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CARDINAL INVARIANTS RELATED TO POROUS SETS

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Abstract. The paper is concerned with cardinal invariants of σ-porous sets and tries to give a picture of relationships between them and the cardial invariants of Cichoń's diagram.

The class of σ-porous sets was introduced by E. P. Dolženko [**5**] in his study of certain exceptional sets in the theory of the cluster sets. Since there have been proved many theorems using porosity and σ-porosity in the cluster sets theory. Even much earlier results in differentiation theory and results concerning some exceptional sets in the theory of trigonometrical series (e.g. [**3**], [**4**]) used notions equivalent to porosity and symmetric porosity. Thus the investigation of properties of the σ-ideal of σ-porous sets (and other similar notions) seem to be important. The topical survey [**17**] of L. Zajíček gives a lot of informations on applications mentioned above.

We are especially interested in cardinal invariants of such σ-ideals those ones which are commonly studied for the σ-ideals $\mathbf{L}$ of Lebesgue measure zero sets and $\mathbf{K}$ of first category sets on the real line. Let $\mathcal{I}$ be an ideal.

$$\begin{aligned}
\operatorname{add}\mathcal{I} &= \min\{|\mathcal{I}_0| : \mathcal{I}_0 \subseteq \mathcal{I} and \bigcup\mathcal{I}_0 \notin \mathcal{I}\},\\
\operatorname{cov}\mathcal{I} &= \min\{|\mathcal{I}_0| : \mathcal{I}_0 \subseteq \mathcal{I} and \bigcup\mathcal{I}_0 = \bigcup\mathcal{I}\},\\
\operatorname{non}\mathcal{I} &= \min\{|A| : A \subseteq \bigcup\mathcal{I} and\ A \notin \mathcal{I}\},\\
\operatorname{cof}\mathcal{I} &= \min\{|\mathcal{I}_0| : \mathcal{I}_0 \subseteq \mathcal{I} and\ (\forall A \in \mathcal{I})(\exists B \in \mathcal{I}_0)\, A \subseteq B\}.
\end{aligned}$$

For the cardinal invariants of the ideals $\mathbf{K}, \mathbf{L}$ were found nice combinatorial characterizations. We can hardly expect that this will succeed with the cardinal invariants of porosity ideals. Partly we can be satisfied when we will be able to examine the relationship between them and the classical cardinal invariants which are perhaps better characterizable.

Let A be a set of reals. The porosity and symmetric porosity of the set A at the real $a \in \mathbf{R}$ are defined by

$$p(A,a) = \limsup_{\varepsilon\to 0+} \frac{\lambda(A,(a-\varepsilon,a+\varepsilon))}{\varepsilon}, \quad \text{and}$$
$$s(A,a) = \limsup_{\varepsilon\to 0+} \frac{\lambda^*(A,(a-\varepsilon,a+\varepsilon))}{\varepsilon}$$

respectively, where $\lambda(A,I)$ denotes the maximal length of an open subinterval of the interval I which is disjoint with A. Similarly $\lambda^*(A,(c,d))$ is the maximal $\delta \geq 0$ such that the set $(c,c+\delta) \cup (d-\delta,d)$ is disjoint with A. Notice that for $a \in A$, $\lambda^*(A,(a-\varepsilon,a+\varepsilon)) \leq \lambda(A,(a-\varepsilon,a+\varepsilon)) \leq \varepsilon$ and $s(A,a) \leq p(A,a) \leq 1$. A is porous (resp. strongly porous) if $p(A,a) > 0$ (resp. $p(A,a) = 1$) for every $a \in A$; symmetrically porous (resp. strongly symmetrically porous) if $s(A,a) > 0$ (resp. $s(A,a) = 1$) for every $a \in A$. A is σ-porous if it is a countable union of porous sets. The notions of σ-strongly porous, σ-symmetrically porous, and σ-strongly symmetrically porous sets ([**17**]) are defined similarly.

Let $\mathbf{P}$, $\mathbf{P}^+$, $\mathbf{S}$, $\mathbf{S}^+$ denote the σ-ideals of σ-strongly porous sets, σ-porous sets, σ-strongly symmetrically porous sets and σ-symmetrically porous sets respectively. Evidently, $\mathbf{S} \subseteq \mathbf{P}$, $\mathbf{S} \subseteq \mathbf{S}^+$, $\mathbf{S}^+ \subseteq \mathbf{P}^+$, $\mathbf{P} \subseteq \mathbf{P}^+$. These four ideals are mutually different (the symmetric perfect sets are counterexamples to all equalities between them, see [**17**] and [**14**]) and it would be interesting to know whether they differ in some cardinal invariants.

