SPACES NOT DISTINGISHING IDEAL CONVERGENCES OF REAL-VALUED FUNCTIONS

MIROSLAV REPICKÝ

ABSTRACT. Taking several ideal convergences of sequences of real-valued functions we consider topological spaces not distinguishing certain pairs of convergences. We classify and compare such properties of spaces and give combinatorial characterizations of the minimal cardinalities of spaces not having a particular property. As counterexamples serve subsets of the Baire space ${}^{\omega}\omega$. We show that these cardinals are connected with the bounding numbers of partial orders of I-to-one functions in ${}^{\omega}\omega$.

1. INTRODUCTION

The study of spaces not distinguishing a pair of convergences of sequences of continuous real-valued functions was initiated in [4] where the notions of a QN-space and a wQN-space were introduced. In the last years this study was extended for ideal convergences. Recently Kwela [9] gave a combinatorial characterization of non(IwQN-space) and obtained nontrivial estimations of this cardinal for a wide class of ideals I on ω. This characterization is similar to the characterization of non(IQN-space) by Šupina [13]. However, there are several approaches how to understand "weak" (expressed by the letter "w") in the notion of an IwQN-space. By Kwela (samely like in [6]) a space X is an IwQN-space if for every sequence $\{f_n\}_n$ of continuous real functions converging to 0 there exists an "increasing" sequence $\{n_k\}_k$ of natural numbers such that $f_{n_k} \xrightarrow{I\mathrm{QN}} 0$ on X. On the other hand Bukovský, Das, and Šupina [3] say that a space X is an (I,J)wQN-space, if for every sequence $\{f_n\}_n$ of continuous real functions I-converging to 0 there exists "arbitrary" sequence $\{n_k\}_k$ of natural numbers such that $f_{n_k} \xrightarrow{J\mathrm{QN}} 0$ on X. The latter approch is more handy because sometimes it is usefull to consider ideals on countable sets different from ω and then it may be not obvious what should mean "increasing". Then without loss of generality we can require the sequence $\{n_k\}_k$ to be J-to-one. A bit stronger condition is the existence of an ideal $J' \leq_{\mathrm{K}} J$ (in the Katětov partial ordering of ideals) such that $f_n \xrightarrow{J'\mathrm{QN}} 0$. This condition we call $\leq_{\mathrm{K}} J$QN-convergence and applying it we get another weakening of the notion of (I,J)QN-space that is stronger than (I,J)wQN-space in the sense of [3].

In the present paper we consider four convergences of sequences of functions: I-convergence, IQN-convergence, $\leq_{\mathrm{K}} I$-convergence, and $\leq_{\mathrm{K}} I$QN-convergence. For each pair of convergences of this kind we consider the property of spaces such that

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the first convergence in the pair implies the second one. In Section 3 excluding trivial properties and identifying equivalent properties we obtain nine nontrivial cases and implications between them. Here we introduce the notation for the properties a bit different from the notation used in the cited papers (in particular we write $(I, J\mathrm{QN})$-space instead of $(I, J)\mathrm{QN}$-space and $\mathrm{w}(I, J\mathrm{QN})$-space instead of $(I, J)\mathrm{wQN}$-space because we work with pairs of convergences). In Section 4 for every of the remaining nine properties we characterize the existence of a space (in fact a set of reals) not having this property. We also get combinatorial characterization of minimal cardinalities of such spaces. In Section 6 we express these cardinal invariants as invariants of a partial ordering of the family of I-to-one functions and compare them with other $\mathfrak{b}$-like invariants of partial orderings of ${}^{\omega}\omega$ and ${}^{\omega}I$. Some results of Kwela [9] have applications in this context and his result on ideals contained in F_σ ideals we prove for capacitous ideals. Section 5 collects the results about heredity of the properties with respect to the Katětov partial ordering $\leq_{\mathrm{K}}$ of ideals. In particular this allows non-tall ideals to be replaced by Fin.

