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GENERALIZED EGOROFF'S THEOREM

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ABSTRACT. This note is closely related to the paper [R. Pinciroli: *On the independence of a generalized statement of Egoroff's theorem from ZFC after T. Weiss,* Real Anal. Exchange **32** (2006–2007), 225–232] and it presents slight improvements of its results. Theorem 1.13 shows a connection with Galois-Tukey embeddings; Corollary 1.14 presents another inequality which is dual to the previously known one; Corollary 3.5 shows that there is no distinction between positive outer measure and full outer measure in the given context; and Corollary 4.3 unifies the known counterexamples.

Notation. For $x, y \in {}^{\omega}\mathbb{R}$, $x \le y$ means that $x(n) \le y(n)$ for all $n \in \omega$ and $x \le^* y$ means that $x(n) \le y(n)$ for all but finitely many $n \in \omega$ (shortly written by $\forall^\infty n$). The set of all sequences of real numbers converging to 0 is denoted by c_0; c_0^+ is the set of all positive sequences from c_0. Clearly, $c_0 \subseteq {}^{\omega}\mathbb{R}$ and ${}^{\omega}\omega \subseteq {}^{\omega}\mathbb{R}$; the restriction of $\le^*$ to c_0 and ${}^{\omega}\omega$ will be denoted by the same symbol. The σ-ideal on ${}^{\omega}\omega$ generated by compact subsets of ${}^{\omega}\omega$ is denoted by $\mathcal{K}_\sigma$. It is known that a set $A \subseteq {}^{\omega}\omega$ is in $\mathcal{K}_\sigma$ if and only if there is $y \in {}^{\omega}\omega$ such that $x \le^* y$ for all $x \in A$. The values of the composition of functions $f \circ g$ are computed by $f \circ g(x) = f\big(g(x)\big)$.

If $\mathcal{I} \subseteq \mathcal{P}(X)$, then

$$\begin{aligned}
\operatorname{add}(\mathcal{I}) &= \min\big\{\mathcal{F} \subseteq \mathcal{I} : \textstyle\bigcup \mathcal{F} \notin \mathcal{I}\big\},\\
\operatorname{non}(\mathcal{I}) &= \min\big\{|Y| : Y \in \mathcal{P}(X) \setminus \mathcal{I}\big\},\\
\operatorname{cov}(\mathcal{I}) &= \min\big\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ and} \textstyle\bigcup \mathcal{F} = X\big\},\\
\operatorname{cof}(\mathcal{I}) &= \min\big\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ and} (\forall\, Y \in \mathcal{I})(\exists\, Z \in \mathcal{F})\, Y \subseteq Z\big\}.
\end{aligned}$$

Let us recall that

$$\mathfrak{b} = \operatorname{add}(\mathcal{K}_\sigma) = \operatorname{non}(\mathcal{K}_\sigma) \quad \text{and} \quad \mathfrak{d} = \operatorname{cov}(\mathcal{K}_\sigma) = \operatorname{cof}(\mathcal{K}_\sigma)$$

is the unbounding number and the dominating number, respectively.

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1. Generalized Egoroff's theorem

Let us recall that a sequence of real-valued functions $\langle f_n : n \in \omega \rangle$ *quasinormally* converges to f on X (shortly written by $f_n \xrightarrow{\mathrm{QN}} f$), if $(\exists\, \varepsilon \in c_0)(\forall\, x \in X)$ $(\forall^\infty n \in \omega)$ $|f_n(x) - f(x)| < \varepsilon(n)$. The uniform convergence is denoted as usual by $\rightrightarrows$. Let us recall that the convergence on X is quasinormal if and only if X can be partitioned into countably many sets on which the convergence is uniform. Conversely, if X can be partitioned to $< \mathfrak{b}$ sets on which the convergence is uniform, then the convergence is quasinormal on X (see [4]).

Definition 1.1. Let $\mathcal{F} \subseteq \mathcal{P}(X)$. Let us consider the following statements:

$E(\mathcal{F})$ Given a sequence $\langle f_n \colon n \in \omega \rangle$ of functions $f_n \colon X \to \mathbb{R}$ converging pointwise to 0, there exists a set $A \in \mathcal{F}$ such that $f_n \xrightarrow{\mathrm{QN}} 0$ on A.

$E^*(\mathcal{F})$ Given a sequence $\langle f_n \colon n \in \omega \rangle$ of functions $f_n \colon X \to \mathbb{R}$ converging pointwise to 0, there exists a set $A \in \mathcal{F}$ such that $f_n \rightrightarrows 0$ on A.

Remark 1.2.

(a) The sequence of functions $f_n \colon X \to \mathbb{R}$, $n \in \omega$ that pointwise converges to 0 on X can be identified with a single function $F \colon X \to c_0$ by setting $F(x)(n) = f_n(x)$ (see [8]). Sometimes when we speak about F, we will mean the sequence $\langle f_n : n \in \omega \rangle$. This convention will be applied mainly in the context of uniform and quasinormal convergence. In particular, $E(\mathcal{F})$ can be rephrased as follows: For every $F \colon X \to c_0$ there exists $A \in \mathcal{F}$ such that $F \xrightarrow{\mathrm{QN}} 0$ on A.

(b) The definition of $E(\mathcal{F})$ can be equivalently restricted to monotone systems of functions because $f_n \xrightarrow{\mathrm{QN}} 0$ if and only if $f'_n \xrightarrow{\mathrm{QN}} 0$, where $f'_n(x) = \sup\bigl\{|f_k(x)| : k \geq n\bigr\}$.

(c) If $\overline{\mathcal{F}}$ is the closure of $\mathcal{F}$ under supersets, then $E(\overline{\mathcal{F}})$ is equivalent to $E(\mathcal{F})$, and $E^*(\overline{\mathcal{F}})$ is equivalent to $E^*(\mathcal{F})$.

Definition 1.3 ([8])**.** We define $\alpha \colon c_0 \to {}^\omega\omega$, $\beta \colon {}^\omega\omega \to c_0$, $\theta \colon {}^\omega\omega \to {}^\omega\omega$, and ${}^\to \colon {}^\omega\omega \to {}^\omega\omega$ by

$$\begin{aligned} \alpha(x)(n) &= \min\bigl\{m \in \omega : (\forall\, k \geq m)\ x(k) \leq 2^{-n}\bigr\}, \\ \theta(y)(n) &= \max\Bigl(\{n\} \cup \bigl\{y(k) : k < n\bigr\}\Bigr), \\ \beta(y)(k) &= \min\bigl\{2^{-n} \colon \theta(y)(n) \leq k\bigr\}, \\ y^{\to}(n) &= \max\bigl\{y(0), y(n+1)\bigr\} \end{aligned}$$

for $x \in c_0$, $y \in {}^\omega\omega$, and $n, k \in \omega$.

Lemma 1.4. *$\alpha \circ \beta = \theta$.*

Proof.
$$\begin{aligned} \alpha\big(\beta(y)\big)(n) &= \min\big\{m : (\forall\, k \geq m)\, \beta(y)(k) \leq 2^{-n}\big\} \\ &= \min\big\{m : \beta(y)(m) \leq 2^{-n}\big\} \\ &= \min\big\{m : \theta(y)(n) \leq m\big\} \\ &= \theta(y)(n). \end{aligned}$$
□

Lemma 1.5. *If $Y \subseteq {}^{\omega}\omega$, then $Y \in \mathcal{K}_\sigma$ if and only if $\theta(Y) \in \mathcal{K}_\sigma$.*

Proof. If $Y \in \mathcal{K}_\sigma$, then $\theta(Y) \in \mathcal{K}_\sigma$ because θ is continuous. Conversely, let us assume that $\theta(Y) \in \mathcal{K}_\sigma$, i.e., there is $z \in {}^{\omega}\omega$ such that $\theta(y) \leq^* z$ for all $y \in Y$. Then, $y \leq^* z^{\rightarrow}$ for all $y \in Y$ and hence, $Y \in \mathcal{K}_\sigma$. □

Lemma 1.6.

1. *$(\forall\, x \in c_0)(\exists\, y \in {}^{\omega}\omega)\ x \leq^* \beta(y)$; in fact $x \leq \beta\big(\alpha(2x)\big)$.*
2. *$(\forall\, x \in c_0)(\forall\, z \in c_0^+$ monotone$)\ x \leq^* z \Rightarrow \alpha(x) \leq^* \alpha(z) \Rightarrow x \leq^* 2z$.*
3. *$y \leq \theta(y^{\rightarrow})$ for $y \in {}^{\omega}\omega$.*
4. *$(\forall\, y \in {}^{\omega}\omega)(\forall z \in {}^{\omega}\omega$ monotone unbounded$)\ y \leq^* z \Leftrightarrow \beta(y) \leq^* \beta(z)$.*

Proof.