Every porous set is nowhere dense and it follows from the Lebesgue density theorem that it has Lebesgue measure zero. By I. Recław [**11**] every γ-set is σ-strongly symmetrically porous while ([**8**]) the minimal cardinality of a set which is not a γ-set is the cardinal number $\mathbf{p}$ (the minimal cardinality of a centered subfamily of $[\omega]^\omega$ which has no infinite pseudo-intersection, [**6**]). Hence we have the following:

THEOREM 1.

a) $\mathbf{p} \leq \operatorname{non}\mathbf{S} \leq \operatorname{non}\mathbf{P}^+ \leq \min\{\operatorname{non}\mathbf{K}, \operatorname{non}\mathbf{L}\}$.
b) $\operatorname{cov}\mathbf{P} \geq \max\{\operatorname{cov}\mathbf{K}, \operatorname{cov}\mathbf{L}\}$.

The next theorem gives another lower bound for $\operatorname{non}\mathbf{S}$ which is independent with the previous one.

THEOREM 2 ([**12**]). $\operatorname{add}\mathbf{L} \leq \operatorname{non}\mathbf{S}$ *and* $\operatorname{cov}\mathbf{S} \leq \operatorname{cof}\mathbf{L}$.

The proof of Theorem 2 uses the Cantor expansion of reals through a function $\rho \in {}^\omega\omega$ with $\lim_{n\to\infty} \rho(n) = \infty$. The reals $a \in \langle 0,1\rangle$ are represented exactly by sequences $x \in {}^\omega\omega$ with $x < \rho$ (i.e. $x(n) < \rho(n)$ for all n) such that

$$a = \varphi_\rho(x) = \sum_{n\in\omega} \frac{x(n)}{\rho(0)\rho(1)\dots\rho(n)}.$$

Let $T_\rho = \{s \in {}^{<\omega}\omega : (\forall n \in \operatorname{dom} s)\, s(n) < \rho(n)\}$. A set $A \subseteq \langle 0,1\rangle$ is said to be ρ-small iff there exists a function $h : T_\rho \to [\omega]^{<\omega}$ with

$$\lim_{n\to\infty} \frac{\sup\{|h(s)| : s \in T_\rho \textit{and } |s| = n\}}{\log \rho(n)} = 0$$

such that $A \subseteq \varphi_\rho(\{x < \rho : (\exists m)(\forall n > m)\, x(n) \in h(x\restriction n)\})$. The family $\mathbf{Sm}_\rho$ of all ρ-small sets is a σ-ideal containing all singletons and $\mathbf{Sm}_\rho \subseteq \mathbf{S}$. At last, $\operatorname{add}\mathbf{L} \le \operatorname{add}\mathbf{Sm}_\rho$ and $\operatorname{cof}\mathbf{Sm}_\rho \le \operatorname{cof}\mathbf{L}$, which concludes the proof of Theorem 2.

Let $\mathcal{P}_n$ be the family of all partitions of ω into n infinite sets. Let us recall that for sets x, y, $x \subseteq^* y$ means $x - y$ is finite. Let us define

$$\mathbf{r}_n = \min\{|\mathcal{X}| : \mathcal{X} \subseteq [\omega]^\omega \textit{and } (\forall A \in \mathcal{P}_n)(\exists x \in \mathcal{X})(\exists y \in A)\, x \subseteq^* y\},$$
$$\mathbf{s}_n = \min\{|\mathcal{X}| : \mathcal{X} \subseteq \mathcal{P}_n \textit{and } (\forall x \in [\omega]^\omega)(\exists A \in \mathcal{X})(\forall y \in A)\, x \not\subseteq^* y\}.$$

The reaping number $\mathbf{r}_n$ was defined by J. van Mill and B. Balcar has shown that $\mathbf{r}_n = \mathbf{r}_2 = \mathbf{r}$ for all $n \ge 2$ ([**15**]). Similar arguments prove that each $\mathbf{s}_n$ is the splitting number $\mathbf{s} = \mathbf{s}_2$ ([**6**]).

THEOREM 3. $\mathbf{s} \le \operatorname{non}\mathbf{S}^+$, $\operatorname{cov}\mathbf{S}^+ \le \mathbf{r}$.