2. Preliminaries

Definition 2.1. Let I and $J \subseteq I$ be ideals on ω and let $f \in {}^{\omega}({}^{X}\mathbb{R})$. We define the following convergences of the sequence f on X:

(i) $f \xrightarrow{I} 0$ if $(\forall x \in X)(\forall \varepsilon > 0)\ \{n \in \omega : |f_n(x)| \geq \varepsilon\} \in I$.
(ii) $f \xrightarrow{I\mathrm{QN}} 0$ if $(\exists \varepsilon \in {}^{\omega}[0, \infty))[\varepsilon \xrightarrow{I} 0$ and $(\forall x \in X)\ \{n \in \omega : |f_n(x)| \geq \varepsilon_n\} \in I]$.
(iii) $f \xrightarrow{IJ\mathrm{QN}} 0$ if $(\exists \varepsilon \in {}^{\omega}[0, \infty))[\varepsilon \xrightarrow{J} 0$ and $(\forall x \in X)\ \{n \in \omega : |f_n(x)| \geq \varepsilon_n\} \in I]$.

The I-convergence and the $I\mathrm{QN}$-convergence are same as in [3, 6]. The $IJ\mathrm{QN}$-convergence was introduced in [7] under the name "the (I, J)-equal convergence" while the "I-equal convergence" was used before with the meaning of (I, J)-equal convergence either with $J = \mathrm{Fin}$ or with $J = I$. Obviously, $I\mathrm{QN}$-convergence is stronger than I-convergence. By [7, Proposition 4.4], $IJ\mathrm{QN}$-convergence is stronger than I-convergence (and $I\mathrm{QN}$-convergence) if and only if $J \subseteq I$.

For an ideal I on ω and a function $\varphi \in {}^{\omega}\omega$ we define

$$\begin{aligned}
\varphi^{\rightarrow}(I) &= \{a \subseteq \omega : \varphi^{-1}(a) \in I\},\\
\varphi^{\leftarrow}(I) &= \{a \subseteq \omega : \varphi(a) \in I\},\\
F(I) &= \{\alpha \in {}^{\omega}\omega : (\forall n \in \omega)\ \alpha^{-1}(\{n\}) \in I\}.
\end{aligned}$$

Recall that $\mathrm{Fin} = [\omega]^{<\omega}$, $I^{+} = \mathcal{P}(\omega) \setminus I$, and $I^{*} = \{\omega \setminus a : a \in I\}$. Note that $\varphi^{\rightarrow}(I)$ is an ideal on ω if and only if $\varphi \in F(I)$; and $\varphi^{\leftarrow}(I)$ is an ideal on ω if and only if $\mathrm{rng}(\varphi) \in I^{+}$. If $\mathrm{rng}(\varphi) \in I^{*}$, then $\varphi^{\rightarrow}(\varphi^{\leftarrow}(I)) = I$. Observe that $F(\mathrm{Fin})$ is a dense $F_{\sigma\delta}$ subset of ${}^{\omega}\omega$, $F(\mathrm{Fin}) \subseteq F(I)$, and $\varphi \in F(\varphi^{\leftarrow}(I))$ for any I and φ.

Let I and J be ideals on ω and let $\kappa \leq \mathfrak{c}$.

$$\begin{aligned}
&I \leq_{\mathrm{RK}} J \Leftrightarrow (\exists \varphi \in F(J))\ I = \varphi^{\rightarrow}(J).\\
&I \leq_{\mathrm{RB}} J \Leftrightarrow (\exists \varphi \in F(\mathrm{Fin}))\ I = \varphi^{\rightarrow}(J).\\
&I \leq_{\mathrm{K}} J \Leftrightarrow (\exists \varphi \in F(J))\ I \subseteq \varphi^{\rightarrow}(J).\\
&I \leq_{\mathrm{KB}} J \Leftrightarrow (\exists \varphi \in F(\mathrm{Fin}))\ I \subseteq \varphi^{\rightarrow}(J).\\
&I \leq_{\mathrm{K}}^{\kappa} J \Leftrightarrow (\forall X \in [I]^{<\kappa})(\exists \varphi \in F(J))\ X \subseteq \varphi^{\rightarrow}(J).\\
&I \leq_{\mathrm{KB}}^{\kappa} J \Leftrightarrow (\forall X \in [I]^{<\kappa})(\exists \varphi \in F(\mathrm{Fin}))\ X \subseteq \varphi^{\rightarrow}(J).
\end{aligned}$$

