrm{Fin}))\ X \setminus \varphi^{\rightarrow}(J) \neq \emptyset\}, \quad \text{if } I \not\leq_{\mathrm{KB}} J.
\end{aligned}$$

If $I \leq_{\mathrm{K}} J$, then we let $\mathfrak{k}_{I,J} = \infty$; if $I \leq_{\mathrm{KB}} J$, then we let $\mathfrak{k}^*_{I,J} = \infty$. Hence, $\mathfrak{k}_{I,J} < \infty \Leftrightarrow \mathfrak{k}_{I,J} \leq \mathfrak{c} \Leftrightarrow I \not\leq_{\mathrm{K}} J$ and $\mathfrak{k}^*_{I,J} < \infty \Leftrightarrow \mathfrak{k}^*_{I,J} \leq \mathfrak{c} \Leftrightarrow I \not\leq_{\mathrm{K}} J$. These invariants are studied by Šupina [16] in connection with sequence selection properties of topological spaces under names $\mathfrak{p}_{\mathrm{K}}(I,J)$ and $\mathfrak{p}_{\mathrm{KB}}(I,J)$.

Following [10] we define $\mathrm{cov}^*(I) = \min\{X \subseteq I : X \text{ generates a tall ideal}\}$, if I is a tall ideal, and $\mathrm{cov}^*(I) = \infty$, if the ideal I is not tall. Kwela [11] proved that a topological space of cardinality $< \mathrm{cov}^*(I)$ is a w(Fin, IQN)-space if and only if it is a wQN-space.

Lemma 3.10. *Let I and J be ideals on ω.*

(a) $\mathfrak{p} \leq \mathrm{cov}^*(I) = \mathfrak{k}^*_{I,\mathrm{Fin}} = \mathfrak{k}_{I,\mathrm{Fin}} \leq \mathfrak{k}^*_{I,J} \leq \mathfrak{k}_{I,J}$.

(b) $\mathfrak{k}_{I,J} = \min\{|X| : X \subseteq I$ *and* $(\forall \varphi \in {}^{\omega}\omega)\ X \setminus \varphi^{\rightarrow}(J) \neq \emptyset\}$.

(c) *If J is a P-ideal, then* $\mathfrak{k}^*_{I,J} = \mathfrak{k}_{I,J}$.

Proof. (a) The minimal size of a set $X \subseteq I$ generating a tall ideal is $\geq \mathfrak{p}$ and therefore $\mathfrak{p} \leq \mathrm{cov}^*(I)$. If $X \subseteq I$ does not generate a tall ideal, then there is a one-to-one function φ such that $X \subseteq \varphi^{\rightarrow}(\mathrm{Fin})$. This proves that $\mathrm{cov}^*(I) \leq \mathfrak{k}^*_{I,\mathrm{Fin}} \leq \mathfrak{k}_{I,\mathrm{Fin}}$. The inequalities $\mathfrak{k}^*_{I,\mathrm{Fin}} \leq \mathfrak{k}^*_{I,J} \leq \mathfrak{k}_{I,J}$ are obvious. If $X \subseteq I$ and there is φ such that $X \subseteq \varphi^{\rightarrow}(\mathrm{Fin}) \subseteq \{a \subseteq \omega : a \cap \mathrm{rng}(\varphi) \in \mathrm{Fin}\}$ and hence X does not generate a tall ideal. This proves $\mathfrak{k}_{I,\mathrm{Fin}} \leq \mathrm{cov}^*(I)$.

(b) By (a), $\mathfrak{k}_{I,J} \geq \omega_1$ and $\varphi \in F(J)$ if and only if $\mathrm{Fin} \subseteq \varphi^{\rightarrow}(J)$.

