Topology and its Applications 41 (1991) 25-40
North-Holland

Spaces not distinguishing pointwise and quasinormal convergence of real functions

Lev Bukovský
Department of Mathematics, University of P.J. Šafárik, 041 54 Košice, Czechoslovakia

Ireneusz Recław
Institute of Mathematics, University of Gdańsk, 80952 Gdańsk, Poland

Miroslav Repický
Mathematical Institute of Slovak Academy of Sciences, 041 54 Košice, Czechoslovakia

Received 28 March 1990

Abstract

Bukovský, L., I. Recław and M. Repický, Spaces not distinguishing pointwise and quasinormal convergence of real functions, Topology and its Applications 41 (1991) 25-40.

A topological space X is said to be a wQN-space if from every sequence of continuous real functions converging pointwise to zero on X one can choose a quasinormally converging subsequence. Some properties of this and related notions are studied.

Keywords: Quasinormal (equal) convergence, QN-space, wQN-space, γ-set, weakly distributive.

AMS (MOS) Subj. Class.: Primary: 54G99, 54A35, secondary: 03E35, 54C30.

Introduction

This work initiated in trigonometrical series theory. In [5], Cholščevnikova proved that every set $E \subseteq \langle 0, 1\rangle$ of cardinality smaller than $\mathfrak{m}$ (for definition see e.g. [10]) is an N_0-set, i.e. there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of natural numbers such that the series

$$\sum_{k=0}^{\infty} |\sin 2\pi n_k x|$$

converges for every $x \in E$.

0166-8641/91/$03.50

For a real $x \in \mathbb{R}$, let $\|x\|$ denote the distance of x to the nearest integer, i.e.

$$\|x\| = \min\{x-[x], [x]+1-x\},$$

where $[x]$ is the integer part of x. In [2], Bukovská introduced the notion of a D-set: $E \subseteq \langle 0, 1\rangle$ is a D-set if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of natural numbers such that the sequence $\{\|n_k x\|\}_{k=0}^{\infty}$ quasinormally converges to zero on E (see Section 1 below). Then she shows that every set $E \subseteq \langle 0, 1\rangle$ of cardinality less than $\mathfrak{p}$ (see [10]) is a D-set. This is an extension of the above-mentioned result by Cholščevnikova, since one can easily see that a D-set is an N_0-set (for details see [2]).

In [12], Gerlits and Nagy introduced the notion of a γ-set. Galvin and Miller [11] observed that

$$\mathfrak{p} = \min\{|X| : X \subseteq \langle 0, 1\rangle \text{ is not a } \gamma\text{-set}\}. \tag{1}$$

So it was very natural to ask about the relationship between γ-sets and D-sets. We have observed two simple facts.

(i) Let us recall a simple consequence of classical Dirichlet–Minkowski theorem (see [4]): if $x_1, \ldots, x_k$ are reals, $\varepsilon > 0$ then there is a natural number n such that $\|nx_i\| < \varepsilon$ for $i = 1, 2, \ldots, k$. This statement is equivalent to the following one: the zero function belongs to the closure (in the topology of $C_p(X) \subseteq {}^X\mathbb{R}$, see [12]) of the set $\{\|nx\| : n > 0\}$ for any $X \subseteq \mathbb{R}$. Hence, if $C_p(X)$ is Fréchet (i.e. X is a γ-set) then there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that $\{\|n_k x\|\}_{k=0}^{\infty}$ pointwise converges to zero on X.

(ii) Using the characterization of γ-sets in terminology of ω-covers given in [12], one can show that (see Theorem 6.1 below) every sequence of continuous real valued functions converging pointwise to zero on X, contains a quasinormally converging subsequence, provided that X is a γ-set.

Thus as a consequence we obtain that a γ-set is a D-set[1] and the above-mentioned results of [2, 5] follow by (1).

On the other hand, in [2] it is shown that in order to solve the old problem of distinguishing so-called R- and N-sets one has to construct a set $X \subseteq \langle 0, 1\rangle$ such that $\{\|n_k x\|\}_{k=0}^{\infty}$ pointwise converges to zero on X for some increasing sequence $\{n_k\}_{k=0}^{\infty}$, however no such sequence quasinormally converges to zero on X.

These were the main inspiration for introducing and investigating the notion of a QN-set and related notions.

1. QN-space

We begin with recalling some notions and facts. Let $f: X \to \mathbb{R}$, $f_n: X \to \mathbb{R}$, $n = 0, 1, \ldots$ be real-valued functions. We say that the sequence $\{f_n\}_{n=0}^{\infty}$ *quasinormally*

[1] This result has been presented by the first author at Logic Colloquium '89 in Berlin.

converges to f on X, written

$$f_n \xrightarrow{\text{QN}} f \quad \text{on } X,$$

if there exists a sequence of positive reals $\{\varepsilon_n\}_{n=0}^{\infty}$ converging to zero (*witnessing the quasinormal convergence*) such that

$$(\forall x \in X)\,(\exists k)\,(\forall n \geq k)\,|f_n(x) - f(x)| < \varepsilon_n. \tag{2}$$

This notion has been introduced and investigated in [6, 7, 2] (implicitly also in [5]). Császar and Laczkovich use the words "equal convergence". We decided to use "quasinormal convergence" because it seems to us that "QN-set" sounds better than "E-set".

We shall need some simple properties of the quasinormal convergence. For the proofs see [3, 6, 7] (Theorem 1.2 is proved in [3]).

Theorem 1.1. *Let f, f_n, $n = 0, 1, \ldots$ be real-valued functions defined on a set X. Then the following are equivalent*:

(i) $f_n \xrightarrow{\text{QN}} f$ *on* X;

(ii) $X = \bigcup_k X_k$ *and* $f_n \rightrightarrows f$ *on* X_k *for every* k;

(iii) $X = \bigcup_k X_k$, $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ *and* $f_n \rightrightarrows f$ *on* X_k *for every* k.