(1) $\beta\big(\alpha(2x)\big)(k) = \min\big\{2^{-n} : \theta(\alpha(2x))(n) \leq k\big\} = \min\big\{2^{-n} : n \leq k$ and $\alpha(2x)(n-1) \leq k\big\} \geq \min\big\{2^{-n} : (\forall\, i \geq k)\ 2x(i) \leq 2^{-(n-1)}\big\} \geq x(k)$.

(2) As $z \in c_0^+$, then $x \leq^* z$ if and only if there is n_0 such that $z(k) \leq 2^{-n_0}$ implies $x(k) \leq z(k)$ and then $\alpha(x)(n) \leq \alpha(z)(n)$ for $n \geq n_0$. If $\alpha(x)(n) \leq \alpha(z)(n)$ for $n \geq n_0$, then as z is monotone, for every $k \in \omega$ and every $n \geq n_0$, $z(k) \leq 2^{-n}$ implies $x(k) \leq 2^{-n}$. Hence, $2^{-n-1} < z(k) \leq 2^{-n}$ implies $x(k) \leq 2^{-n} < 2z(k)$. Consequently, $x \leq^* 2z$.

(4) $y \leq^* z$ implies $\theta(y) \leq^* \theta(z)$. Let n_0 be such that $\theta(y)(n) \leq \theta(z)(n)$ for all $n \geq n_0$. Then for $k \geq \theta(z)(n_0)$, $\big\{n : \theta(y)(n) \leq k\big\} \supseteq \big\{n : \theta(z)(n) \leq k\big\}$ and so $\beta(y)(k) \leq \beta(z)(k)$. Conversely, if $\beta(y) \leq^* \beta(z)$, then there is k_0 such that $\theta(z)(n) = k$ implies $\theta(y)(n) \leq k$ for $k \geq k_0$ and so, $\theta(y) \leq^* \theta(z)$ (notice that $\theta(z)$ is monotone unbounded). Now, as z is monotone unbounded, we can easily see that $y \leq^* z$. □

Lemma 1.7. *Let $F \colon X \to c_0$ and $G \colon X \to {}^{\omega}\omega$.*

(a) *$F \xrightarrow{QN} 0$ on X if and only if $(\alpha \circ F)(X) \in \mathcal{K}_\sigma$.*

(b) *$G(X) \in \mathcal{K}_\sigma$ if and only if $\beta \circ G \xrightarrow{QN} 0$ on X.*

Proof. In the following equivalences, the assertions of Lemma 1.6 are applied:

$$\begin{aligned}
F \xrightarrow{\mathrm{QN}} 0 \text{ on } X &\Leftrightarrow (\exists\, \varepsilon \in c_0)(\forall\, x \in X)\, F(x) \le^* \varepsilon \\
&\Leftrightarrow (\exists\, y \in {}^{\omega}\omega)(\forall\, x \in X)\, F(x) \le^* \beta(y) && \text{by (1),} \\
&\Leftrightarrow (\exists\, y \in {}^{\omega}\omega)(\forall\, x \in X)\, \alpha \circ F(x) \le^* \theta(y) && \text{by (2) and Lemma 1.4,} \\
&\Leftrightarrow (\exists\, y \in {}^{\omega}\omega)(\forall\, x \in X)\, \alpha \circ F(x) \le^* y && \text{backward use (3).} \\
\beta \circ G \xrightarrow{\mathrm{QN}} 0 \text{ on } X &\Leftrightarrow (\exists\, y \in {}^{\omega}\omega)(\forall\, x \in X)\ \beta \circ G(x) \le^* \beta(y) && \text{by (1),} \\
&\Leftrightarrow (\exists\, y \in {}^{\omega}\omega)(\forall\, x \in X)\ G(x) \le^* y && \text{by (4).}
\end{aligned}$$

□

Theorem 1.8 ([8]). *Let $\mathcal{F} \subseteq \mathcal{P}(X)$. The following conditions are equivalent:*

(1) *$E(\mathcal{F})$ holds.*

(2) $(\forall\, F\colon X \to c_0)(\exists\, Y \in \mathcal{F})\ \alpha \circ F(Y) \in \mathcal{K}_\sigma$.

(3) $(\forall\, \varphi\colon X \to {}^{\omega}\omega)(\exists\, Y \in \mathcal{F})\ \varphi(Y) \in \mathcal{K}_\sigma$.

Proof.

(1) ⇔ (2) holds by Lemma 1.7 (a).

(2) ⇒ (3): If $\varphi\colon X \to {}^{\omega}\omega$, then $\beta \circ \varphi\colon X \to c_0$ and so, by (2), there is $Y \in \mathcal{F}$ such that $\theta \circ \varphi(Y) = \alpha \circ \beta \circ \varphi(Y) \in \mathcal{K}_\sigma$. Hence, $\varphi(Y) \in \mathcal{K}_\sigma$ by Lemma 1.5.

The implication (3) ⇒ (2) is trivial because $\alpha \circ F\colon X \to {}^{\omega}\omega$ in (2). □

We denote $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$ for an ideal $\mathcal{I} \subseteq \mathcal{P}(X)$.

Corollary 1.9. *Let $\mathcal{F} \subseteq \mathcal{P}(X)$. Then $E(\mathcal{F})$ holds if and only if $E(\mathcal{I}^+)$ holds for every σ-ideal $\mathcal{I} \subseteq \mathcal{P}(X) \setminus \mathcal{F}$.*

Proof. If $E(\mathcal{F})$ does not hold, then by (3) in Theorem 1.8 there exists $\varphi\colon X \to {}^{\omega}\omega$ such that $\varphi(Y) \notin \mathcal{K}_\sigma$ for all $Y \in \mathcal{F}$. This means that $\varphi^{-1}(\mathcal{K}_\sigma) \subseteq \mathcal{P}(X) \setminus \mathcal{F}$ is a σ-ideal on X, where $\varphi^{-1}(\mathcal{K}_\sigma) = \{Y \subseteq X : \varphi(Y) \in \mathcal{K}_\sigma\}$. Then φ witnesses that $E\big((\varphi^{-1}(\mathcal{K}_\sigma))^+\big)$ does not hold. Conversely, if there exists an ideal $\mathcal{I} \subseteq \mathcal{P}(X) \setminus \mathcal{F}$ such that $E(\mathcal{I}^+)$ does not hold, then $E(\mathcal{F})$ does not hold because $\mathcal{F} \subseteq \mathcal{I}^+$. □

Corollary 1.10. *If $|X| \le \mathfrak{c}$ and $\mathcal{F} \subseteq \mathcal{P}(X)$, then $E(\mathcal{F})$ is equivalent to each of the following conditions:*

(4) $\Big(\forall\, \varphi\colon X \xrightarrow{1\text{–}1} {}^{\omega}\omega\Big)(\exists\, Y \in \mathcal{F})\ \varphi(Y) \in \mathcal{K}_\sigma$.

(5) $\Big(\forall\, \varphi\colon X \xrightarrow{1\text{–}1} {}^{\omega}\omega\Big)(\exists\, Z \in \mathcal{K}_\sigma)\ \varphi^{-1}(Z) \in \mathcal{F}$.

Proof. Let us fix a one-to-one enumeration $\{x_\alpha\colon \alpha < |X|\}$ of the set X and let us assume that (4) holds. We verify (3) in Theorem 1.8. Let $\varphi\colon X \to {}^{\omega}\omega$ be arbitrary. By induction on $\alpha < |X|$, we define $\varphi'(x_\alpha) = \varphi(x_\alpha) + y_\alpha$, where

$y_\alpha \in {}^\omega 2$ is such that $\varphi'(x_\alpha) \neq \varphi'(x_\beta)$ for all $\beta < \alpha$. So, $\varphi' \colon X \to {}^\omega\omega$ is one-to-one and $\varphi(x) \leq \varphi'(x) \leq \varphi(x) + 1$. By (4) there is $Y \in \mathcal{F}$ such that $\varphi'(Y) \in \mathcal{K}_\sigma$ and then also $\varphi(Y) \in \mathcal{K}_\sigma$.