(c) Let $X \subseteq I$ be arbitrary such that $|X| < \mathfrak{k}_{I,J}$ and let $\varphi \in {}^{\omega}\omega$ be such that $X \cup \mathrm{Fin} \subseteq \varphi^{\rightarrow}(J)$. Then $\varphi \in F(J)$ and since J is a P-ideal, there is a set $c \in J$ such that $\|\varphi = n\| \setminus c \in \mathrm{Fin}$ for all $n \in \omega$. Define $\psi \in F(\mathrm{Fin})$ by $\psi(k) = \varphi(k)$, if $k \in \omega \setminus c$, and $\psi(k) = k$, if $k \in c$. Then $\varphi^{\rightarrow}(J) \subseteq \psi^{\rightarrow}(J)$ because $\psi^{-1}(a) \subseteq \varphi^{-1}(a) \cup c$ for all $a \subseteq \omega$. This proves that $\mathfrak{k}_{I,J} \leq \mathfrak{k}^*_{I,J}$. □

Theorem 3.11. *For all ideals I and J on ω and all reasonable classes of spaces,*

$$\begin{aligned}
&\mathrm{non}(\mathrm{w}(I,J)\textit{-space}) = \mathrm{non}(\mathrm{w}(I\mathrm{QN},J)\textit{-space}) = \mathrm{non}(\mathrm{w}(I\mathrm{QN},J\mathrm{QN})\textit{-space})\\
&\quad = \mathrm{non}((I\mathrm{QN},\leq_{\mathrm{K}}J\mathrm{QN})\textit{-space}) = \mathrm{non}((\leq_{\mathrm{K}}I\mathrm{QN},\leq_{\mathrm{K}}J\mathrm{QN})\textit{-space}) = \mathfrak{k}_{I,J}.
\end{aligned}$$

Proof. If $I \leq_{\mathrm{K}} J$, then by Theorem 3.9 and by definition of $\mathfrak{k}_{I,J}$ all terms have value ∞ and so the equalities hold. Let $I \not\leq_{\mathrm{K}} J$. By Lemma 2.9,

$$\mathrm{w}(I,J) \Rightarrow \mathrm{w}(I\mathrm{QN},J),$$
$$(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN}) \Rightarrow (I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN}) \Rightarrow \mathrm{w}(I\mathrm{QN}, J\mathrm{QN}) \Rightarrow \mathrm{w}(I\mathrm{QN}, J).$$

Therefore it is enough to prove the inequalities

$$\mathfrak{k}_{I,J} \leq \mathrm{non}(\mathrm{w}(I,J)\text{-space}), \qquad \mathfrak{k}_{I,J} \leq \mathrm{non}((\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})\text{-space}), \qquad \mathrm{non}(\mathrm{w}(I\mathrm{QN},J)\text{-space}) \leq \mathfrak{k}_{I,J}.$$

Let X be arbitrary space of cardinality $< \mathfrak{k}_{I,J}$ and let $f \in {}^{\omega}C(X)$ be arbitrary such that (a) $f \xrightarrow{I} 0$; or (b) $f \xrightarrow{\leq_{\mathrm{K}} I\mathrm{QN}} 0$.

(a) The convergence $f \xrightarrow{I} 0$ is witnessed by a family $I_0 \subseteq I$ of cardinality $\leq |X| \cdot \omega < \mathfrak{k}_{I,J}$. Let $\psi \in {}^{\omega}\omega$ be such that $I_0 \cup \mathrm{Fin} \subseteq \psi^{\rightarrow}(J)$ and denote $I' = I \cap \psi^{\rightarrow}(J)$. Then $I' \leq_{\mathrm{K}} J$ and $f \xrightarrow{I'} 0$. By Theorem 3.9, X is a $\mathrm{w}(I',J)$-space and therefore there is $\varphi \in {}^{\omega}\omega$ such that $\varphi \circ f \xrightarrow{J} 0$.