Moreover, if X is a topological space and all f_n, $n = 0, 1, \ldots$ are continuous, we can assume in (ii) *and* (iii) *that all X_k are closed.*

For the definition of the cardinal $\mathfrak{b}$ see e.g. [10, p. 82].

Theorem 1.2. *Let $X = \bigcup_{s \in S} X_s$, $|S| < \mathfrak{b}$. If $f_n : X \to \mathbb{R}$, $n = 0, 1, \ldots$ are such that $f_n \xrightarrow{\text{QN}} f$ on X_s for every $s \in S$ then $f_n \xrightarrow{\text{QN}} f$ on X.*

If Y, ρ is a metric space, $f, f_n : X \to Y$, $n = 0, 1, \ldots$, then we define the quasinormal convergence $f_n \xrightarrow{\text{QN}} f$ on X in a natural way (depending on the metric ρ!).

Now we shall introduce the central notion of the paper. A topological space X is called a *QN-space* if for any sequence $\{f_n\}_{n=0}^{\infty}$ of continuous real-valued functions pointwise converging to zero on X, also $f_n \xrightarrow{\text{QN}} 0$ on X. X is called a *weak QN-space* (shortly *wQN-space*) if for any sequence $\{f_n\}_{n=0}^{\infty}$ of continuous real-valued functions pointwise converging to zero on X there is an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that $f_{n_k} \xrightarrow{\text{QN}} 0$ on X.

A set $X \subseteq \langle 0, 1\rangle$ is called a *QN-set* (a *wQN-set*) if X with the subspace topology is a QN-space (a wQN-space).

Evidently every QN-space is a wQN-space. A continuous image of a QN-space (a wQN-space) is a QN-space (a wQN-space). Every countable space (more generally, of cardinality less than $\mathfrak{b}$) is a QN-space.

2. QN-subsets of the Baire space

It turned out that the best subject for studying the QN-spaces is the Baire space endowed with the order of eventually dominating.

Baire space is the set ${}^{\omega}\omega$ with Baire metric

$$\delta(x,y)=\begin{cases}\dfrac{1}{\min\{n: x(n)\neq y(n)\}+1}, & \text{if } x\neq y,\\ 0, & \text{otherwise.}\end{cases}$$

The eventually dominating quasiorder $\leqslant^*$ is the relation

$$x\leqslant^* y \equiv (\exists k)\,(\forall n\geqslant k)\; x(n)\leqslant y(n).$$

The Baire space is homeomorphic to the set of irrational numbers from $\langle 0,1\rangle$. We shall often identify subsets of ${}^{\omega}\omega$ with corresponding subsets of $\langle 0,1\rangle$ via canonical homeomorphism (continued fractions).

We introduce some special functions from ${}^{\omega}\omega$ into $\mathbb{R}$. For $n\in\omega$, $x\in{}^{\omega}\omega$ we set

$$\varphi_n(x)=\frac{1}{\min\{k: x(k)+k>n\}+1}.$$

One can easily check that $\varphi_n: {}^{\omega}\omega\to\mathbb{R}$ is continuous for every n, actually the set $\{\varphi_n: n\in\omega\}$ is equicontinuous (if $\delta(x,y)<1/k$, then $|\varphi_n(x)-\varphi_n(y)|<1/k$ for any $n\in\omega$, $x,y\in{}^{\omega}\omega$) and $\varphi_n(x)\geqslant\varphi_{n+1}(x)$ for any $x\in{}^{\omega}\omega$, any $n\in\omega$. Finally,

$$\varphi_n\to 0 \quad \text{on } {}^{\omega}\omega.$$

Theorem 2.1. *Let X be a subset of ${}^{\omega}\omega$. The following are equivalent*:

(a) *X is bounded, i.e. there is $z\in{}^{\omega}\omega$ such that $x\leqslant^* z$ for any $x\in X$*;

(b) $\varphi_n\xrightarrow{\text{QN}}0$ *on X*;

(c) *there is an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that*

$$\varphi_{n_k}\xrightarrow{\text{QN}}0 \text{ on } X.$$

Proof. (b) follows from (c) by the monotonicity. So (b) and (c) are equivalent.

Assume that X is bounded. If $x(k)\leqslant z(k)$ for every $k\geqslant k_0$, then

$$\min\{k: x(k)+k>n\}\geqslant\min\{k: z(k)+k>n\}$$

for every $n\geqslant n_0=\max\{x(i)+i: i<k_0\}$. Thus $\varphi_n(x)\leqslant\varphi_n(z)$ for $n\geqslant n_0$ and the sequence of reals $\{\varphi_n(z)\}_{n=0}^{\infty}$ witnesses the quasinormal convergence of $\{\varphi_n\}_{n=0}^{\infty}$ on X.

Now suppose that $\varphi_n\xrightarrow{\text{QN}}0$ on X and that $\{\varepsilon_n\}_{n=0}^{\infty}$ witnesses the quasinormal convergence. We can suppose that $\{\varepsilon_n\}_{n=0}^{\infty}$ is nonincreasing and $1\geqslant\varepsilon_n\geqslant 1/(n+1)$ for every n.

Denote

$$k_n=\min\{i: 1/(i+1)\leqslant\varepsilon_n\}.$$

Then $k_n \to \infty$, $k_n \leqslant n$ and we can define

$$z(i) = n - i \quad \text{for } k_{n-1} \leqslant i < k_n.$$

By direct computation you can see that z is an upper bound (in the relation of eventually dominating) of the set X. □

Corollary 2.2. *If $X \subseteq {}^\omega\omega$ is a wQN-set then X is bounded.*

Corollary 2.3. *The notions of QN-space and wQN-space are not hereditary.*

Proof. Take any unbounded subset X of the Baire space, X endowed with the discrete topology. Then X is not a wQN-space.