Let $\varphi \colon X \xrightarrow{1\text{–}1} {}^\omega\omega$. If (4) holds and $Y \in \mathcal{F}$ such that $Z = \varphi(Y) \in \mathcal{K}_\sigma$, then $\varphi^{-1}(Z) = Y \in \mathcal{F}$ and so, (5) holds. If (5) holds and $Z \in \mathcal{K}_\sigma$ is such that $Y = \varphi^{-1}(Z) \in \mathcal{F}$, then $\varphi(Y) = Z \in \mathcal{K}_\sigma$ and so, (4) holds. ☐

Corollary 1.11. *If $|X| = \mathfrak{c}$, $\mathcal{F} \subseteq \mathcal{P}(X)$, and there is $A \subseteq X$ of size $\mathfrak{c}$ such that $Y \setminus A \in \mathcal{F}$ for all $Y \in \mathcal{F}$, then $E(\mathcal{F})$ is equivalent to each of the following conditions:*

(6) $(\forall\, \varphi : X \to {}^\omega\omega$ *bijective*$)(\exists\, Y \in \mathcal{F})\ \varphi(Y) \in \mathcal{K}_\sigma$.

(7) $(\forall\, \varphi : {}^\omega\omega \to X$ *bijective*$)(\exists\, Y \in \mathcal{F})\ \varphi^{-1}(Y) \in \mathcal{K}_\sigma$.

(8) $(\forall\, \chi : {}^\omega\omega \to X$ *surjective*$)(\exists\, Z \in \mathcal{K}_\sigma)\ \chi(Z) \in \mathcal{F}$.

Proof. (6) and (7) are equivalent. Obviously condition (4) from Corollary 1.10 implies (6). We prove that (6) implies (4). Let $\varphi \colon X \xrightarrow{1\text{–}1} {}^\omega\omega$ be arbitrary. As $|A| = \mathfrak{c}$, we can find $\psi \colon A \to {}^\omega\omega$ such that the mapping $\varphi' = \varphi\restriction(X \setminus A) \cup \psi$ is a bijection from X onto ${}^\omega\omega$. Applying (6) to φ', we find $Y \in \mathcal{F}$ such that $\varphi'(Y) \in \mathcal{K}_\sigma$. Then $Y \setminus A \in \mathcal{F}$ and $\varphi(Y \setminus A) = \varphi'(Y \setminus A) \subseteq \varphi'(Y) \in \mathcal{K}_\sigma$. Therefore, $E(\mathcal{F})$ is equivalent to (4).

Clearly, (8) implies (6); we prove that (4) from Corollary 1.10 implies (8). If $\chi \colon {}^\omega\omega \to X$ is an arbitrary surjective function, then let φ be its arbitrary right inverse function, i.e., $\chi \circ \varphi = \mathrm{id}_X$. As φ is injective, there is $Y \in \mathcal{F}$ such that $Z = \varphi(Y)$ is in $\mathcal{K}_\sigma$. Then $\chi(Z) = Y \in \mathcal{F}$. ☐

Let us note that we cannot remove "surjective" from (8) because it would be equivalent to the assertion $(\forall A \subseteq X)\ E(\mathcal{F}\restriction A)$ (where $\mathcal{F}\restriction A = \{Y \in \mathcal{F} : Y \subseteq A\}$).

Corollary 1.12. *Let $\mathcal{F} \subseteq \mathcal{P}(X)$. If $\mathcal{F}$ is closed under supersets, then $E(\mathcal{F})$ is equivalent to the assertion:*

(9) $(\forall\, \varphi : X \to {}^\omega\omega)(\exists\, Z \in \mathcal{K}_\sigma)\ \varphi^{-1}(Z) \in \mathcal{F}$.

Proof. The implication from (9) to (3) does not require any assumptions because, if $Z \in \mathcal{K}_\sigma$ and $Y = \varphi^{-1}(Z) \in \mathcal{F}$, then $\varphi(Y) = Z \in \mathcal{K}_\sigma$. Conversely, if $Y \in \mathcal{F}$ and $Z = \varphi(Y) \in \mathcal{K}_\sigma$, like in (3), then $Y \subseteq \varphi^{-1}(Z)$. Hence, if $\mathcal{F}$ is closed under supersets, then $\varphi^{-1}(Z) \in \mathcal{F}$. ☐

Binary relations can be treated as triples $\mathbf{A} = (A_-, A_+, A)$, where A is a binary relation between sets A_- and A_+. A morphism between binary relations $\mathbf{A}$ and $\mathbf{B}$ is a pair of functions $\varphi_- : A_- \to B_-$ and $\varphi_+ : B_+ \to A_+$ such that

$$B\bigl(\varphi_-(a), b\bigr) \quad \text{implies} \quad A\bigl(a, \varphi_+(b)\bigr)$$

for all $a \in A_-$ and $b \in B_+$. The morphism (φ_-, φ_+) is called a *Tukey embedding* or a *Galois-Tukey embedding* (*connection*) (see [1] and [10]). We write $\mathbf{A} \preceq \mathbf{B}$, if there exists a Galois-Tukey embedding between $\mathbf{A}$ and $\mathbf{B}$; we write $\mathbf{A} \simeq \mathbf{B}$, if $\mathbf{A} \preceq \mathbf{B}$ and $\mathbf{B} \preceq \mathbf{A}$.

Let $\mathcal{F} \subseteq \mathcal{P}(X)$ and let $\mathcal{I} = \mathcal{P}(X) \setminus \mathcal{F}$. If $\varphi\colon X \to {}^\omega\omega$ violates (9), then the pair of functions $\varphi\colon X \to {}^\omega\omega$ and $\psi = \varphi^{-1}\colon \mathcal{K}_\sigma \to \mathcal{I}$ form a Galois-Tukey embedding $(X, \mathcal{I}, \in) \preceq ({}^\omega\omega, \mathcal{K}_\sigma, \in)$ in the notation of [1], because $\varphi(x) \in Z$ implies $x \in \psi(Z)$ for $x \in X$ and $Z \in \mathcal{K}_\sigma$. Conversely, if a pair of functions (φ, ψ) is such an embedding, then φ violates (3) provided that $\mathcal{F}$ is closed under supersets because, if $\varphi(Y) = Z \in \mathcal{K}_\sigma$, then $Y \subseteq \psi(Z) \in \mathcal{I}$. A little modification of this proof gives an embedding $(X, \mathcal{I}, \in) \preceq ({}^\omega\omega, {}^\omega\omega, \le^*)$ because $({}^\omega\omega, {}^\omega\omega, \le^*) \simeq ({}^\omega\omega, \mathcal{K}_\sigma, \in)$. This proves the following theorem.

Theorem 1.13. *Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be closed under supersets and let $\mathcal{I} = \mathcal{P}(X) \setminus \mathcal{F}$. Then the following conditions are equivalent:*

(1) *$\neg E(\mathcal{F})$ holds.*
(2) *$(X, \mathcal{I}, \in) \preceq ({}^\omega\omega, \mathcal{K}_\sigma, \in)$.*
(3) *$(X, \mathcal{I}, \in) \preceq ({}^\omega\omega, {}^\omega\omega, \le^*)$.*

Let us note that for the implications (1) → (2) → (3) → (2) we do not need the assumption that $\mathcal{F}$ is closed under supersets and the following cardinal inequalities are consequences of the embedding $(X, \mathcal{I}, \in) \preceq ({}^\omega\omega, {}^\omega\omega, \le^*)$.

Corollary 1.14. *Let $\mathcal{F} \subseteq \mathcal{P}(X)$ and let $\mathcal{I} = \mathcal{P}(X) \setminus \mathcal{F}$. Then $\neg E(\mathcal{F})$ implies $\mathfrak{b} \le \operatorname{non}(\mathcal{I})$ and $\operatorname{cov}(\mathcal{I}) \le \mathfrak{d}$.*

Corollary 1.15. *Let $\mathcal{I} = \mathcal{P}(X) \setminus \mathcal{F}$ and let $L(\mathcal{F})$ denote the statement:*

$$\big(\exists\, Z \subseteq {}^\omega\omega,\ |Z| \ge |X|\big)(\forall\, Y \subseteq Z)\ Y \in \mathcal{K}_\sigma \text{ implies } |Y| < \operatorname{non}(\mathcal{I}).$$

If $|X| \le \mathfrak{c}$, then $L(\mathcal{F})$ implies $\neg E(\mathcal{F})$.

Proof. An easy application of condition (4) from Corollary 1.10. □

Condition (3) of Theorem 1.8 has these easy consequences: $\mathfrak{b}$ is the least cardinal κ such that there is a uniform ultrafilter $\mathcal{F}$ on κ so that $\neg E(\mathcal{F})$ holds; $\mathfrak{b}$ is the least cardinal κ such that $\neg E(\mathcal{F})$ holds for each uniform ultrafilter $\mathcal{F}$ on κ.