PROOF. Let $n \ge 3$. For $g \in {}^\omega n$ let $\varphi_n(g) = \sum_{n\in\omega} g(i) n^{-i-1}$ (i.e. g is an n-adic expansion of the real $\varphi_n(g) \in \langle 0,1\rangle$). One can easily verify that if $x \in [\omega]^\omega$ and $i < n$ then the set

$$A_{x,i} = \varphi_n(\{g \in {}^\omega n : (\forall k \in x)\, g(k) = i\})$$

is symmetrically porous since $s(A_{x,i}, a) \ge (n-2)/(n-1) > 0$ for each $a \in A_{x,i}$. Each $g \in {}^\omega n$ is in fact a partition of ω into n sets. Hence almost immediatelly we get $\mathbf{s}_n \le \operatorname{non}\mathbf{S}^+$ and $\operatorname{cov}\mathbf{S}^+ \le \mathbf{r}_n$. □

We do not know whether Theorem 3 holds true for the ideals $\mathbf{P}, \mathbf{S}$ too.

We have much less informations on the cardinal invariants $\operatorname{add}\mathcal{I}$, $\operatorname{cof}\mathcal{I}$ with $\mathcal{I} = \mathbf{S}, \mathbf{P}, \mathbf{S}^+, \mathbf{P}^+$. We cannot say even if the inequalities $\operatorname{add}\mathcal{I} > \omega_1$, $\operatorname{cof}\mathcal{I} < 2^\omega$ are possible. On the other hand we have:

THEOREM 4 ([**13**]). *Let $\mathcal{I}$ be any ideal with $\mathbf{S} \subseteq \mathcal{I} \subseteq \mathbf{K}$. Then* $\operatorname{add}\mathcal{I} \le \mathbf{b}$, $\mathbf{d} \le \operatorname{cof}\mathcal{I}$.

The cardinals $\mathbf{b}, \mathbf{d}$ are the minimal cardinalities of an unbounded family and a cofinal family of functions respectively (in ${}^\omega\omega$ with eventual dominance). The proof of Theorem 4 is a slight modification of A. W. Miller's proof of $\operatorname{add}\mathbf{K} \le \mathbf{b}$ ([**10**]).

L. Zajíček [**16**] defined porosity in a general metric space taking in the definition of the porosity open balls instead of open intervals. For instance, in the space ${}^\omega\omega$ with the Baire metric, every set

$$A_f = \{g \in {}^\omega\omega : (\exists m)(\forall n > m)\, f(n) \ne g(n)\}, \quad f \in {}^\omega\omega,$$

is σ-strongly porous. Hence, using the following equalities (see [**1**], the symbols $(\exists^\infty n)$, $(\forall^\infty n)$ stand for $(\forall m)(\exists n > m)$, $(\exists m)(\forall n > m)$)

$$\operatorname{non}\mathbf{K} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\omega\omega \textit{and } (\forall h \in {}^\omega\omega)(\exists g \in \mathcal{F})(\exists^\infty n)\, h(n) = g(n)\},$$
$$\operatorname{cov}\mathbf{K} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\omega\omega \textit{and } (\forall h \in {}^\omega\omega)(\exists g \in \mathcal{F})(\forall^\infty n)\, h(n) \neq g(n)\}$$

and the fact that σ-porous sets are meager, we immediately get:

$$\operatorname{non}\mathbf{P}_{^\omega\omega} = \operatorname{non}\mathbf{P}^+_{^\omega\omega} = \operatorname{non}\mathbf{K},$$
$$\operatorname{cov}\mathbf{P}_{^\omega\omega} = \operatorname{cov}\mathbf{P}^+_{^\omega\omega} = \operatorname{cov}\mathbf{K}.$$

Consistency results on non P. We assume that the reader is familiar with Cichoń's diagram (see [**7**]). According to the previous results it remains to compare the cardinals $\operatorname{non}\mathcal{I}$, $\mathcal{I} = \mathbf{S}, \mathbf{P}, \mathbf{S}^+, \mathbf{P}^+$, with the following five cardinals of the diagram: $\operatorname{cov}\mathbf{L}$, $\operatorname{cov}\mathbf{K}$, $\mathbf{b}$, $\mathbf{d}$, $\operatorname{add}\mathbf{K}$. Let us recall that $\mathbf{P}^+$ is the largest and $\mathbf{S}$ is the smallest of these four ideals.

THEOREM 5.