Let $X^* = X \cup \{\infty\}$ be the one-point compactification of X. If $f_n : X^* \to \mathbb{R}$, $n \in \omega$, are continuous and $f_n \to 0$ on X^*, then for every n there is a finite set X_n such that

$$|f_n(x) - f_n(\infty)| < 1/(n+1)$$

for every $x \in X - X_n$. Then $\bigcup_k X_k$ is countable and $\{1/(n+1) + |f_n(\infty)|\}_{n=0}^{\infty}$ witnesses the quasinormal convergence $f_n \xrightarrow{\text{QN}} 0$ on $X - \bigcup_k X_k$. □

Unfortunately this result is not satisfactory because the space X is not perfectly normal and even short of continuous real-valued functions.

3. Cardinal characteristics

We shall restrict our considerations to perfectly normal spaces.

Let non(QN-space) denote the minimal cardinality of a perfectly normal space which is not a QN-space. Similarly we define non(wQN-space), non(QN-set), non(wQN-set).

The cardinal add(QN-space) denotes the minimal cardinal α such that there is a perfectly normal non-QN-space which can be expressed as the union $X = \bigcup_{\xi<\alpha} X_\xi$ where the X_ξ's are QN-spaces. Similarly the remaining three cases.

The cardinal $\mathfrak{h}$ is defined and investigated in [1]—it is the minimal nondistributivity of the Boolean algebra $\mathscr{P}(\omega)/\text{fin}$. It is well known that $\mathfrak{h} \geqslant \mathfrak{b} > \aleph_0$.

Theorem 3.1. add(QN-set) = add(QN-space) = $\mathfrak{b}$.

Proof. The inequalities

$$\text{add(QN-set)} \geqslant \text{add(QN-space)} \geqslant \mathfrak{b}$$

follow from Theorem 1.2. By Corollary 2.2 we have

$$\text{add(QN-set)} \leqslant \text{non(QN-set)} \leqslant \text{non(wQN-set)} = \mathfrak{b}. \quad □$$

Actually we have shown more.

Corollary 3.2.

$$\mathrm{non}(\text{QN-space}) = \mathrm{non}(\text{QN-set}) = \mathrm{non}(\text{wQN-space}) = \mathrm{non}(\text{wQN-set}) = \mathfrak{b}.$$

Theorem 3.3. $\mathfrak{b} \geqslant \mathrm{add}(\text{wQN-set}) \geqslant \mathrm{add}(\text{wQN-space}) \geqslant \mathfrak{h}$.

Proof. The first two inequalities are evident, so we have to prove the third one.

Let $X = \bigcup_{\xi<\alpha} X_\xi$, $\alpha < \mathfrak{h}$, X be a topological space, and let X_ξ be a wQN-space for every $\xi < \alpha$. Let $f_n : X \to \mathbb{R}$ be continuous, $n = 0, 1, \ldots$ and $f_n \to 0$ on X.

If $A \subseteq \omega$ is infinite, we denote by $c(A, n)$ the nth element of A, i.e. $A = \{c(A, n) : n \in \omega\}$ and

$$c(A, 0) < c(A, 1) < \cdots.$$

Let us remark that if $h_n \xrightarrow{\text{QN}} 0$ on some set Y then there is an infinite $A \subseteq \omega$ such that

$$(\forall x \in Y)\,(\exists k)\,(\forall n \geqslant k)\,|h_{c(A,n)}(x)| < 1/(n+1)$$

i.e. the quasinormal convergence of $\{h_{c(A,n)}\}_{n=0}^{\infty}$ is witnessed by $\{1/(n+1)\}_{n=0}^{\infty}$.

Now, since every X_ξ is a wQN-space the set

$$\mathcal{K}_\xi = \{A \in [\omega]^\omega : \{1/(n+1)\}_{n=0}^{\infty} \text{ witnesses the quasinormal convergence of } \{f_{c(A,n)}\}_{n=0}^{\infty}\}$$

is dense in $\mathcal{P}(\omega)/\mathrm{fin}$, i.e. for any infinite $B \subseteq \omega$ there is a set $A \in \mathcal{K}_\xi$ such that $A \subseteq B$. Especially, the Boolean union of $\mathcal{K}_\xi$ in the Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$ is equal to the unit element.

Since $\alpha < \mathfrak{h}$, the Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$ is α-distributive. Hence, there is an $A \in [\omega]^\omega$ such that there are $A_\xi \in \mathcal{K}_\xi$, $\xi < \alpha$ with $A - A_\xi$ finite. Let

$$Y_n = \bigcup \{X_\xi : (\forall m \geqslant n)\,(m \in A \to m \in A_\xi)\}.$$

Then $X = \bigcup_n Y_n$. Since $\{f_{c(A,k)}\}_{k=0}^{\infty}$ quasinormally converges on Y_n with the control $\{1/\max\{1, k-n\}\}_{k=0}^{\infty}$, it also does so on X. □

4. Some properties of QN-spaces

As we shall see later a subset of a QN-space need not be a QN-space, i.e. the notion of a QN-space is not hereditary. However:

Theorem 4.1. *An F_σ-subset of a perfectly normal QN-space (wQN-space) is a QN-space (wQN-space).*

Proof. By Theorems 3.1 and 3.3 it suffices to prove the assertions for closed subsets. So let X be a perfectly normal QN-space, and $A \subseteq X$ a closed subset. Let $f_n : A \to \mathbb{R}$ be continuous, $f_n \to 0$ on A.

Since A is closed in a perfectly normal space, there exist open sets $B_0 \supseteq B_1 \supseteq \cdots$ such that

$$\bigcap_n B_n = A.$$

For every n, let $h_n : X \to \mathbb{R}$ be continuous and such that $h_n | A = f_n$ and $h_n(x) = 0$ for $x \in X - B_n$. Then $h_n \to 0$ on X. Since X is a QN-space, $h_n \xrightarrow{\text{QN}} 0$ on X and therefore $f_n \xrightarrow{\text{QN}} 0$ on A. □

We show that some QN-spaces are small (topologically, in measure, dimension).

Theorem 4.2. *Let X be a metric separable space, A being a subset of X without isolated points. If A is a wQN-space, then A is meager in X.*

Proof. The proof is a modification of a construction given in [8, p. 136]. It was Laczkovich who kindly called our attention to this construction.