We say that κ is an *E-cardinal*, if there exists a uniform ultrafilter $\mathcal{F}$ on κ such that $E(\mathcal{F})$ holds. By Corollary 1.14 each $\kappa < \mathfrak{b}$ is an E-cardinal and every measurable cardinal κ is an E-cardinal because, if $\mathcal{F}$ is a κ-complete ultrafilter on κ, then $\operatorname{cov}\big(\mathcal{P}(\kappa) \setminus \mathcal{F}\big) = \kappa > \mathfrak{d}$.

Question 1.16. When is a cardinal an E-cardinal?

2. Quasinormal versus uniform

Now, we try to compare the assertions $E(\mathcal{F})$ and $E^*(\mathcal{F})$.

Shortly, we will say that $\mathcal{F} \subseteq \mathcal{P}(X)$ is *closed*, if $\mathcal{F}$ is closed under supersets, i.e., if $A \subseteq B \subseteq \mathcal{F}$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$. Let us consider some properties for $\mathcal{F}$. All next properties ensure that $\mathcal{F}$ is closed. Without the closedness assumptions, these properties would be more complex.

- (F0) $\mathcal{F}$ is closed and if $\bigcup_{m\in\omega} A_{n,m} \in \mathcal{F}$ for all $n \in \omega$, then there is $f \in {}^\omega\omega$ such that $\bigcup_{n\in\omega} \bigcup_{m\le f(n)} A_{n,m} \in \mathcal{F}$.
- (F1) $\mathcal{F} = \bigcap_{n\in\omega} \mathcal{F}_n$ with $\mathcal{F}_n \subseteq \mathcal{P}(X)$ closed such that whenever $m \in \omega$ and $A_n \notin \mathcal{F}_m$ for $n \in \omega$, then there is k such that $\bigcup_{n\in\omega} A_n \notin \mathcal{F}_k$.
- (F2) $\mathcal{F} = \bigcap_{x\in[0,1)} \mathcal{F}_x$ with $\mathcal{F}_x \subseteq \mathcal{P}(X)$ closed such that whenever $A_n \subseteq X$ and $A_n \notin \mathcal{F}_{x_n}$ for $n \in \omega$, then for every x with $1 \ge x > \sup_n x_n$ there is $y < x$ such that $\bigcup_{n\in\omega} A_n \notin \mathcal{F}_y$.
- (F3) $\mathcal{F} = \bigcap_{p\in P} \mathcal{F}_p$ where $(P, \le)$ is a partially ordered set and $\langle \mathcal{F}_p : p \in P \rangle$ is a system of closed subsets of $\mathcal{P}(X)$ such that
  1. For all $p, q \in P$, $p \le q$ implies $\mathcal{F}_p \supseteq \mathcal{F}_q$.
  2. If $\bigcup_{n\in\omega} Y_n \in \mathcal{F}_p$ for all $p \in P$, then for every $p \in P$ there is $n \in \omega$ such that $Y_n \in \mathcal{F}_p$.

Example 2.1.

(a) In [8], the case when $\mathcal{F} = \{Y \subseteq X : \mu^*(Y) = \mu^*(X)\}$ for some finite upward continuous monotone outer measure μ^* on X is considered. Then $\mathcal{F}$ satisfies (F2) with $\mathcal{F}_x = \{Y \subseteq X : \mu^*(Y)/\mu^*(X) \ge x\}$. The corresponding system $\mathcal{I}$ for Theorem 1.13 is the family of all sets $A \subseteq X$ such that $\mu^*(A) < \mu^*(X)$. Vice versa, if $\mathcal{F}_x$ for $x \in [0,1]$ are as in (F2), then $\mu^*(Y) = \max\{x \in [0,1] : Y \in \bigcap_{y<x} \mathcal{F}_y\}$ is an outer measure with the stated properties. Notice that for condition (F1), the formula $\mu^*(Y) = \sup\{1 - 2^{-n} : Y \in \bigcap_{m<n} \mathcal{F}_m\}$ defines a finite monotone outer measure which is upward continuous (only) at the value 1.

(b) Let X be a topological space and let P be a π-base of open sets in X. Let $\mathcal{F}_p$ for $p \in P$ be the system of all sets $A \subseteq X$ such that $A \cap p$ is not meager. Then $\mathcal{F} = \bigcap_{p\in P} \mathcal{F}_p$ has property (F3) and $\mathcal{I}$ is the system of all sets $A \subseteq X$ which are somewhere meager, i.e., there is $p \in P$ such that $p \cap A$ is meager.

(c) The previous example is connected with Baire category. The measurability is involved in a similar way. Let P be the system of all perfect subsets of $\mathbb{R}$ with positive measure and let $\mathcal{F}_p$ be the system of all sets $A \subseteq \mathbb{R}$ such that $A \cap q \neq \emptyset$ for every $q \in P$ with $q \subseteq p$. Then $\mathcal{F} = \bigcap_{p\in P} \mathcal{F}_p$ has the property (F3) and $\mathcal{I}$ is the system of all sets in $\mathbb{R}$ which are disjoint from a set of positive measure.

It is easy to see that the implications (F2) ⇒ (F1) ⇒ (F0) hold. Clearly, (F3) generalizes (F2) and (F1).

Lemma 2.2. *Let* $\mathcal{F} = \bigcap_{p\in P} \mathcal{F}_p$ *where* $(P, \leq)$ *is a partially ordered set with a cofinal subset of size* $< \mathfrak{b}$ *and* $\langle \mathcal{F}_p : p \in P\rangle$ *is a system of closed subsets of* $\mathcal{P}(X)$ *with property* (F3)*. Then* $E(\mathcal{F})$ *holds if and only if* $E^*(\mathcal{F}_p)$ *holds for all* $p \in P$.

Proof. Let $D \subseteq P$ be a cofinal subset of P of size $< \mathfrak{b}$ and let $F\colon X \to c_0$. If for every $p \in D$ there is $A_p \in \mathcal{F}_p$ such that $F \rightrightarrows 0$ on A_p, then $Y = \bigcup_{p\in D} A_p$ belongs to $\bigcap_{p\in D} \mathcal{F}_p = \bigcap_{p\in P} \mathcal{F}_p = \mathcal{F}$. Then $F \xrightarrow{\mathrm{QN}} 0$ on Y, since $|D| < \mathfrak{b}$. It follows that $E(\mathcal{F})$ holds.

Conversely, if $F\colon X \to c_0$, then as we assume $E(\mathcal{F})$, there is $Y \in \mathcal{F}$ such that $F \xrightarrow{\mathrm{QN}} 0$ on Y. Let $Y_n \subseteq X$ for $n \in \omega$ be such that $Y = \bigcup_{n\in\omega} Y_n$ and $F \rightrightarrows 0$ on Y_n for all $n \in \omega$. By condition (2), for every $p \in P$, there is $n \in \omega$ such that $Y_n \in \mathcal{F}_p$. Consequently, $E^*(\mathcal{F}_p)$ holds for all $p \in P$. ☐

Lemma 2.3. *Let* $\langle \mathcal{F}_p : p \in P\rangle$ *be a system of closed subsets of* $\mathcal{P}(X)$ *such that* $\mathcal{F} = \bigcap_{p\in P} \mathcal{F}_p = \bigcap_{p\in D} \mathcal{F}_p$ *for some set* $D \subseteq P$ *of size* $< \mathfrak{b}$*. If* $E^*(\mathcal{F}_p)$ *holds for all* $p \in D$*, then* $E(\mathcal{F})$ *holds.*

3. The case of measure and category

Definition 3.1. Let $\mathcal{I} \subseteq \mathcal{P}(X)$ be an ideal and let $\mathcal{B} \subseteq \mathcal{P}(X)$ be a set algebra.

(1) We say that $\mathcal{B}$ is a *covering* of $\mathcal{P}(X)$ modulo $\mathcal{I}$, if for every $Y \subseteq X$ there exists $B \in \mathcal{B}$ such that $Y \subseteq B$ and $(\forall B' \in \mathcal{B})(Y \subseteq B'$ implies $B \setminus B' \in \mathcal{I})$. B is said to be a *cover* of Y.