For $f \in {}^\omega({}^X\mathbb{R})$ and $\varphi \in {}^\omega\omega$ we define $\varphi \circ f \in {}^\omega({}^X\mathbb{R})$ by $(\varphi \circ f)_k(x) = f_{\varphi(k)}(x)$ and $\varphi * f \in {}^\omega({}^X\mathbb{R})$ by $(\varphi * f)_n(x) = \sup\{|f_k(x)| : k \in \varphi^{-1}(\{n\})\}$, if $n \in \operatorname{rng}(\varphi)$, and $(\varphi * f)_n(x) = 0$, otherwise. Note that $\varphi * (\varphi \circ f)\restriction\operatorname{rng}(\varphi) = |f|\restriction\operatorname{rng}(\varphi)$.

Lemma 2.2. *Let I be an ideal on ω and let $\varphi \in F(I)$ and $f \in {}^\omega({}^X\mathbb{R})$.*

(a) $f \xrightarrow{\varphi^{\to}(I)} 0$ *if and only if* $\varphi \circ f \xrightarrow{I} 0$.
(b) *If* $f \xrightarrow{\varphi^{\to}(I)\mathrm{QN}} 0$, *then* $\varphi \circ f \xrightarrow{I\mathrm{QN}} 0$.
(c) *If* $\varphi * f \xrightarrow{\varphi^{\to}(I)} 0$, *then* $f \xrightarrow{I} 0$.
(d) *If* $\varphi * f \xrightarrow{\varphi^{\to}(I)\mathrm{QN}} 0$, *then* $f \xrightarrow{I\mathrm{QN}} 0$.

Proof. (a) For every $\varepsilon > 0$ and $x \in X$, $\{k \in \omega : |f_{\varphi(k)}(x)| \geq \varepsilon\} = \varphi^{-1}(\{n \in \omega : |f_n(x)| \geq \varepsilon\})$. Therefore $\{n \in \omega : |f_n(x)| \geq \varepsilon\} \in \varphi^{\to}(I)$ if and only if $\{k \in \omega : |f_{\varphi(k)}(x)| \geq \varepsilon\} \in I$.

(b) Let $\varepsilon \in {}^\omega[0,\infty)$ be such that $\varepsilon \xrightarrow{\varphi^{\to}(I)} 0$ and $\{n \in \omega : |f_n(x)| \geq \varepsilon_n\} \in \varphi^{\to}(I)$ for all $x \in X$. Then $\varphi \circ f \xrightarrow{I\mathrm{QN}} 0$ because for every $x \in X$, $\{k \in \omega : |f_{\varphi(k)}(x)| \geq \varepsilon_{\varphi(k)}\} = \varphi^{-1}(\{n \in \omega : |f_n(x)| \geq \varepsilon_n\}) \in I$ and, by (a), $\varphi \circ \varepsilon \xrightarrow{I} 0$.

(c) If for every $\varepsilon > 0$ and $x \in X$, $b_{x,\varepsilon} = \{n \in \omega : (\varphi * f)_n(x) \geq \varepsilon\} \in \varphi^{\to}(I)$, then $\{k \in \omega : |f_k(x)| \geq \varepsilon\} \subseteq \varphi^{-1}(b_{x,\varepsilon}) \in I$.

(d) Let $\varepsilon \in {}^\omega[0,\infty)$ be such that $\varepsilon \xrightarrow{\varphi^{\to}(I)} 0$ and $b_x = \{n \in \omega : (\varphi * f)_n(x) \geq \varepsilon_n\} \in \varphi^{\to}(I)$ for all $x \in X$. Then $f \xrightarrow{I\mathrm{QN}} 0$ because $\{k \in \omega : |f_k(x)| \geq \varepsilon_{\varphi(k)}\} \subseteq \varphi^{-1}(b_x) \in I$ and, by (a), $\varphi \circ \varepsilon \xrightarrow{I} 0$. $\square$

Lemma 2.3. *Let I be an ideal on ω, let $\varphi \in {}^\omega\omega$ be such that $\operatorname{rng}(\varphi) \in I^+$, and let $f \in {}^\omega({}^X\mathbb{R})$.*

(a) $f \xrightarrow{\varphi^{\leftarrow}(I)} 0$ *if and only if* $\varphi * f \xrightarrow{I} 0$.
(b) *If* $f \xrightarrow{\varphi^{\leftarrow}(I)\mathrm{QN}} 0$, *then* $\varphi * f \xrightarrow{I\mathrm{QN}} 0$.