Let $B = \{r_n : n \in \omega\}$ be a countable dense subset of $\bar{A}$. For every $n \in \omega$, let $x_{n,m} \in A$, $m = 0, 1, \ldots$ be such that $x_{n,m} \to r_n$, $x_{n,m} \neq r_n$ for each $m \in \omega$. Let $f_{n,m} : X \to \langle 0, 1/2^n \rangle$ be a continuous function, $f_{n,m}(x_{n,m}) = 1/2^n$ and $f_{n,m}(x) = 0$ for $\rho(x, x_{n,m}) \geq \frac{1}{2}\rho(r_n, x_{n,m})$. Denote

$$h_m(x) = \sum_{n=0}^{\infty} f_{n,m}(x), \qquad x \in X, \quad m = 0, 1, \ldots.$$

Then every h_m is a continuous function from X into $\langle 0, 2 \rangle$ and $h_m \to 0$ on X (pointwise).

Assume now that A is not meager in X and that there exists a subsequence $\{h_{m_k}\}_{k=0}^{\infty}$ quasinormally converging on A. By Theorem 1.1 there exist closed sets $A_l \subseteq X$, $l = 0, 1, \ldots$, $A \subseteq \bigcup_l A_l$ such that

$$h_{m_k} \rightrightarrows 0 \quad \text{on } A \cap A_l, \qquad l = 0, 1, 2, \ldots. \tag{4}$$

Moreover, we can assume that $A_l \subseteq \bar{A}$ (otherwise replace A_l by $A_l \cap \bar{A}$). Since A is not meager there exists an index p such that $\operatorname{Int} A_p \neq \emptyset$. Since B is dense in $\bar{A}$ there is $r_n \in \operatorname{Int} A_p$. Then $x_{n,m} \in \operatorname{Int} A_p$ for $m \geq m_0$. Thus, for $m \geq m_0$ we obtain

$$\sup\{h_m(x) : x \in A \cap A_p\} \geq h_m(x_{n,m}) \geq f_{n,m}(x_{n,m}) = 1/2^n.$$

This is a contradiction with (4). □

Corollary 4.3. *If A is a wQN-subspace of a metric separable space, then A is perfectly meager. Especially, any wQN-set is perfectly meager.*

Proof. If P is a perfect set, then $P \cap A$ can be decomposed

$$P \cap A = A_0 \cup A_1,$$

where A_0 is countable and A_1 is dense in itself and closed in A. Then both A_0, A_1 are meager. □

Corollary 4.4. *If A is a wQN-set, then for every Radon measure ν on $\langle 0, 1\rangle$, the inner measure $\nu_*(A)$ is zero.*

Corollary 4.5. *If X is a wQN-set, then X is zero-dimensional.*

Proof. X cannot contain any closed interval. □

Corollary 4.6. *If X is a completely regular wQN-space, then X has a clopen basis. Moreover, if X is also perfectly normal then every open subset of X is a countable union of clopen sets.*

Proof. Let $x \in A \subseteq X$, A being open. Then there is a continuous function $f: X \to \langle 0, 1\rangle$, $f(x)=0$, $f(y)=1$ for $y \in X-A$. By Corollary 4.5 there is a clopen set $U \subseteq f(X)$, $0 \in U$, $1 \notin U$. Then $f^{-1}(U)$ is a clopen subset of A containing x.

In the case of a perfectly normal space take f such that $A=\{x \in X: f(x)<1\}$. □

However this need not be true for the outer measure. Let $\alpha \leqslant \mathfrak{c}$ be a regular cardinal. A set $X \subseteq \langle 0, 1\rangle$ is called an *α-Sierpiński set* if $|X| \geqslant \alpha$ and for every measure zero set A, $|A \cap X| < \alpha$. Martin axiom implies the existence of a $\mathfrak{c}$-Sierpiński set.

Theorem 4.7. *If X is a $\mathfrak{b}$-Sierpiński set, then every subset of X is a QN-set.*

Proof. Let $A \subseteq X$, $f_n: A \to \mathbb{R}$ being continuous and $f_n \to 0$ on A. We can assume that all f_n are defined and continuous on a G_δ-set $G \supseteq A$. Let $C \subseteq G$ be the Borel set of those $x \in G$ for which $f_n(x) \to 0$. Evidently $A \subseteq C$. By the Egoroff Theorem (see e.g. [9]) there is a set $H \subseteq C$ such that $f_n \xrightarrow{\text{QN}} 0$ on H and $C-H$ has measure zero. Since $|A \cap (C-H)| < \mathfrak{b}$ the assertion follows from Theorem 1.2. □

A continuous image of a QN-space (wQN-space) is also a QN-space (wQN-space). We can show stronger result.

Theorem 4.8. *Let $f: X \to Y$ be a mapping from a QN-space (a wQN-space) X into a metric space Y. If f is a quasinormal limit of a sequence of continuous mappings then $f(X) \subseteq Y$ is a QN-space (a wQN-space).*

Proof. Let $f_n: X \to Y$, $n=0, 1, \ldots$ be continuous and such that $f_n \xrightarrow{\text{QN}} f$ on X. Then there are closed sets X_k, $k=0, 1, \ldots,$ such that $X=\bigcup_k X_k$ and $f_n \rightrightarrows f$ on X_k, $k=0, 1, \ldots.$ Thus $f: X_k \to Y$ is continuous and $f(X_k) \subseteq Y$ is a QN-space. Since

$$f(X)=\bigcup_k f(X_k),$$

the theorem follows by the σ-additivity. □

5. A distributive law in topological spaces

Let X be a nonempty set, $\mathscr{A} \subseteq \mathscr{P}(X)$ being a family of subsets. $\mathscr{A}$ is called *weakly distributive* if for any system $A_{n,m} \in \mathscr{A}$, $n, m \in \omega$ such that

$$\bigcap_n \bigcup_m A_{n,m} = X \tag{5}$$

there exists a function $\varphi \in {}^\omega\omega$ such that

$$\bigcup_k \bigcap_{n \geq k} \bigcup_{m \leq \varphi(n)} A_{n,m} = X. \tag{6}$$

Let us remark that the weak distributive law is preserved in certain sense by countable unions, e.g. the family of closed sets is weakly distributive if and only if the family of F_σ-sets is weakly distributive. We shall use this fact later.