(2) We say that an ideal $\mathcal{I}$ is $\mathcal{B}$*-homogeneous*, if for every set $B \in \mathcal{B}$ such that $B \notin \mathcal{I}$ there exists a function $f\colon X \to X$ such that $f(X \setminus B) \subseteq B$ and $Y \in \mathcal{I}$ if and only if $f(Y) \in \mathcal{I}$ for all $Y \subseteq X$. (If we define $f'\colon X \to B$ by $f'(x) = x$ for $x \in B$ and $f'(x) = f(x)$ for $x \in X \setminus B$, then f' works, too.)

(3) Let $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$ and $\mathcal{I}^{++} = \{A \subseteq X : (\forall B \in \mathcal{B} \setminus \mathcal{I})\ B \cap A \notin \mathcal{I}\}$.

Let us note that the condition in the definition of a $\mathcal{B}$-homogeneous ideal $\mathcal{I}$ is enough to verify only for $B \in \mathcal{B}$ such that $B \notin \mathcal{I}$ and $X \setminus B \notin \mathcal{I}$ (because, if $x_0 \in B \in \mathcal{B}$ and $X \setminus B \in \mathcal{I}$, then the condition is trivially fulfilled for the function f defined by $f(x) = x$ for $x \in B$ and $f(x) = x_0$ for $x \in X \setminus B$).

It is well-known that the σ-algebra of Borel sets is a covering of $\mathcal{P}(\mathbb{R})$ modulo the σ-ideal of meager sets $\mathcal{M}$ as well as modulo the σ-ideal of null sets $\mathcal{N}$. The factor algebras Borel$/\mathcal{M}$ and Borel$/\mathcal{N}$ have both c.c.c.

Lemma 3.2. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be ideals on X and let $\mathcal{B} \subseteq \mathcal{P}(X)$ be a set algebra.*

1. *If $\mathcal{B}$ is a covering of $\mathcal{P}(X)$ modulo $\mathcal{I}_1$ as well as modulo SI_2, then $\mathcal{B}$ is a covering of $\mathcal{P}(X)$ modulo $\mathcal{I}_1 \cap \mathcal{I}_2$.*
2. *If $\mathcal{I}_1$ and $\mathcal{I}_2$ are $\mathcal{B}$-homogeneous and $\mathcal{B}$-orthogonal (i.e., there exists $A \in \mathcal{B}$ such that $A \in \mathcal{I}_1$ and $X \setminus A \in \mathcal{I}_2$), then $\mathcal{I}_1 \cap \mathcal{I}_2$ is $\mathcal{B}$-homogeneous.*

Proof. (1) is trivial. We prove (2). Let us fix $A \in \mathcal{B}$ such that $A \in \mathcal{I}_1$ and $X \setminus A \in \mathcal{I}_2$. Let $B \in \mathcal{B}$ be arbitrary such that $B \notin \mathcal{I}_1 \cap \mathcal{I}_2$. Then $B \cap A \notin \mathcal{I}_2$ and $B \setminus A \notin I$, and so there are functions $f_1\colon X \to B \setminus A$ and $f_2\colon X \to B \cap A$ such that for every $Y \subseteq X$,

$$Y \in \mathcal{I}_i \quad \text{if and only if} \quad f_i(Y) \in \mathcal{I}_i, \qquad i = 1, 2.$$

Let us define $f\colon X \to B$ by $f = \bigl(f_1{\restriction}(X \setminus A)\bigr) \cup (f_2{\restriction}A)$.

If $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$, then $f(Y) = f_1(Y \setminus A) \cup f_2(Y \cap A)$. Now, $f_1(Y \setminus A) \subseteq X \setminus A \in \mathcal{I}_2$ and $f_2(Y \cap A) \subseteq A \in \mathcal{I}_1$. Also, $f_1(Y \setminus A) \in \mathcal{I}_1$ and $f_2(Y \cap A) \in \mathcal{I}_2$ because $Y \setminus A \in \mathcal{I}_1$ and $Y \cap A \in \mathcal{I}_2$. Therefore, $f(Y) \in \mathcal{I}_1 \cap \mathcal{I}_2$.

Conversely, if $f(Y) \in \mathcal{I}_1 \cap \mathcal{I}_2$, then $f_1(Y \setminus A) \in \mathcal{I}_1$ and $f_2(Y \cap A) \in \mathcal{I}_2$. It follows that $Y \setminus A \in \mathcal{I}_1$ and $Y \cap A \in \mathcal{I}_2$, and consequently, $Y = (Y \cap A) \cup (Y \setminus A) \in \mathcal{I}_1 \cap \mathcal{I}_2$. □

From now on, if μ denotes a measure, then it denotes the Lebesgue measure.

Lemma 3.3. *The ideals $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{M} \cap \mathcal{N}$ are* Borel-*homogeneous ideals.*

Proof.

(a) The case of $\mathcal{M}$. Let B be a nonmeager Borel set such that $B' = \mathbb{R} \setminus B$ is nonmeager. There are meager sets $M \subseteq B$ and $M' \subseteq B'$ of size $\mathfrak{c}$ such that $B \setminus M$ and $B' \setminus M'$ are homeomorphic to the Baire space ${}^{\omega}\omega$. Let $g\colon B \to B'$ be any bijection such that the restriction $g{\restriction}(B \setminus M) : B \setminus M \to B' \setminus M'$ is a homeomorphism. Then $f = g \cup g^{-1}$ is a bijective mapping on $\mathbb{R}$ preserving the Baire category and $f(\mathbb{R} \setminus B) = B$.

(b) The case of $\mathcal{N}$. We say that a set $A \subseteq \mathbb{R}$ is *nowhere null*, if $\mu(P \cap I) > 0$ for every interval I with $A \cap I \neq \emptyset$.

If $P, Q \subseteq \mathbb{R}$ are perfect nowhere null nowhere dense sets of finite measure, then there is a homeomorphism $f\colon P \to Q$ such that $\mu\bigl(f(U)\bigr)/\mu(Q) = \mu(U)/\mu(P)$ for every relatively open set $U \subseteq P$. To see this, let us define $f(x) = y$ if and only if $\mu\bigl(P \cap (-\infty, x)\bigr)/\mu(P) = \mu\bigl(Q \cap (-\infty, y)\bigr)/\mu(Q)$.

Now, let $B \subseteq \mathbb{R}$ be a Borel set of positive measure such that $A = \mathbb{R} \setminus B$ has positive measure. We can find infinite systems of disjoint perfect nowhere null nowhere dense sets of finite measure $\{A_n : n \in \omega\}$ and $\{B_n : n \in \omega\}$ such that $A_n \subseteq A$, $B_n \subseteq B$ for $n \in \omega$, and $A' = A \setminus \bigcup_{n\in\omega} A_n$ and $B' = B \setminus \bigcup_{n\in\omega} B_n$ are null sets of size $\mathfrak{c}$. For $n \in \omega$, let $f_n\colon A_n \to B_n$ be the measure preserving homeomorphisms defined in the previous paragraph and $g\colon A' \to B'$

be any bijection. Then the function $f = g \cup g^{-1} \cup \bigcup_{n\in\omega} f_n \cup f_n^{-1}$ is a bijective mapping preserving null sets and $f(\mathbb{R} \setminus B) = B$.

(c) The homogeneity of $\mathcal{M} \cap \mathcal{N}$ follows by Lemma 3.2. $\square$

For $X \subseteq \mathbb{R}$, let $\mathcal{N}(X)$ and $\mathcal{M}(X)$ denote the ideal of measure zero subsets of X and the ideal of meager subsets X, respectively. We can ask about those X for which $\mathcal{N}(X)$ or $\mathcal{M}(X)$ is homogeneous with respect to relatively Borel subsets of X. Obviously, the proof of Theorem 3.4 works also if X is measurable or has the Baire property, respectively. On the other hand, there may be (at least consistently with ZFC) also other such sets because, if $\mathcal{I}$ is an ideal on X such that $\operatorname{add}(\mathcal{I}) = \operatorname{cof}(\mathcal{I})$, then $\mathcal{I}$ is $\mathcal{P}(X)$-homogeneous.

Theorem 3.4. *Let us assume that $\mathcal{I} \subseteq \mathcal{P}(X)$ is a σ-ideal and $\mathcal{B} \subseteq \mathcal{P}(X)$ is a σ-algebra which is a covering of $\mathcal{P}(X)$ modulo $\mathcal{I}$ such that $\mathcal{B}/\mathcal{I}$ has c.c.c. If $\mathcal{I}$ is a $\mathcal{B}$-homogeneous ideal, then $E(\mathcal{I}^+)$ implies $E(\mathcal{I}^{++})$, i.e., the following two conditions are equivalent:*

(1) $(\forall \varphi\colon X \to {}^\omega\omega)(\exists Y \in \mathcal{I}^+)\ \varphi(Y) \in \mathcal{K}_\sigma$.