If $\operatorname{rng}(\varphi) \in I^*$, *then:*

(c) $\varphi \circ f \xrightarrow{\varphi^{\leftarrow}(I)} 0$ *if and only if* $f \xrightarrow{I} 0$.
(d) *If* $\varphi \circ f \xrightarrow{\varphi^{\leftarrow}(I)\mathrm{QN}} 0$, *then* $f \xrightarrow{I\mathrm{QN}} 0$.

Proof. (a) For every $x \in X$ and $\varepsilon > 0$, $\{n \in \omega : (\varphi * f)_n(x) \geq 2\varepsilon\} \subseteq \varphi(\{k \in \omega : |f_k(x)| \geq \varepsilon\}) \subseteq \{n \in \omega : (\varphi * f)_n(x) \geq \varepsilon\}$.

(b) Let $\varepsilon \in {}^\omega[0,\infty)$ be such that $\varepsilon \xrightarrow{\varphi^{\leftarrow}(I)} 0$ and $b_x = \{k \in \omega : |f_k(x)| \geq \varepsilon_k\} \in \varphi^{\leftarrow}(I)$ for every $x \in X$. Then $\varphi * f \xrightarrow{I\mathrm{QN}} 0$ because $\{n \in \omega : (\varphi * f)_n(x) \geq (\varphi * \varepsilon)_n + 2^{-n}\} \subseteq \varphi(b_x) \in I$ and, by (a), $\varphi * \varepsilon \xrightarrow{I} 0$.

(c)–(d) Substitute $\varphi \circ f$ for f into (a) and (b) and then replace $\varphi * (\varphi \circ f)_k$ by f; the latter is possible because $\{k \in \omega : \varphi * (\varphi \circ f)_k \neq |f|_k\} \subseteq \omega \setminus \operatorname{rng}(\varphi) \in I$. $\square$

We will use the following two reductions between the functions $\alpha \in {}^\omega\omega$ and the sequences of reals $\xi = \langle \xi_k : k \in \omega \rangle \in {}^\omega\mathbb{R}$

$$\sigma : {}^\omega\omega \to {}^\omega(0,1] \quad \text{and} \quad \tau : {}^\omega\mathbb{R} \to {}^\omega\omega$$

defined by

$$\sigma_k(\alpha) = \sigma(\alpha)(k) = 2^{-\alpha(k)},$$
$$\tau(\xi)(k) = \min\{n \in \omega : |\xi_k| + 2^{-k} > 2^{-n}\}.$$

The functions $\sigma_k : {}^\omega\omega \to \mathbb{R}$, $k \in \omega$, are continuous.

If ξ and η are sequences of reals (or natural numbers), then we denote

$$\|\xi \le \eta\| = \{k \in \omega : \xi(k) \le \eta(k)\} = \{k \in \omega : \xi_k \le \eta_k\}.$$

In the same way we define $\|\xi < \eta\|$ and $\|\xi = \eta\|$.

Lemma 2.4. *Let I be an ideal on ω and let $\alpha, \varphi \in {}^\omega\omega$ and $\xi \in {}^\omega\mathbb{R}$.*

(1) $\sigma(\alpha) \xrightarrow{I} 0$ *if and only if* $\alpha \in F(I)$.
(2) $\tau(\xi) \in F(I)$ *if and only if* $\xi \xrightarrow{I} 0$.
(3) $\{k \in \omega : |\xi_k| \ge \sigma_k(\alpha)\})\| \subseteq \|\tau(\xi) \le \alpha\|$.
(4) $\{k \in \omega : \sigma_k(\alpha) \ge |\xi_k|\} \supseteq \|\alpha < \tau(\xi)\|$.
(5) $\{k \in \omega : |\xi_{\varphi(k)}| \ge \sigma_k(\alpha)\} \subseteq \|\varphi \circ \tau(\xi) \le \alpha\|$.
(6) $\{k \in \omega : \sigma_{\varphi(k)}(\alpha) \ge |\xi_k|\} \supseteq \|\varphi \circ \alpha < \tau(\xi)\|$.