Theorem 5.1. *Let X be a topological space. If the family of closed subsets of X is weakly distributive then X is a QN-space.*

Proof. Let $f_n : X \to \mathbb{R}$, $n = 0, 1, \ldots$, be continuous and $f_n \to 0$ on X. The sets

$$A_{n,m} = \{x \in X : (\forall k \geq m) |f_k(x)| \leq 1/(n+1)\}$$

are closed and (5) holds true. Let $\varphi \in {}^\omega\omega$ be such that (6) holds true. We can assume that φ is increasing. Set

$$\varepsilon_l = \begin{cases} 2/(n+1), & \text{for } \varphi(n) \leq l < \varphi(n+1), \\ 3, & \text{for } l < \varphi(0) \text{ (if any)}. \end{cases}$$

Then $\varepsilon_l \to 0$ and for any $x \in X$ there is a k such that

$$|f_n(x)| < \varepsilon_n$$

for every $n \geq k$. □

Theorem 5.2. *Let X be a perfectly normal topological space. If the family of closed subsets of X is weakly distributive then* $F_\sigma(X) = G_\delta(X)$, i.e. *$X$ is a σ-space (see* [15]).

Proof. It suffices to show that $G_\delta(X) \subseteq F_\sigma(X)$. So, suppose $A \in G_\delta(X)$, $A = \bigcap_n A_n$, A_n open, $A_{n+1} \subseteq A_n$, $n = 0, 1, \ldots$. Let

$$A_n = \bigcup_m F_{n,m}$$

where $F_{n,m}$ are closed. Then

$$X = \bigcap_n \left(\bigcup_m F_{n,m} \cup (X - A) \right).$$

By the weak distributivity there is a function $\varphi \in {}^\omega\omega$ such that

$$X = \bigcup_k \bigcap_{n \geq k} \left(\bigcup_{m \leq \varphi(n)} F_{n,m} \cup (X - A) \right).$$

Since $\bigcap_{n\geq k}\bigcup_{m\leq\varphi(n)}F_{n,m}\subseteq A$ we obtain

$$A=\bigcup_k\bigcap_{n\geq k}\bigcup_{m\leq\varphi(n)}F_{n,m}. \quad\square$$

Corollary 5.3. *Let X be a perfectly normal space. Then the family of closed subsets of X is weakly distributive if and only if the family of Borel subsets of X is weakly distributive.*

Corollary 5.4. *Let X be a perfectly normal topological space. If the family of closed subsets is weakly distributive then every subset of X is a QN-space.*

Proof. By Theorem 5.1, X is a QN-space. Let $A\subseteq X$; $f_n:A\to\mathbb{R}$, $n=0,1,\ldots$ being continuous, $f_n\to 0$ on A. There exists a G_δ-set $B\supseteq A$ such that every f_n can be continuously extended to B. By Theorem 5.2 the Borel set $C=\{x\in B: f_n(x)\to 0\}$ is an F_σ-set and therefore, $f_n\xrightarrow{QN}0$ on $C\supseteq A$ (by Theorem 4.1). □

Corollary 5.5. *Let X be a perfectly normal space. Every subset of X is a QN-space if and only if X is simultaneously a QN- and a σ-space.*

Corollary 5.4 can be equivalently reformulated as:

Corollary 5.6. *Let X be a perfectly normal space. If for every* F_σ*-measurable mapping* $f:X\to{}^\omega\omega$ *(i.e.* $f^{-1}(U)\in F_\sigma$ *for open* U*) the image* $f(X)\subseteq{}^\omega\omega$ *is bounded then every subset of X is a QN-space.*

Proof. Let $A_{n,m}$, $n,m=0,1,\ldots$, be F_σ-sets satisfying (5). By Reduction Theorem (see e.g. [15]) we can assume that $A_{n,m}\cap A_{n,k}=\emptyset$ for $m\neq k$. We define $f(x)=\psi$ where $\psi(n)=m$ iff $x\in A_{n,m}$. One can easily see that $f:X\to{}^\omega\omega$ is F_σ-measurable. Thus, $f(X)$ is bounded by some $\varphi\in{}^\omega\omega$. Then we obtain that (6) holds true. □

Replacing F_σ-sets by clopen sets in this proof, you will obtain a proof of the following.

Lemma 5.7. *If every image* $f(X)$ *of X by a continuous mapping* $f:X\to{}^\omega\omega$ *is bounded then the family of clopen subsets of X is weakly distributive.*

Theorem 5.8. *If X is a perfectly normal wQN-space, then the family of open subsets of X is weakly distributive.*

Proof. The proof is immediate by Lemma 5.7 and Corollary 4.6. □

Hurewicz [13] investigated a property E^{**}, which in case of a Lindelöf space is exactly the weak distributivity of open sets. Miller and Fremlin [16] showed that every Sierpiński set has Hurewicz property. We have obtained a little more.

Corollary 5.9. *If X is a perfectly normal Lindelöf wQN-space, then X has Hurewicz property.*

6. γ-property

A family of open subsets of X is called an *ω-cover* iff every finite subset of X is contained in an element of the family. A space X has *γ-property* (see [12]) iff for every ω-cover $\mathscr{A}$ of X there are $A_n \in \mathscr{A}$, $n = 0, 1, \ldots$ such that

$$X = \bigcup_n \bigcap_{m \geq n} A_m.$$

A set $X \subseteq \langle 0, 1\rangle$ with γ-property is called a *γ-set.*

The main result of [12] says that X has γ-property if and only if $C_p(X)$ (i.e. $C(X)$ equipped with the subspace topology of the product space ${}^X\mathbb{R}$) is Fréchet.