(2) $(\forall \varphi\colon X \to {}^\omega\omega)(\exists Y \in \mathcal{I}^{++})\ \varphi(Y) \in \mathcal{K}_\sigma$.

Proof. Let us assume that (1) holds, and we prove (2). Let $\varphi\colon X \to {}^\omega\omega$.

We claim that for every $B \in \mathcal{B} \setminus \mathcal{I}$ there exists $Y \subseteq B$ such that $Y \in \mathcal{I}^+$ and $\varphi(Y) \in \mathcal{K}_\sigma$. To see this, let us fix $B \in \mathcal{B} \setminus \mathcal{I}$. Let $f\colon X \to X$ be such that $f(X \setminus B) \subseteq B$ and $Y \in \mathcal{I}$ if and only if $f(Y) \in \mathcal{I}$ for all $Y \subseteq X$. Let us define $\varphi'(x) = \varphi(x)$ for $x \in B$ and $\varphi'(x) = \varphi\big(f(x)\big)$ for $x \in X \setminus B$. By (1) there is $Z \in \mathcal{I}^+$ such that $\varphi'(Z) \in \mathcal{K}_\sigma$. If $Z \cap B \in \mathcal{I}^+$, then we set $Y = Z \cap B$. Otherwise, $Z \setminus B \in \mathcal{I}^+$ and then we set $Y = f(Z \setminus B)$. In both cases, $Y \subseteq B$, $Y \in \mathcal{I}^+$, and $\varphi(Y) \in \mathcal{K}_\sigma$.

Now, let F be a maximal system of pairs (Z, B), where $Z \in \mathcal{I}^+$, $\varphi(Z) \in \mathcal{K}_\sigma$, B is a cover of Z in $\mathcal{B}$ modulo $\mathcal{I}$, and $B \cap B' \in \mathcal{I}$ for distinct $(Z, B), (Z', B') \in F$. Let $Y = \bigcup_{(Z,B)\in F} Z$. Then $Y \in \mathcal{I}^{++}$ and, as $\mathcal{B}/\mathcal{I}$ has c.c.c., F is countable and so, $\varphi(Y) \in \mathcal{K}_\sigma$. $\square$

Obviously, the ideals $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{M} \cap \mathcal{N}$ satisfy the assumptions of Theorem 3.4. Now, we can say a bit more than Theorem 1.13 says for the assertion $E(\mathcal{N}^{++})$.

Corollary 3.5.

(a) *The following conditions are equivalent:*

(1) $\neg E(\mathcal{N}^+)$.

(2) $\neg E(\mathcal{N}^{++})$.

(3) $\big([0,1], \mathcal{N}, \in\big) \preceq ({}^\omega\omega, {}^\omega\omega, \leq^*)$.

(4) $\big([0,1], \{A \subseteq [0,1] : \mu^*(A) < 1\}, \in\big) \preceq ({}^\omega\omega, {}^\omega\omega, \leq^*)$.

(b) *The following conditions hold:*

(1) $\neg E(\mathcal{M}^+)$.

(2) $\neg E(\mathcal{M}^{++})$.

(3) $\big([0,1],\mathcal{M},\in\big) \preceq ({}^{\omega}\omega,{}^{\omega}\omega,\leq^*)$.

(4) $\big([0,1],\{A\subseteq[0,1] : A\cap U\in\mathcal{M}$ *for some open* $U\neq\emptyset\},\in\big) \preceq ({}^{\omega}\omega,{}^{\omega}\omega,\leq^*)$.

(c) *The following conditions are equivalent:*

(1) $\neg E\big((\mathcal{M}\cap\mathcal{N})^+\big)$.

(2) $\neg E\big((\mathcal{M}\cap\mathcal{N})^{++}\big)$.

(3) $\big([0,1],\mathcal{M}\cap\mathcal{N},\in\big) \preceq ({}^{\omega}\omega,{}^{\omega}\omega,\leq^*)$.

(4) $\big([0,1],\{A\subseteq[0,1] : A\cap U\in\mathcal{M}\cap\mathcal{N}$ *for some open* $U\neq\emptyset\},\in\big) \preceq ({}^{\omega}\omega,{}^{\omega}\omega,\leq^*)$.

Proof. The equivalences hold by Theorem 3.4 and by Theorem 1.13 in all three cases. It is well-known that there exists a Galois-Tukey morphism $\big([0,1],\mathcal{M},\in\big)\preceq({}^{\omega}\omega,{}^{\omega}\omega,\leq^*)$ (see [1], [6]). □

Corollary 3.6. *$E(\mathcal{N}^+)$ holds if and only if $E\big((\mathcal{M}\cap\mathcal{N})^+\big)$ holds.*

Proof. Let us assume that $E\big((\mathcal{M}\cap\mathcal{N})^+\big)$ holds and let $F\colon\mathbb{R}\to c_0$ be given. As $\neg E(\mathcal{M}^+)$ holds, there is $H\colon\mathbb{R}\to c_0$ such that for every $Y\subseteq\mathbb{R}$, if $H\xrightarrow{\text{QN}}0$ on Y, then $Y\in\mathcal{M}$. Let $G=\max\{F,H\}$. Then there is $Y\in(\mathcal{M}\cap\mathcal{N})^+$ such that $G\xrightarrow{\text{QN}}0$ on Y. As $F\leq G$ also $F\xrightarrow{\text{QN}}0$ on Y. Then $Y\in\mathcal{M}$ by the choice of H, and so $Y\in\mathcal{N}^+$. It follows that $E(\mathcal{N}^+)$ holds. The inverse implication is a consequence of the inclusion $\mathcal{N}^+\subseteq(\mathcal{M}\cap\mathcal{N})^+$. □

Question 3.7. If $\mathcal{F}_1\subseteq\mathcal{F}_2\subseteq\mathcal{P}(X)$, then $E(\mathcal{F}_1)$ implies $E(\mathcal{F}_2)$. This fact can be expressed by a Galois-Tukey embedding $\big(X,\mathcal{P}(X)\setminus\mathcal{F}_1,\in\big)\preceq\big(X,\mathcal{P}(X)\setminus\mathcal{F}_2,\in\big)$ given by the pair of identity functions. Under what conditions does an inverse embedding exist?

Question 3.8. Which implications of the form $E(\mathcal{F}_1)\Rightarrow E(\mathcal{F}_2)$ with $\mathcal{F}_1,\mathcal{F}_2\subseteq\mathcal{P}(X)$ do imply a Galois-Tukey embedding

$$\big(X,\mathcal{P}(X)\setminus\mathcal{F}_2,\in\big)\preceq\big(X,\mathcal{P}(X)\setminus\mathcal{F}_1,\in\big)?$$

The assertion $E(\mathcal{N}^{++})$ is denoted by (GES) in [8]. By Corollary 3.5, ¬(GES) is equivalent to $\neg E(\mathcal{N}^+)$ which by Corollary 1.11 is equivalent to the condition

$$(\exists\,\varphi\colon[0,1]\to{}^{\omega}\omega\text{ bijective})(\forall\,Y\in\mathcal{N}^+)\ \varphi(Y)\notin\mathcal{K}_\sigma.$$

Condition $L(\mathcal{N}^+)$ from Corollary 1.15 states the existence of a non$(\mathcal{N})$-$\mathcal{K}_\sigma$-Luzin set of cardinality $\mathfrak{c}$ (see Definition 4.1 below). Let us consider another condition:

(M) There exists a non-atomic real-valued σ-additive measure ν on a σ-algebra of subsets of ${}^{\omega}\omega$ such that $\nu({}^{\omega}\omega) = 1$ and $\nu(K) = 0$ for all $K \in \mathcal{K}_\sigma$.

Now, it is obvious that $L(\mathcal{N}^+)$ implies $\neg$(GES) and $\neg$(GES) implies (M).

4. Other questions related to $E(\mathcal{F})$

Definition 4.1.

(1) A sequence $\langle f_\xi : \xi < \kappa \rangle$ in ${}^{\omega}\omega$ is a *B-sequence*, if for every $f \in {}^{\omega}\omega$ there is $\eta < \kappa$ such that $f_\xi \not\leq^* f$ for all $\xi > \eta$. A cardinal κ is a *B-cardinal* (i.e., $\mathfrak{b}$ *like cardinal number*), if there exists a B-sequence of the length κ.