(1a) $\operatorname{Con}(\operatorname{non}\mathbf{P}^+ < \operatorname{cov}\mathbf{L})$	(1b) $\operatorname{Con}(\operatorname{non}\mathbf{S} > \operatorname{cov}\mathbf{L})$
(2a) $\operatorname{Con}(\operatorname{non}\mathbf{P}^+ < \operatorname{cov}\mathbf{K})$	(2b) $\operatorname{Con}(\operatorname{non}\mathbf{S}^+ > \operatorname{cov}\mathbf{K})$
(3a) $\operatorname{Con}(\operatorname{non}\mathbf{P}^+ < \mathbf{b})$	(3b) $\operatorname{Con}(\operatorname{non}\mathbf{S} > \mathbf{b})$
(4a) $\operatorname{Con}(\operatorname{non}\mathbf{P}^+ < \mathbf{d})$	(4b) ?
(5a) ?	(5b) $\operatorname{Con}(\operatorname{non}\mathbf{S} > \operatorname{add}\mathbf{K})$

PROOF. The proof is a consequence of known consistency results:

(1a) $\operatorname{Con}(\operatorname{cov}\mathbf{L} > \operatorname{non}\mathbf{L})$ and $\operatorname{non}\mathbf{L} \geq \operatorname{non}\mathbf{P}^+$.

(1b) $\operatorname{Con}(\operatorname{cov}\mathbf{L} < \mathbf{p})$ and $\mathbf{p} \leq \operatorname{non}\mathbf{S}$.

(2a) $\operatorname{Con}(\operatorname{cov}\mathbf{K} > \operatorname{non}\mathbf{K})$ and $\operatorname{non}\mathbf{K} \geq \operatorname{non}\mathbf{P}^+$.

(2b) $\operatorname{non}\mathbf{S}^+ \geq \mathbf{s}$ and $\operatorname{Con}(\mathbf{s} > \operatorname{cov}\mathbf{K})$ (Mathias reals – [**10**]).

(3a), (4a) $\mathbf{d} \geq \mathbf{b}$, $\operatorname{Con}(\mathbf{b} > \operatorname{non}\mathbf{L})$ and $\operatorname{non}\mathbf{L} \geq \operatorname{non}\mathbf{P}^+$.

(3b), (5b) Assuming CH iterate meager forcing (introduced in [**9**]) with finite support. Meager forcing makes "old reals" ρ-small for an appropriate $\rho \in {}^\omega\omega \cap V[G]$ (hence σ-strongly symmetrically porous, see [**12**]) and iterations of meager forcing preserve unbounded families of functions ([**9**]). Therefore in the extension $\operatorname{add}\mathbf{K} = \mathbf{b} = \omega_1 < \operatorname{non}\mathbf{S}$. □

We do not know whether the inequalities $\operatorname{non}\mathbf{S} > \operatorname{cov}\mathbf{K}$, $\operatorname{non}\mathbf{P} > \operatorname{cov}\mathbf{K}$ are consistent with ZFC.

Consistency results on cov P. It remains to compare the cardinals $\operatorname{cov}\mathcal{I}$, $\mathcal{I} = \mathbf{S}$, $\mathbf{P}, \mathbf{S}^+, \mathbf{P}^+$, with the following five cardinals of Cichoń's diagram: $\operatorname{non}\mathbf{L}$, $\operatorname{non}\mathbf{K}$, $\mathbf{b}$, $\mathbf{d}$, $\operatorname{cof}\mathbf{K}$.

THEOREM 6.

(1a) $\operatorname{Con}(\operatorname{cov}\mathbf{P}^+ > \operatorname{non}\mathbf{L})$	(1b) $\operatorname{Con}(\operatorname{cov}\mathbf{S} < \operatorname{non}\mathbf{L})$
(2a) $\operatorname{Con}(\operatorname{cov}\mathbf{P}^+ > \operatorname{non}\mathbf{K})$	(2b) $\operatorname{Con}(\operatorname{cov}\mathbf{S} < \operatorname{non}\mathbf{K})$
(3a) $\operatorname{Con}(\operatorname{cov}\mathbf{P}^+ > \mathbf{d})$	(3b) $\operatorname{Con}(\operatorname{cov}\mathbf{S} < \mathbf{d})$
(4a) $\operatorname{Con}(\operatorname{cov}\mathbf{P}^+ > \mathbf{b})$	(4b) $\operatorname{Con}(\operatorname{cov}\mathbf{S} < \mathbf{b})$

(5a) ? (5b) Con($\mathrm{cov}\,\mathbf{S} < \mathrm{cof}\,\mathbf{K}$)

PROOF. (1a) $\mathrm{cov}\,\mathbf{P}^+ \geq \mathrm{cov}\,\mathbf{L}$ and Con($\mathrm{cov}\,\mathbf{L} > \mathrm{non}\,\mathbf{L}$).

(2a) $\mathrm{cov}\,\mathbf{P}^+ \geq \mathrm{cov}\,\mathbf{K}$ and Con($\mathrm{cov}\,\mathbf{K} > \mathrm{non}\,\mathbf{K}$).