(b) If $f \xrightarrow{\leq_{\mathrm{K}} I\mathrm{QN}} 0$, then $f \xrightarrow{\eta^{\rightarrow}(I)\mathrm{QN}} 0$ for some $\eta \in F(I)$ and this convergence is witnessed by a family $I_0 \subseteq \eta^{\rightarrow}(I)$ of cardinality $< \mathfrak{k}_{I,J}$. Let $\psi \in {}^{\omega}\omega$ be such that $\{\eta^{-1}(a) : a \in I_0\} \cup \mathrm{Fin} \subseteq \psi^{\rightarrow}(J)$ and denote $I' = I \cap \psi^{\rightarrow}(J)$. Then $I' \leq_{\mathrm{K}} J$ and $f \xrightarrow{\leq_{\mathrm{K}} I'\mathrm{QN}} 0$ because $I_0 \subseteq \eta^{\rightarrow}(I')$. By Theorem 3.9, X is a $(\leq_{\mathrm{K}} I'\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$-space and therefore there is $J' \leq_{\mathrm{K}} J$ such that $f \xrightarrow{J'\mathrm{QN}} 0$.

(c) Assume that a set $X \subseteq I$ as a subset of the Cantor space $\mathcal{P}(\omega)$ is a $\mathrm{w}(I\mathrm{QN},J)$-space. Let $f \in {}^{\omega}C(X)$ be defined by $f_n(x) = 1$, if $n \in x$, $f_n(x) = 0$, if $n \in \omega \setminus x$ (see the proof of Theorem 3.9). Then $f \xrightarrow{I\mathrm{QN}} 0$ because $\{n \in \omega : f_n(x) \geq 2^{-n}\} = x \in I$. Since X is a $\mathrm{w}(I\mathrm{QN},J)$-space there is $\varphi \in F(J)$ such that $\varphi \circ f \xrightarrow{J} 0$. Then for every $x \in X$, $\varphi^{-1}(x) = \{k \in \omega : f_{\varphi(k)}(x) \geq 1\} \in J$ and hence $X \subseteq \varphi^{\rightarrow}(J)$. This proves $\mathrm{non}(\mathrm{w}(I\mathrm{QN},J)\text{-space}) \leq \mathfrak{k}_{I,J}$. □

Theorem 3.12. *For all ideals $I \subseteq J$ on ω and all reasonable classes $\mathcal{X}$,*

$$\mathrm{add}((J,J\mathrm{QN})\textit{-space}) = \mathrm{non}((J,J\mathrm{QN})\textit{-space}) \leq \mathrm{add}_{\mathcal{X}}((I,J\mathrm{QN})\textit{-space}) \leq \mathrm{non}((I,J\mathrm{QN})\textit{-space}).$$

Proof. By Corollary 3.8, $\mathrm{non}_{\mathcal{X}}((I,J\mathrm{QN})\text{-space}) = \mathrm{non}((I,J\mathrm{QN})\text{-space})$ for all $\mathcal{X}$. Let $\mathcal{X}$ be fixed. Assume that $\kappa < \mathrm{non}((J,J\mathrm{QN})\text{-space})$, $X = \bigcup_{\xi<\kappa} X_\xi$ is in $\mathcal{X}$, $f \in {}^{\omega}C(X)$, and sequences $\varepsilon^{\xi} \in {}^{\omega}[0,\infty)$, $\xi < \kappa$, that J-converge to 0 control the $J\mathrm{QN}$-convergence of f on X_ξ. Define $g_n : \kappa \to X$ by $g_n(\xi) = \varepsilon^{\xi}_n$.

Since $g \xrightarrow{J} 0$ on κ, it follows that $g \xrightarrow{J\mathrm{QN}} 0$ on κ and some $\delta \in {}^{\omega}[0,\infty)$ that J-converges to 0 controls this convergence. Every $x \in X$ belongs to some X_ξ and for any such x and ξ we have

$$\{n \in \omega : |f_n(x)| \geq \delta_n\} \subseteq \{n \in \omega : |f_n(x)| \geq \varepsilon_n^\xi\} \cup \{n \in \omega : g_n(\xi) \geq \delta_n\} \in J.$$

This proves $\mathrm{non}((J, J\mathrm{QN})\text{-space}) \leq \mathrm{add}_{\mathcal{X}}((I, J\mathrm{QN})\text{-space})$. The inequality $\mathrm{add}_{\mathcal{X}}((I, J\mathrm{QN})\text{-space}) \leq \mathrm{non}((I, J\mathrm{QN})\text{-space})$ is obvious and for $I = J$ we get the equality. □

Question 3.13. Is $\mathrm{add}_{\mathcal{X}}((I, J\mathrm{QN})\text{-space})$ the same for all reasonable classes $\mathcal{X}$, if $I \neq J$?