(2) Let $\mathcal{I} \subseteq \mathcal{P}(X)$. A set $Y \subseteq X$ is a *κ-$\mathcal{I}$-Luzin set*, if $|Y| \geq \kappa$ and $|Y \cap A| < \kappa$ for all $A \in \mathcal{I}$.

(3) Let $F \subseteq \mathcal{P}(\omega)$ be a filter. For $f, g \in {}^{\omega}\omega$ we define
$$f \leq_F g \equiv (\exists\, A \in F)(\forall\, n \in A)\ f(n) \leq g(n).$$
Let $\mathfrak{b}_F$ and $\mathfrak{d}_F$ denote, respectively, the unbounding number and the dominating number for this quasi-ordering of ${}^{\omega}\omega$.

Let us note that $\mathfrak{b} = \mathfrak{b}_F$ and $\mathfrak{d} = \mathfrak{d}_F$ for Fréchet filter F. If $F \subseteq F'$ are filters that extend Fréchet filter, then $\mathfrak{b}_F \leq \mathfrak{b}_{F'} \leq \mathfrak{d}_{F'} \leq \mathfrak{d}_F$. The reader can find more information on cardinals $\mathfrak{b}_F$ and $\mathfrak{d}_F$ in [3].

Here, we list several facts on B-cardinals and $\mathcal{K}_\sigma$-Luzin sets:

Theorem 4.2.

(1) *$\mathfrak{b}$ is the least B-cardinal and if a filter F contains Fréchet filter, then $\mathfrak{b}_F$ and $\mathfrak{d}_F$ are B-cardinals.*

(2) *κ is a B-cardinal if and only if $\operatorname{cf}\kappa$ is a B-cardinal.*

(3) *If κ is a regular B-cardinal, then $\mathfrak{b} \leq \kappa \leq \mathfrak{d}$.*

(4) *If $\kappa \leq \mathfrak{c}$ is a B-cardinal, then there exists a κ-$\mathcal{K}_\sigma$-Luzin set $X \subseteq {}^{\omega}\omega$ such that κ is the least cardinal such that X is a κ-$\mathcal{K}_\sigma$-Luzin set.*

(5) *If there exists a κ-$\mathcal{K}_\sigma$-Luzin set $X \subseteq {}^{\omega}\omega$ of size λ, then $\mathfrak{b} \leq \kappa$ and every regular cardinal μ with $\kappa \leq \mu \leq \lambda$ is a B-cardinal. If, moreover, $\kappa < \lambda$, then $\lambda \leq \mathfrak{d}$.*

(6) *A cardinal κ is a B-cardinal if and only if there exists a $\operatorname{cf}(\kappa)$-$\mathcal{K}_\sigma$-Luzin set in ${}^{\omega}\omega$.*

(7) *If there exists a κ-Luzin set X of size λ, then $\operatorname{non}(\mathcal{M}) \leq \kappa$ and $\mu \leq \operatorname{cov}(\mathcal{M})$ for every regular cardinal μ with $\kappa \leq \mu \leq \lambda$. If, moreover, $\kappa < \lambda$, then $\mathfrak{b} = \operatorname{add}(\mathcal{M}) \leq \operatorname{non}(\mathcal{M}) \leq \kappa < \lambda \leq \operatorname{cov}(\mathcal{M}) \leq \operatorname{cof}(\mathcal{M}) = \mathfrak{d}$.*

(8) *If $\operatorname{cov}(\mathcal{M}) = \operatorname{cof}(\mathcal{M}) = \kappa$, then there exists a κ-Luzin set of size κ.*

Proof. (1) Let $\langle f_\xi : \xi < \mathfrak{b}\rangle$ be an $\leq_F$-unbounded system of functions such that $f_\xi \leq_F f_\eta$ for $\xi < \eta < \mathfrak{b}$ and let $\langle g_\xi : \xi < \mathfrak{d}\rangle$ be a $\leq_F$-dominating system of functions such that $g_\xi \not\geq_F g_\eta$ for $\xi < \eta < \mathfrak{d}$. Both these sequences are B-sequences.

For (2) it is enough to realize that elements in a B-sequence can repeat (not cofinally many times) and every cofinal subsequence of a B-sequence is a B-sequence.

(3) If κ is a regular B-cardinal, then $\mathfrak{b} \leq \kappa$ by (1). We prove that also $\kappa \leq \mathfrak{d}$; by (2), $\operatorname{cf}\mathfrak{d}$ is a regular B-cardinal. Let $\langle f_\xi : \xi < \kappa\rangle$ be any B-sequence and let $D \subseteq {}^\omega\omega$ be a dominating family of size $\mathfrak{d}$. For $f \in D$ let $X_f = \{\xi < \kappa : f_\xi \leq^* f\}$. Clearly, $|X_f| < \kappa$ and $\kappa = \bigcup_{f\in D} X_f$. It follows that $|D| \geq \kappa$ because κ is regular.

(4) Let $\langle f_\xi : \xi < \operatorname{cf}\kappa\rangle$ be a B-sequence and let $D_\xi \subseteq {}^\omega 2$ for $\xi < \operatorname{cf}\kappa$ be such that $|D_\xi| < \kappa$ and $|\bigcup_{\xi<\operatorname{cf}\kappa} D_\xi| = \kappa$. Define $X = \big\{f + g : (\exists\, \xi < \operatorname{cf}\kappa)\ f = f_\xi$ and $g \in D_\xi\big\}$. Clearly $|X| = \kappa$ and X is a κ-$\mathcal{K}_\sigma$-Luzin set with κ minimal.

(5) If $\sigma = \langle f_\xi : \xi < \mu\rangle$ is a sequence of distinct elements of X, then σ is a B-sequence whenever $\operatorname{cf}\mu \geq \kappa$. The inequality $\mathfrak{b} \leq \kappa$ holds because every subset of X of size $< \mathfrak{b}$ is bounded, and hence has size $< \kappa$. By an argument similar to the proof of (3), we can see that if $\kappa < \lambda$ then $\lambda \leq \kappa\mathfrak{d} = \mathfrak{d}$.

(6) By (2), we can restrict to regular cardinals. For regular cardinals, the assertion follows by (4) and (5).

(7) $\operatorname{non}(\mathcal{M}) \leq \kappa$ because every subset of X of size $< \operatorname{non}(\mathcal{M})$ is meager and hence has size $< \kappa$. Let us assume that $\kappa \leq \mu \leq \lambda$ and μ is regular. If C is a system of meager subsets of X covering X, then $|\bigcup C| \geq \mu$ and, as each set in C has size $< \mu$ and μ is regular, it follows that $|C| \geq \mu$. Therefore, $\operatorname{cov}(\mathcal{M}) \geq \mu$. So, if $\kappa < \lambda$, then $\operatorname{cov}(\mathcal{M}) \geq \lambda$, and the rest is a consequence of the equalities $\operatorname{add}(\mathcal{M}) = \min\big\{\operatorname{cov}(\mathcal{M}), \mathfrak{b}\big\}$ and $\operatorname{cof}(\mathcal{M}) = \max\big\{\operatorname{non}(\mathcal{M}), \mathfrak{d}\big\}$ (see [6], [2]).

(8) Let $\langle M_\alpha : \alpha < \kappa\rangle$ be an enumeration of a base of $\mathcal{M}$. By induction, choose $x_\alpha \in \mathbb{R} \setminus \bigcup_{\beta<\alpha} M_\beta$. Then $X = \{x_\alpha : \alpha < \kappa\}$ is a κ-Luzin set. □

Let us note that κ is a B-cardinal if and only if there exists a Galois-Tukey morphism $(\kappa, \kappa, \leq) \preceq ({}^\omega\omega, {}^\omega\omega, \leq^*)$. For example, if $\langle f_\xi : \xi < \kappa\rangle$ is a B-sequence, then the pair of functions $\varphi\colon \kappa \to {}^\omega\omega$ and $\psi\colon {}^\omega\omega \to \kappa$ defined by $\varphi(\xi) = f_\xi$ and $\psi(f) = \sup\{\xi : f_\xi \leq^* f\}$ is a morphism because $\varphi(\xi) \leq^* f$ implies $\xi \leq \psi(f)$. This fact gives another argument for conditions (2) and (3). We do not know whether supremum of regular B-cardinals can be strictly smaller than $\mathfrak{d}$ (in this case, by (1), $\mathfrak{d}$ must be singular).