Theorem 6.1. *If X has γ-property, then X is a wQN-space.*

Proof. Let X be an infinite topological space with γ-property and let $f_n : X \to \mathbb{R}$, $n = 0, 1, \ldots$, be continuous, $f_n \to 0$ on X pointwise. Choose a sequence $y_n \in X$, $n \in \omega$ of pairwise different elements of X. We denote

$$U_{n,m} = f_m^{-1}((-1/(n+1), 1/(n+1))) - \{y_n\},$$

$$\mathscr{A}_n = \{U_{n,m} : m \in \omega, m \geq n\},$$

$$\mathscr{A} = \bigcup_n \mathscr{A}_n.$$

The family $\mathscr{A}$ is an ω-cover of X. Thus there are $A_n \in \mathscr{A}$, $n = 0, 1, \ldots$ such that

$$X = \bigcup_n \bigcap_{m \geq n} A_m.$$

Let $m(k)$, $n(k)$ be those integers for which

$$A_k = U_{n(k),m(k)}.$$

Thus $m(k) \geq n(k)$. Since every element of $\mathscr{A}_n$ does not contain y_n, only finitely many of the A_k's belongs to $\mathscr{A}_n$.

By induction we choose an increasing sequence k_i, $i = 0, 1, \ldots$, such that $m(k_i) < n(k_{i+1})$ for every i. Then both sequences $n(k_i)$, $m(k_i)$, $i = 0, 1, \ldots$, are increasing and for every $x \in X$ there exists an index i_0 such that

$$|f_{m(k_i)}(x)| < 1/n(k_i)$$

for every $i \geq i_0$. $\square$

A topological space X is said to be *α-concentrated* on a subset $A \subseteq X$ if $|X - U| < \alpha$ for every open $U \supseteq A$. Assuming $\mathfrak{p} = \mathfrak{c}$, Galvin and Miller [11] have constructed a γ-set of cardinality $\mathfrak{c}$ that is $\mathfrak{c}$-concentrated on a countable subset $A \subseteq X$. We show that this set is not a hereditary wQN-set.

Lemma 6.2. *Let X be a perfectly normal space α-concentrated on a countable set A, α being uncountable regular cardinal. If $Y \subseteq X - A$ is a wQN-space, then $|Y| < \alpha$.*

Proof. Since Y is a G_δ-subset of the wQN-space $Y \cup A$ we have

$$Y = \bigcap_n U_n,$$

where $U_n \supseteq U_{n+1}$ are open subsets of $Y \cup A$. By Corollary 4.6, every U_n is a countable union of clopen subsets of $Y \cup A$:

$$U_n = \bigcup_m V_{n,m}.$$

Then

$$Y = \bigcap_n \bigcup_m (V_{n,m} \cap Y) = \bigcap_n \bigcup_m V_{n,m}$$

and by Theorem 5.8 there exists a function $\varphi \in {}^\omega\omega$ such that

$$Y = \bigcup_k \bigcap_{n \geq k} \bigcup_{m \leq \varphi(n)} (V_{n,m} \cap Y) = \bigcup_k \bigcap_{n \geq k} \bigcup_{m \leq \varphi(n)} V_{n,m},$$

i.e.

$$Y = \bigcup_k B_k,$$

where the B_k's are closed in $Y \cup A$. Since $(Y \cup A) - B_k$ is open set containing A we obtain

$$|(Y \cup A) \cap B_k| < \alpha$$

and therefore

$$|Y| \leq \sum_k |(Y \cup A) \cap B_k| < \alpha. \quad \square$$

Theorem 6.3. *If $\mathfrak{p} = \mathfrak{c}$, then there exists a γ-set of cardinality $\mathfrak{c}$ that is $\mathfrak{c}$-concentrated on a countable subset A and such that every wQN-subset of $X - A$ has cardinality less than $\mathfrak{c}$.*

Proof. Take the Galvin–Miller γ-set X $\mathfrak{c}$-concentrated on a countable subset A and use Lemma 6.2. $\square$

Modifying the Galvin–Miller construction we shall construct (assuming $\mathfrak{p} = \mathfrak{c}$) a γ-set that is not a QN-set. More precisely

Theorem 6.4. *If $\mathfrak{p} = \mathfrak{c}$ then there exists a γ-set X of cardinality $\mathfrak{c}$ such that every QN-subset of X has cardinality less than $\mathfrak{c}$. Moreover, X is $\mathfrak{c}$-concentrated on a countable subset.*

We shall identify the space ${}^\omega 2$ with $\mathcal{P}(\omega)$ via characteristic functions. For $Y \subseteq \omega$ we denote

$$Y^* = \{X \subseteq \omega \colon Y - X \text{ is finite}\}.$$

In [11, Lemma 1.2] the following auxilliary result is proved.

Lemma 6.5. *Suppose $X \in [\omega]^\omega$ and $\mathcal{A}$ is an open ω-cover of $[\omega]^{<\omega}$. Then there are $D_n \in \mathcal{A}$, $n = 0, 1, \ldots$, and $Y \in [X]^\omega$ such that*

$$Y^* \subseteq \bigcup_n \bigcap_{m \geq n} D_m.$$

We shall need another rather technical statement. We start with some notations. For $A \subseteq \omega$, $s \in [\omega]^{<\omega}$, $k \in \omega$ we set

$$f_{s,k}(A) = \begin{cases} 1/(\max s), & \text{if } A \cap (k+1) = s \cup \{k\}, \\ 0, & \text{otherwise.} \end{cases}$$

Every $f_{s,k} \colon \mathcal{P}(\omega) \to \mathbb{R}$ is continuous and $f_{s,k}(A) \to 0$ when $\max(s \cup \{k\}) \to \infty$, A fixed. Let $\pi \colon [\omega]^{<\omega} \times \omega \to \omega$ be a bijection and

$$g_n = f_{s,k}, \quad \text{if} \quad n = \pi(s, k).$$

Lemma 6.6. *Let $X \in [\omega]^\omega$, ε_n, $n = 0, 1, \ldots$, be positive reals, $\varepsilon_n \to 0$. Then there exists a subset $Y \in [X]^\omega$ such that*

$$(\forall Z \in Y^*)\, (\forall m)\, (\exists n > m)\, g_n(Z) > \varepsilon_n.$$