4 Reductions of the properties with respect to $\leq_{\mathrm{K}}$

In this section we present results on monotonicity of the investigated properties of spaces with respect to the Katětov partial quasi-ordering. As a consequence we show that in some cases non-tall ideals can be equivalently replaced by Fin.

Lemma 4.1. *Let I and J be ideals on ω and $\varphi \in {}^{\omega}\omega$.*

(a) *If $\varphi \in F(I)$ and $\mathrm{rng}(\varphi) \in J^*$, then $(I, \varphi^{\leftarrow}(J)\mathrm{QN}) \Rightarrow (\varphi^{\rightarrow}(I), J\mathrm{QN})$.*

(b) *If $\varphi \in F(\mathrm{Fin})$ and $\mathrm{rng}(\varphi) \in I^+$, then $(I, \varphi^{\rightarrow}(J)\mathrm{QN}) \Rightarrow (\varphi^{\leftarrow}(I), J\mathrm{QN})$.*

(c) *If φ is injective, then $(\varphi^{\rightarrow}(I), \varphi^{\rightarrow}(J)\mathrm{QN}) \Rightarrow (I, J\mathrm{QN})$.*

(d) *If $\mathrm{rng}(\varphi) \in I^*$, then $(\varphi^{\leftarrow}(I), \varphi^{\leftarrow}(J)\mathrm{QN}) \Rightarrow (I, J\mathrm{QN})$.*

(e) *If $\varphi \in F(\mathrm{Fin})$ and $\mathrm{rng}(\varphi) \in I^*$, then $(\varphi^{\leftarrow}(I), \varphi^{\leftarrow}(J)\mathrm{QN}) \Leftrightarrow (I, J\mathrm{QN})$.*

Proof. (a) Let X be an $(I, \varphi^{\leftarrow}(J)\mathrm{QN})$-space and let $f \in {}^{\omega}C(X)$ be arbitrary such that $f \xrightarrow{\varphi^{\rightarrow}(I)} 0$. By Lemma 1.2 (a), $\varphi \circ f \xrightarrow{I} 0$, and then $\varphi \circ f \xrightarrow{\varphi^{\leftarrow}(J)\mathrm{QN}} 0$. Since $\mathrm{rng}(\varphi) \in J^+$, by Lemma 1.3 (d), $f \xrightarrow{J\mathrm{QN}} 0$.

(b) Let X be an $(I, \varphi^{\rightarrow}(J)\mathrm{QN})$-space and let $f \in {}^{\omega}C(X)$ be such that $f \xrightarrow{\varphi^{\leftarrow}(I)} 0$. Then $\varphi * f \in {}^{\omega}C(X)$ because φ is finite-to-one. By Lemma 1.3 (a), $\varphi * f \xrightarrow{I} 0$, therefore $\varphi * f \xrightarrow{\varphi^{\rightarrow}(J)\mathrm{QN}} 0$, and then, by Lemma 1.2 (d), $f \xrightarrow{J\mathrm{QN}} 0$.

(c) This is a consequence of (b) because $\varphi^{\leftarrow}(\varphi^{\rightarrow}(I)) = I$, if φ is injective.