By adding Cohen reals, we obtain a model in which $\mathfrak{b} = \omega_1 < \mathfrak{c}$ and there is an ω_1-$\mathcal{K}_\sigma$-Luzin set of cardinality $\mathfrak{c}$. In this model every regular cardinal with $\omega_1 \leq \kappa \leq \mathfrak{c}$ is the cofinality $\mathfrak{d}_U$ of an ultraproduct ${}^\omega\omega/U$ for some ultrafilter U on ω (see [5]).

Using B-cardinals, we can rewrite Corollary 1.15 as follows (compare with [8, Proposition 8]):

Corollary 4.3. *Let* $|X| \leq \mathfrak{c}$, *let* $\mathcal{F} \subseteq \mathcal{P}(X)$, *and let* $\mathcal{I} = \mathcal{P}(X) \setminus \mathcal{F}$.

(1) *If there exists a* $\operatorname{non}(\mathcal{I})$-$\mathcal{K}_\sigma$-*Luzin set of size* $|X|$, *then* $\neg E(\mathcal{F})$ *holds.*

(2) *If* $|X|$ *is a* B-*cardinal and* $\operatorname{non}(\mathcal{I}) = |X|$, *then* $\neg E(\mathcal{F})$ *holds.*

Notice that (2) is a special case of (1).

In the case of measure, Corollary 4.3 states:

Corollary 4.4 ([8, Proposition 8])**.**

(1) *If there exists a* $\operatorname{non}(\mathcal{N})$-$\mathcal{K}_\sigma$-*Luzin set of size* $\mathfrak{c}$, *then* $\neg$(GES) *holds.*

(2) *If* $\mathfrak{c}$ *is a* B-*cardinal and* $\operatorname{non}(\mathcal{N}) = \mathfrak{c}$, *then* $\neg$(GES) *holds.*

Let us recall a theorem of W. Sierpiński ([9, Proposition P_3]): CH holds if and only if there are functions $f_n\colon \mathbb{R} \to \mathbb{R}$ for $n \in \omega$ such that $\big(\forall A \in [\mathbb{R}]^{\geq \omega_1}\big)$ $(\forall^\infty n \in \omega)$ $f_n(A) = \mathbb{R}$. This theorem was a motivation for the next characterization of κ-$\mathcal{K}_\sigma$-Luzin sets.

Theorem 4.5. *The following conditions are equivalent for any* $\kappa \leq \lambda \leq \mathfrak{c}$ *with* $\operatorname{cf} \kappa \geq \omega_1$:

(1) *There is a* κ-$\mathcal{K}_\sigma$-*Luzin set of size* λ.

(2) *There are functions* $f_n\colon \lambda \to \omega$ *for* $n \in \omega$ *such that* $\big(\forall A \in [\lambda]^\kappa\big)(\forall^\infty n \in \omega)$ $|f_n(A)| = \omega$.

(3) *There are functions* $f_n\colon \lambda \to \omega$ *for* $n \in \omega$ *such that* $\big(\forall A \in [\lambda]^\kappa\big)(\exists\, n \in \omega)$ $|f_n(A)| = \omega$.

Proof. (1) $\Rightarrow$ (3): Let $L \subseteq {}^\omega\omega$ be a κ-$\mathcal{K}_\sigma$-Luzin set of size λ and let $\{g_\xi\colon \xi < \lambda\}$ be a one-to-one enumeration of L. Let us define $f_n\colon \lambda \to \omega$ by $f_n(\xi) = g_\xi(n)$. To obtain a contradiction, let us assume that there is $A \in [\lambda]^\kappa$ such that $|f_n(A)| < \omega$ for all $n \in \omega$. Hence, there is $h \in {}^\omega\omega$ such that $f_n(\xi) \leq h(n)$ for all n. Then $\{g_\xi : \xi \in A\} \in \mathcal{K}_\sigma$ which is a contradiction.

(3) $\Rightarrow$ (1): Let $f_n\colon \lambda \to \omega$ for $n \in \omega$ satisfy (3). Let us define $g_\xi \in {}^\omega\omega$ by $g_\xi(n) = f_n(\xi)$. By induction on $\xi < \lambda$, let us define $h_\xi = g_\xi + y_\xi$ where $y_\xi \in {}^\omega 2$ is such that $h_\xi \neq h_\eta$ for all $\eta < \xi$. Then the set $L = \{h_\xi : \xi < \lambda\}$ has size λ and we prove that it is a κ-$\mathcal{K}_\sigma$-Luzin set. To obtain a contradiction let us assume that we have $A \in [\lambda]^\kappa$ such that $\{h_\xi : \xi \in A\} \in \mathcal{K}_\sigma$. As $\operatorname{cf} \kappa > \omega$, there is $B \in [A]^\kappa$ and $h \in {}^\omega\omega$ such that $h_\xi \leq h$ for all $\xi \in B$. Then $f_n(\xi) \leq h_\xi(n) \leq h(n)$ for all $\xi \in B$ and $n \in \omega$ which contradicts condition (3).

(3) $\Rightarrow$ (2): If $f_n\colon \lambda \to \omega$ for $n \in \omega$ satisfy (3), then $f'_n(\xi) = \max\big\{f_i(\xi) : i \leq n\big\}$ for $n \in \omega$ satisfy (2). The implication (2) $\Rightarrow$ (3) is trivial. $\square$

For a while, let us consider a special case of Corollary 4.3:

Let $\mathcal{F} \subseteq \mathcal{P}(\mathbb{R})$ *and* $\mathcal{I} = \mathcal{P}(\mathbb{R}) \setminus \mathcal{F}$ *be such that* $\mathcal{F} \cap [\mathbb{R}]^{\leq\omega} = \emptyset$*. Then CH implies* $\neg E(\mathcal{F})$. (Because $\operatorname{non}(\mathcal{I}) = \mathfrak{c}$ and there exists a $\mathfrak{c}$-$\mathcal{K}_\sigma$-Luzin set of size $\mathfrak{c}$.)

This special case has this application: If $\mathcal{F} \subseteq \mathcal{P}(X)$ does not contain countable sets and the definition of $\mathcal{F}$ does not contradict either CH or $|X| = \omega_1$, then $E(\mathcal{F})$ is not provable in ZFC, i.e., $E(\mathcal{F})$ is independent from ZFC if and only if $E(\mathcal{F})$ is consistent with ZFC.

Examples.

1. Let $\mathcal{E} \subseteq \mathcal{P}(\mathbb{R})$ be the σ-ideal generated by closed sets of measure 0. Then $\mathcal{E} \subseteq \mathcal{M} \cap \mathcal{N}$ and $E\big((\mathcal{M} \cap \mathcal{N})^+\big)$ is consistent with ZFC by Corollary 3.6 (because $E(\mathcal{N}^{++})$ is consistent, see [8] or [11]). As $(\mathcal{M} \cap \mathcal{N})^+ \subseteq \mathcal{E}^+$, $E(\mathcal{E}^+)$ is consistent with ZFC and hence independent from ZFC.

2. Let s^0 be the Marczewski ideal. It is well-known that $\mathfrak{d} < \operatorname{cov}(s^0)$ holds in the forcing extension of a model of ZFC + CH via a countable support iteration of Sacks forcing of length ω_2 (see [1] and [7]). Therefore, $E\big((s^0)^+\big)$ is consistent by Corollary 1.14 and consequently, $E\big((s^0)^+\big)$ is independent from ZFC.

3. Let $(s^0)^{++}$ be the family of all sets $Y \subseteq \mathbb{R}$ such that $Y \cap P \neq \emptyset$ for all perfect subsets $P \subseteq \mathbb{R}$. We prove $\neg E\big((s^0)^{++}\big)$: Let $\varphi\colon \mathbb{R} \to {}^{\omega}\omega$ be such that the restriction $\varphi{\restriction}\mathbb{I}\mathrm{r}\colon \mathbb{I}\mathrm{r} \to {}^{\omega}\omega$ is a homeomorphism from the set of irrational numbers onto the Baire space. Now, if there is $f \in {}^{\omega}\omega$ such that $\varphi(y) \leq^* f$ for all $y \in Y$, i.e., $\varphi(Y) \in \mathcal{K}_\sigma$, then the set $Z = \big\{x \in {}^{\omega}\omega : (\forall\, n \in \omega)(x(n) = f(n) + 1$ or $x(n) = f(n) + 2)\big\}$ is compact perfect subset of ${}^{\omega}\omega$ and $\varphi^{-1}(Z) \cap \mathbb{I}\mathrm{r}$ is a perfect set disjoint from Y. Therefore, $Y \notin (s^0)^{++}$. It follows that $\neg E\big((s^0)^{++}\big)$ holds.

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