(3a), (4a) $\mathrm{cov}\,\mathbf{P}^+ \geq \mathrm{cov}\,\mathbf{L}$, Con($\mathrm{cov}\,\mathbf{L} > \mathbf{d}$) and $\mathbf{d} \geq \mathbf{b}$.

(1–5b) Assuming CH iterate ω_2 Mathias reals with countable support. Then in the extension $\mathrm{non}\,\mathbf{L} = \mathrm{non}\,\mathbf{K} = \mathbf{b} = \mathbf{d} = \mathrm{cof}\,\mathbf{K} = \omega_2$. We will show that $\mathrm{cov}\,\mathbf{S} = \omega_1$.

A. W. Miller [10] proved that for arbitrary $\rho \in {}^\omega\omega \cap V$ and arbitrary $x \in {}^\omega\omega \cap V[G]$, $x < \rho$ there exists $f \in {}^\omega([\omega]^{<\omega})$ such that

$$(\forall n \in \omega)(|f(n)| \leq 2^{n^2} \textit{and } x(n) \in f(n)$$

(actually he did it for $\rho(n) = 2^{n^3}$ but the same proof works also in this general case). Let $\rho \in {}^\omega\omega \cap V$ be such that $\lim_{n\to\infty} 2^{n^2}/\rho(n) = 0$. Each function $f \in {}^\omega([\omega]^{<\omega})$ with $|f(n)| \leq 2^{n^2}$ corresponds to a ρ-small set

$$A_{\rho,f} = \varphi_\rho(\{x < \rho : (\exists m)(\forall n > m)\, x(n) \in f(n)\})$$

and so each real from $\langle 0,1\rangle \cap V[G]$ is an element of a Borel σ-strongly symmetrically porous set coded in V. □

In the last generic extension $\mathrm{cov}\,\mathbf{S} < \mathrm{non}\,\mathbf{S}^+$ holds true. Another interesting generic extension is the model [2] in which every two nonprincipal ultrafilters have a common image via a finite-to-one function. There $\omega_1 = \mathbf{r} < \mathbf{s} = \omega_2$.

References

[1] Bartoszyński T., *Combinatorial aspects of measure and category*, Fund. Math. **127** (1987), no. 3, 225–239.

[2] Blass A., Shelah S., *There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed*, Ann. Pure Appl. Logic **33** (1987), no. 3, 213–243.

[3] Denjoy A., *Sur une propriété des séries trigonométriques*, Verlag v.d.G.V. der Wis-en Natuur. Afd., 30 Oct. 1920.

[4] Denjoy A., *Leçons sur le calcul des coefficients d'une série trigonométrique*, 2e partie, Métrique et topologie des ensembles parfaits et des fonctions, Gauthier-Villars, Paris 1941.

[5] Dolženko E. P., *Boundary properties of arbitrary functions*, (in Russian), Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), 3–14.

[6] van Douwen E., *The handbook of set-theoretic topology*, (K. Kunen and J. Vaughan, editors), North-Holland, Amsterdam, 1984.

[7] Fremlin D. H., *Cichoń's diagram*, Publ. Math. Univ. Pierre Marie Curie 66, Semin. Initiation Anal. 23ème Année – 1983/84 Exp. No5 (1984), 13p.

[8] Galvin F., Miller A. W., *γ-sets and other singular sets of real line*, Topology Appl. **17** (1984), 145–155.

[9] Ihoda J., Shelah S., *The Lebesgue measure and the Baire property: Laver's reals, preservation theorems for forcing, completing a chart of Kunen-Miller*, J. Symb. Logic **55** (1990), 909–927.

[10] Miller A. W., *Some properties of measure and category*, Trans. Amer. Math. Soc. **266** (1981), 93–114.

[11] Recław I., *A note on the σ-ideal of σ-porous sets*, Real Analysis Exchange **12** (1986–87), no. 2, 455–457.

[12] Repický M., *Porous sets and additivity of Lebesgue measure*, Real Analysis Exchange **15** (1989–90), no. 1, 282–298.

[13] Repický M., *Additivity of porous sets*, Real Analysis Exchange **16** (1990–91), no. 1, 340–343.

[14] Repický M., *An example which discerns porosity and symmetric porosity*, Real Analysis Exchange 17 (1991-92), no. 1.
[15] Steprāns J., *Some cardinal invariants of the continuum related to splitting finite sets*, preliminary version 1990.
[16] Zajíček L., *Sets of σ-porosity and sets of σ-porosity (q)*, Časopis Pěst. Mat. **101** (1976), 350–359.
[17] Zajíček L., *Porosity and σ-porosity*, Real Analysis Exchange **13** (1987–88), no. 2, 314–350.