(d)–(e) By Lemma 2.11 we can assume that $I \subseteq J$. Hence $\mathrm{rng}(\varphi) \in I^* \subseteq J^*$. Let X be an $(\varphi^{\leftarrow}(I), \varphi^{\leftarrow}(J)\mathrm{QN})$-space. If $f \xrightarrow{I} 0$, then by Lemma 1.3 (c), $\varphi \circ f \xrightarrow{\varphi^{\leftarrow}(I)} 0$, then $\varphi \circ f \xrightarrow{\varphi^{\leftarrow}(J)\mathrm{QN}} 0$, and by Lemma 1.3 (d), $f \xrightarrow{J\mathrm{QN}} 0$. This finishes the proof of (d). The implication $\Leftarrow$ in (e) follows by (b) because $\varphi^{\rightarrow}(\varphi^{\leftarrow}(J)) = J$ due to $\mathrm{rng}(\varphi) \in J^*$. □

Proposition 4.2. *Let I and J be ideals on ω.*

(a) $(I, J\mathrm{QN}) \Rightarrow (\forall I' \subseteq I)(\forall J' \supseteq J)\ (I', J'\mathrm{QN})$.

(b) $(I, J\mathrm{QN}) \Rightarrow (I \vee \langle \omega \setminus b\rangle, J \vee \langle \omega \setminus b\rangle\mathrm{QN}) \Leftrightarrow (I{\restriction}b, J{\restriction}b\mathrm{QN})$ *for* $b \in J^+$.

(c) $(I, J\mathrm{QN}) \Leftrightarrow (I \vee \langle a\rangle, J\mathrm{QN})$ *for all* $a \in J$.

(d) $(\mathrm{Fin}, J\mathrm{QN}) \Rightarrow (\forall J' \geq_{\mathrm{KB}} J$ *on* $\omega)\ (\mathrm{Fin}, J'\mathrm{QN})$.

Proof. To prove (a) and (b) verify definition of $(I, J\mathrm{QN})$. In case (b) replace f_n, $n \in \omega \setminus b$, by zero functions. (c) is a consequence of (a) and (b) for $b = \omega \setminus a$.

(d) Let J' be an ideal on ω such that $J \subseteq \varphi^{\rightarrow}(J')$ for a finite-to-one function $\varphi \in {}^{\omega}\omega$. If X is a $(\mathrm{Fin}, J\mathrm{QN})$-space, then X is a $(\mathrm{Fin}, \varphi^{\rightarrow}(J')\mathrm{QN})$-space, and by Lemma 4.1 (b), X is a a $(\mathrm{Fin}, J'\mathrm{QN})$-space because $\varphi^{\leftarrow}(\mathrm{Fin}) = \mathrm{Fin}$. □

A variant of assertion (d) for the invariant non($(\mathrm{Fin}, J\mathrm{QN})$-space) has a bit stronger form (see [13, the inequality $\mathrm{B}_{\mathrm{Fin},J} \preccurlyeq \mathrm{B}_{\mathrm{Fin},J'}$ in Lemma 2.7 (d) and Theorem 2.2 (a)]): If $J \leq_{\mathrm{K}} J'$, then

$$\mathrm{non}((\mathrm{Fin}, J\mathrm{QN})\text{-space}) \leq \mathrm{non}((\mathrm{Fin}, J'\mathrm{QN})\text{-space}).$$

Obviously, $(I, \leq_{\mathrm{K}} J\mathrm{QN}) \Rightarrow (I, \leq_{\mathrm{K}} J'\mathrm{QN})$ for all $J' \geq_{\mathrm{K}} J$. By Proposition 4.3 (a) below it seems that the notion of a $\mathrm{w}(I, J\mathrm{QN})$-space has stronger closure properties than the notion of an $(I, \leq_{\mathrm{K}} J\mathrm{QN})$-space.

Proposition 4.3. *Let I and J be ideals on ω.*

(a) $\mathrm{w}(I, J\mathrm{QN}) \Rightarrow (\forall I' \leq_{\mathrm{K}} I)(\forall J' \geq_{\mathrm{K}} J)\ \mathrm{w}(I', J'\mathrm{QN})$.

(b) $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN}) \Rightarrow (\forall I' \leq_{\mathrm{K}} I)(\forall J' \geq_{\mathrm{K}} J)\ \mathrm{w}(I'\mathrm{QN}, J'\mathrm{QN})$.