Proof. Denote $\delta(s, k) = \varepsilon_{\pi(s,k)}$. By induction we can easily find $k_n \in X$ such that $\delta(s, k_n) < 1/(\max s)$ for all $s \subseteq k_{n-1} + 1$. Put $Y = \{k_0, k_1, \ldots\}$. If $Z \in Y^*$, then

$$f_{Z \cap k_n, k_n}(Z) > \delta(Z \cap k_n, k_n)$$

for all but finitely many $k_n \in Z$. □

Proof of Theorem 6.4 (cf. [11, proof of Theorem 1]). Let $\mathcal{A}_\alpha$, $\alpha < \mathfrak{c}$ be all of the countable families of open subsets of $\mathcal{P}(\omega)$ and let $\{\varepsilon_n^\alpha \colon n \in \omega\}$ for $\alpha < \mathfrak{c}$ be all of the sequences of positive reals converging to zero.

We construct $X_\alpha \in [\omega]^\omega$ for $\alpha < \mathfrak{c}$ so that $|X_\beta - X_\alpha| < \aleph_0$ for every $\alpha < \beta < \mathfrak{c}$. Let $X_0 = \omega$.

For α limit take any X_α such that $|X_\beta - X_\alpha| < \aleph_0$ for every $\beta < \alpha$ (remind that $\mathfrak{p} = \mathfrak{c}$).

Do the same if α is not a limit, $\alpha = \beta + 1$ and $\mathcal{A}_\beta$ is not an ω-cover of the set

$$\{X_\xi \colon \xi \leq \beta\} \cup [\omega]^{<\omega}.$$

Assume now $\alpha = \beta + 1$ and $\mathcal{A}_\beta$ is an ω-cover of the set $\{X_\xi \colon \xi \leq \beta\} \cup [\omega]^{<\omega}$. Since this set is a γ-set (since it has cardinality $|\beta| + \aleph_0 < \mathfrak{p}$), there are $D_n \in \mathcal{A}_\beta$, $n = 0, 1, \ldots$ such that

$$\{X_\xi \colon \xi \leq \beta\} \cup [\omega]^{<\omega} \subseteq \bigcup_n \bigcap_{m \geq n} D_m.$$

However this implies that $\{D_n : n \in \omega\}$ is also an ω-cover of the same set. By Lemma 6.5 there are $k_n \in \omega$, $n = 0, 1, \ldots$ and $Y \in [X_\beta]^\omega$ such that

$$Y^* \subseteq \bigcup_n \bigcap_{m \geq n} D_{k_m}.$$

By Lemma 6.6 there is $X_\alpha \in [Y]^\omega$ such that

$$(\forall Z \in X_\alpha^*)\,(\forall m)\,(\exists n > m)\; g_n(Z) > \varepsilon_n^\beta.$$

One can easily check that the set

$$\{X_\alpha : \alpha < \mathfrak{c}\} \cup [\omega]^{<\omega}$$

is the desired one. □

7. A consistency result

By Theorems 6.3 and 6.4 one cannot prove that QN-sets or wQN-sets are hereditary. Also it is consistent to assume that there is a γ-set, hence a wQN-set, that is not a QN-set. If $\mathfrak{p} < \mathfrak{b}$ (and it is consistent), then there is a QN-set of cardinality $\mathfrak{p}$ that is not a γ-set. So the picture of relationships between QN, wQN and γ is complete.

One can easily construct a perfect D-set. According to Corollary 4.3, this set is not a wQN-set (so neither a γ-set). On the other hand a Sierpiński set is a QN-set that is not a D-set (a D-set has Lebesgue measure zero). Even cardinal invariants of these notions may differ.

Theorem 7.1. *Let $\aleph_0 < \alpha < \beta$ be regular cardinals of a transitive countable model $\mathcal{M}$ of ZFC. Then there is a ccc generic extension $\mathcal{N}$ of $\mathcal{M}$ such that*

$$\mathcal{N} \models \text{non}(\gamma\text{-set}) = \aleph_1 \quad \textit{and}\ \text{non}(D\text{-set}) = \alpha \quad \textit{and}\ 2^{\aleph_0} = \beta.$$

Proof. Without loss of generality we can assume that

$$\mathcal{M} \models \beta = 2^{\aleph_0} \quad \text{and}\ \mathfrak{b} = \aleph_1.$$

Consider the following forcing notion:

$$p \in P \quad \text{iff} \quad p = \langle s, A\rangle \text{ where } s \in {}^{<\omega}\omega \text{ is increasing and } A \in [\langle 0, 1\rangle]^{<\omega},$$

$$\langle s, A\rangle \leq \langle t, B\rangle \quad \text{iff} \quad t \subseteq s \quad \text{and}\ B \subseteq A \quad \text{and}$$
$$(\forall i \in \text{dom}(s - t))\,(\forall x \in B)\,\|s(i)x\| < 1/(i+1).$$

One can easily see that P is σ-centered and by the consequence of Dirichlet-Minkowski Theorem mentioned in the Introduction,

$$\Vdash_P \text{“} V \cap \langle 0, 1\rangle \text{ is a } D\text{-set”}.$$

In $\mathcal{M}$ we construct the finite support iterated forcing system $\langle P_\xi : \xi \leq \alpha\rangle$ such that

$$P_{\xi+1} = P_\xi * Q_\xi \quad \text{for } \xi < \alpha,$$

where Q_ξ is a P_ξ-name of P. Let G be an $\mathcal{M}$-generic filter over P_α. We set $\mathcal{N} = \mathcal{M}[G]$. Evidently

$$\mathcal{N} \models 2^{\aleph_0} = \beta \quad \text{and non}(D\text{-set}) \geqslant \alpha.$$

The non-meager set of Cohen reals added on limit steps ensures the equality non(D-set) $= \alpha$ in $\mathcal{N}$. We prove that $\mathfrak{b} = \mathfrak{p} = \aleph_1$ in $\mathcal{N}$.

Ihoda and Shelah [14] have proved the following: the property "to be an unbounded subfamily of ${}^\omega\omega$" is preserved by a finite support iteration of forcings preserving this property. So it remains to prove. □

Lemma 7.2. *If $F \subseteq {}^\omega\omega$ is unbounded, then*

$$\Vdash_P \text{"}\check{F} \text{ is unbounded"}.$$

We need a technical result.

Lemma 7.3. *Let $k \in \omega$, $s \in {}^{<\omega}\omega$ be increasing and let τ be a P-name of an integer. Then there is an integer $m = m(k, s, \tau)$ such that*

$$(\forall A \in [\langle 0, 1\rangle]^k)\ (\exists \langle t, B\rangle \in P)$$

$$(\langle t, B\rangle \leqslant \langle s, A\rangle \quad \textit{and } \langle t, B\rangle \textit{ decides } \tau \quad \textit{and } (\forall i \in \text{dom}(t))\ t(i) < m).$$

Proof. For $r = \langle t, B\rangle \in P$, $t \supseteq s$ denote

$$U_r = \{x \in \langle 0, 1\rangle \colon (\forall i \in \text{dom}(t - s)) \|t(i)x\| < 1/(i+1)\}.$$

Then U_r is open and

$$\langle 0, 1\rangle^k \subseteq \bigcup \{U_r^k \colon r \leqslant \langle s, \emptyset\rangle \quad \text{and } r \text{ decides } \tau\}.$$

Since $\langle 0, 1\rangle^k$ is compact there are $r_1, \ldots, r_n \in P$ such that

$$\langle 0, 1\rangle^k \subseteq U_{r_1}^k \cup \cdots \cup U_{r_n}^k.$$

Now it suffices to put

$$m = \max\{t_i(j) \colon i = 1, 2, \ldots, n,\ j \in \text{dom}(t_i)\} + 1,$$

where $r_i = \langle t_i, A_i\rangle$, $i = 1, 2, \ldots, n$. □

Proof of Lemma 7.2 (Compare [14]). Suppose g is a P-name of a function from ω into ω and for some $p \in P$

$$p \Vdash (\forall f \in \check{F})\ (\exists k)\ (\forall n \geqslant k)\ f(n) \leqslant g(n). \tag{7}$$

The finite set

$$C_{s,k} = \{t \in {}^{<\omega}\omega \colon t \text{ is increasing and } t \supseteq s \text{ and}$$
$$(\exists B)\langle t, B\rangle \text{ decides } g(k) \text{ and}$$
$$(\forall j \in \text{dom}(t))\ t(j) < m(k, s, g(k))\}$$

is nonempty by Lemma 7.3. Therefore the set

$$\{i\in\omega\colon (\exists\langle t, B\rangle\in P)t\in C_{s,k} \text{ and } \langle t, B\rangle \Vdash g(k)=i\}$$

is nonempty and finite. We denote by $h_s(k)$ its maximum. Let $f\in F$ be such that for every increasing $s\in {}^{<\omega}\omega$ for infinitely many n, $h_s(n)<f(n)$. By (7) there is $\langle s, A\rangle\leqslant p$ and k_0 such that

$$\langle s, A\rangle \Vdash (\forall k\geqslant k_0)f(k)\leqslant g(k).$$

Let $k>k_0$, $k>|A|$ be such that $h_s(k)<f(k)$. Then by Lemma 7.3 there is $\langle t, B\rangle\leqslant \langle s, A\rangle$, $\langle t, B\rangle$ decides $g(k)$ and $t\in C_{s,k}$. Therefore

$$\langle t, B\rangle \Vdash g(k)\leqslant h_s(k)<f(k)\leqslant g(k),$$

a contradiction. □

References

[1] B. Balcar, J. Pelant and P. Simon, The space of ultrafilters on N covered by nowhere dense sets, Fund. Math. 110 (1980) 11-24.

[2] Z. Bukovská, Thin sets in trigonometrical series and quasinormal convergence, Math. Slovaca 40 (1990) 53-62.

[3] Z. Bukovská, Quasinormal convergence, Math. Slovaca 41 (1991) 137-146.

[4] J.W.S. Cassels, An Introduction to Diophantine Approximation (Cambridge Univ. Press, Cambridge, 1965).

[5] N.N. Cholščevnikova, O nesčetnych R- i N-množestvach, Mat. zametki 38 (1985) 270-277.

[6] Á. Császár and M. Laczkovich, Discrete and equal convergence, Stud. Sci. Math. Hungar. 10 (1975) 463-472.

[7] Á. Császár and M. Laczkovich, Some remarks on discrete Baire classes, Acta Math. Acad. Sci. Hungar. 33 (1979) 51-70.

[8] J. Dieudonné, Foundations of Modern Analysis (Academic Press, New York, 1969).

[9] G.B. Folland, Real Analysis (Wiley, New York, 1984).

[10] D.H. Fremlin, Consequences of Martin's Axiom (Cambridge Univ. Press, Cambridge, 1984).

[11] F. Galvin and A.W. Miller, γ-sets and other singular sets of real numbers, Topology Appl. 17 (1984) 145-155.

[12] J. Gerlits and Z. Nagy, Some properties of C(X),I, Topology Appl. 14 (1982) 151-161.

[13] W. Hurewicz, Über Folgen stetiger Funktionen, Fund. Math. 9 (1927) 193-204.

[14] J.I. Ihoda and S. Shelah, The Lebesgue measure and Baire property: Laver's reals, preservation theorems for forcing, completing a chart of Kunen-Miller, to appear.

[15] K. Kuratowski, Topologie I (PWN, Warsaw, 1958).

[16] A.W. Miller and D.H. Fremlin, On some properties of Hurewicz, Menger and Rothberger, Fund. Math. 129 (1988) 17-33.