(c) $\mathrm{w}(I, J) \Rightarrow (\forall I' \leq_{\mathrm{K}} I)(\forall J' \geq_{\mathrm{K}} J)\ \mathrm{w}(I', J')$.

(d) $\mathrm{w}(I\mathrm{QN}, J) \Rightarrow (\forall I' \leq_{\mathrm{K}} I)(\forall J' \geq_{\mathrm{K}} J)\ \mathrm{w}(I'\mathrm{QN}, J')$.

Proof. (a)–(d) The implications follow by Lemma 2.8. For example, by (c) and (b) of Lemma 2.8 we have $\mathrm{w}(I, J\mathrm{QN}) \Rightarrow (\forall I' \leq_{\mathrm{K}} I)\ \mathrm{w}(I', J\mathrm{QN})$ because $\mathrm{w}(I, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(\leq_{\mathrm{K}} I, J\mathrm{QN}) \Rightarrow \mathrm{w}(I', J\mathrm{QN})$; and $\mathrm{w}(I, J\mathrm{QN}) \Rightarrow (\forall J' \geq_{\mathrm{K}} J)$ $\mathrm{w}(I, J'\mathrm{QN})$ because $\mathrm{w}(I, J\mathrm{QN}) \Rightarrow \mathrm{w}(I, \leq_{\mathrm{K}} J'\mathrm{QN}) \Leftrightarrow \mathrm{w}(I, J'\mathrm{QN})$. □

Corollary 4.4. *Let I and J be ideals on ω and let $a \in I^+$ and $b \in J^+$.*

$$\mathrm{w}(I, J\mathrm{QN}) \Rightarrow \mathrm{w}(I, J \vee \langle \omega \setminus b\rangle\mathrm{QN}) \Rightarrow \mathrm{w}(I, J{\restriction}b\mathrm{QN}),$$
$$\mathrm{w}(I{\restriction}a, J\mathrm{QN}) \Rightarrow \mathrm{w}(I \vee \langle \omega \setminus a\rangle, J\mathrm{QN}) \Rightarrow \mathrm{w}(I, J\mathrm{QN}).$$

Similar implications hold for $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN})$, $\mathrm{w}(I, J)$, $\mathrm{w}(I\mathrm{QN}, J)$.

Proof. This is a consequence of Proposition 4.3 because $I \leq_{\mathrm{KB}} I \vee \langle \omega \setminus a \rangle \leq_{\mathrm{KB}} I \restriction a$. In fact, $I \leq_{\mathrm{KB}} I \vee \langle \omega \setminus a \rangle$ holds due to the inclusion $I \subseteq I \vee \langle \omega \setminus a \rangle$ and $I \vee \langle \omega \setminus a \rangle \leq_{\mathrm{KB}} I \restriction a$ is witnessed by the identity function $i : a \to \omega$. □

Statements (a1) and (c1) of the next corollary are proved in [2, Theorem 3.3].

Corollary 4.5. *Let I and J be ideals on ω.*

(a) (1) *If J is not tall, then* $(\mathrm{Fin}, J\mathrm{QN}) \Leftrightarrow \mathrm{QN}$.

(2) *If J is not tall, then* $(I, \leq_{\mathrm{K}} J\mathrm{QN}) \Leftrightarrow \mathrm{w}(I, \mathrm{FinQN})$.

(3) *If J is not tall, then* $(I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN}) \Leftrightarrow \mathrm{w}(I\mathrm{QN}, \mathrm{FinQN})$.

(4) *If J is not tall, then* $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN}) \Leftrightarrow \mathrm{w}(I, \mathrm{FinQN})$.

(5) *If J is not tall, then* $(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN}) \Leftrightarrow \mathrm{w}(I\mathrm{QN}, \mathrm{FinQN})$.

(b) (1) *If I is not tall, then* $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN}) \Leftrightarrow (\leq_{\mathrm{K}} \mathrm{Fin}, \leq_{\mathrm{K}} J\mathrm{QN})$.

(2) *If I is not tall, then* $(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN}) \Leftrightarrow (\leq_{\mathrm{K}} \mathrm{FinQN}, \leq_{\mathrm{K}} J\mathrm{QN})$.

(c) (1) *If J is not tall, then* $\mathrm{w}(I, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(I, \mathrm{FinQN})$*;*
if I is not tall, then $\mathrm{w}(I, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(\mathrm{Fin}, J\mathrm{QN})$.

(2) *If J is not tall, then* $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(I\mathrm{QN}, \mathrm{FinQN})$*;*
if I is not tall, then $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(\mathrm{FinQN}, J\mathrm{QN})$.

(3) *If J is not tall, then* $\mathrm{w}(I, J) \Leftrightarrow \mathrm{w}(I, \mathrm{Fin})$*;*
if I is not tall, then $\mathrm{w}(I, J) \Leftrightarrow \mathrm{w}(\mathrm{Fin}, J)$.

(4) *If J is not tall, then* $\mathrm{w}(I\mathrm{QN}, J) \Leftrightarrow \mathrm{w}(I\mathrm{QN}, \mathrm{Fin})$*;*
If I is not tall, then $\mathrm{w}(I\mathrm{QN}, J) \Leftrightarrow \mathrm{w}(\mathrm{FinQN}, J)$.

(d) *If I is not tall, then every space is* $(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$, $(I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$, $\mathrm{w}(I, J)$, $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN})$, *and* $\mathrm{w}(I\mathrm{QN}, J)$.

Proof. (a)–(c) If an ideal $K = J$ or $K = I$ is not tall, then there is $a \in [\omega]^\omega$ such that $K \subseteq \mathrm{Fin} \vee \langle \omega \setminus a \rangle$. Then (i) $K \leq_{\mathrm{KB}} \mathrm{Fin} \leq_{\mathrm{KB}} K$ because $K \subseteq \varphi^{\rightarrow}(\mathrm{Fin})$ for a one-to-one function $\varphi \in {}^{\omega}\omega$ with $\mathrm{rng}(\varphi) = a$; and (ii) $K' \leq_{\mathrm{K}} K \Leftrightarrow K' \leq_{\mathrm{K}} \mathrm{Fin}$. The instances of K can be replaced by Fin, in (a1), by (i) and Proposition 4.2 (a) and (d); in (c1)–(c4), by (i) and Proposition 4.3; in (b1)–(b2) by (ii). In (a2)–(a5), in addition to replacing all instances of a non-tall ideal J by Fin due to (ii) we use the following equivalences (applications of Lemma 2.3 and Lemma 2.7 and two equivalences related to the equivalence classes 7 and 6 in Section 2):

(a2) $(I, \leq_{\mathrm{K}}\mathrm{FinQN}) \Leftrightarrow \mathrm{w}(I, \mathrm{FinQN})$.

(a3) $(I\mathrm{QN}, \leq_{\mathrm{K}}\mathrm{FinQN}) \Leftrightarrow \mathrm{w}(I\mathrm{QN}, \mathrm{FinQN})$.

(a4) $(\leq_{\mathrm{K}}I, \leq_{\mathrm{K}}\mathrm{FinQN}) \Leftrightarrow \mathrm{w}(\leq_{\mathrm{K}}I, \mathrm{FinQN}) \stackrel{7}{\Leftrightarrow} \mathrm{w}(I, \mathrm{FinQN})$.

(a5) $(\leq_{\mathrm{K}}I\mathrm{QN}, \leq_{\mathrm{K}}\mathrm{FinQN}) \Leftrightarrow \mathrm{w}(\leq_{\mathrm{K}}I\mathrm{QN}, \mathrm{FinQN}) \stackrel{6}{\Leftrightarrow} \mathrm{w}(I\mathrm{QN}, \mathrm{FinQN})$.

(d) If I is not tall, $I \leq_{\mathrm{K}} J$ for every ideal J because $I \leq_{\mathrm{K}} \mathrm{Fin} \leq_{\mathrm{K}} J$ and hence by Theorem 3.9 every space satisfies the properties in (d). ☐

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