Proof. (1) and (2) hold by definitions; (3) and (4) are special cases of (5) and (6).

(5) $\{k \in \omega : |\xi_{\varphi(k)}| \ge \sigma_k(\alpha)\} \subseteq \{k \in \omega : |\xi_{\varphi(k)}| + 2^{-\varphi(k)} > 2^{-\alpha(k)}\} = \{k \in \omega : \tau(\xi)(\varphi(k)) \le \alpha(k)\} = \|\varphi \circ \tau(\xi) \le \alpha\|$.

(6) $\{k \in \omega : \sigma_{\varphi(k)}(\alpha) \ge |\xi_k|\} \supseteq \{k \in \omega : 2^{-\alpha(\varphi(k))} \ge |\xi_k| + 2^{-k}\} = \{k \in \omega : \alpha(\varphi(k)) < \tau(\xi)(k)\} = \|\varphi \circ \alpha < \tau(\xi)\|$. □

3. (β, γ)-space and w(β, γ)-space

In this section and only in this section the symbols β and γ denote convergences of sequences of functions. We consider the following ideal convergences:

$$\begin{array}{ccc} I\text{-convergence} & \Rightarrow & \le_{\mathrm{K}} I\text{-convergence} \\ \Uparrow & & \Uparrow \\ I\mathrm{QN}\text{-convergence} & \Rightarrow & \le_{\mathrm{K}} I\mathrm{QN}\text{-convergence} \end{array}$$

where I is an ideal on ω and where we say that a sequence f $\le_{\mathrm{K}} I$-converges to 0, if $(\exists J \le_{\mathrm{K}} I)\ f \xrightarrow{J} 0)$. Similarly, f $\le_{\mathrm{K}} I$QN-converges to 0, if $(\exists J \le_{\mathrm{K}} I)\ f \xrightarrow{J\mathrm{QN}} 0$.

Following [3] but in accordance with the denotation in the paper [5] we say that a space X is an $\beta\gamma$-space, where β and γ are two convergences of sequences of functions, if for every $f \in {}^\omega C(X)$, if $f \xrightarrow{\beta} 0$, then $f \xrightarrow{\gamma} 0$. We say that X is a w$\beta\gamma$-space (the letter 'w' means 'weak'), if for every $f \in {}^\omega C(X)$, if $f \xrightarrow{\beta} 0$, then there is $\varphi \in {}^\omega\omega$ such that $\varphi \circ f \xrightarrow{\gamma} 0$. Since β and γ may be represented by strings of several letters, we prefer the denotation (β, γ)-space and w(β, γ)-space.

For example, QN $\Leftrightarrow$ (Fin, FinQN) and wQN $\Leftrightarrow$ w(Fin, FinQN), where Fin and FinQN stand for I-convergence and IQN-convergence with $I = \mathrm{Fin}$. Hence, (I, J)QN-space and (I, J)wQN-space from [3] mean $(I, J\mathrm{QN})$-space and w$(I, J\mathrm{QN})$-space, respectively.

In this paper, like in [3], the subsequence $\varphi \circ f$ need not be strictly increasing or one-to-one (unlike [6] and [9] where the 'subsequence of f' means $\varphi \circ f$ with φ strictly increasing). This allows to extend the definition of a w(β, γ)-space for convergences of sequences with arbitrary countable set of indices. Then the set of indices of γ-convergent sequences may differ from the set of indices of β-convergent sequences in the definition of a w(β,γ)-space, but may not be different in the definition of an (β,γ)-space. In the definition of a w(β,γ)-space for convergences γ stronger than the J-convergence we can require $\varphi \in F(J)$: Consider $h_n(x) = |f_n(x)| + 2^{-n}$ instead of f_n. If $\varphi \circ h \xrightarrow{J} 0$, then $2^{-\varphi(n)} \xrightarrow{J} 0$, i.e., $\varphi \in F(J)$.

The following two lemmas exclude trivial properties.

Lemma 3.1. *Let I and J be ideals on ω. The following conditions are equivalent:*

(1) $I \subseteq J$.
(2) *Every space is an (I,J)-space.*
(3) *Every space is an $(I\mathrm{QN}, J\mathrm{QN})$-space.*
(4) *Every space is an $(I\mathrm{QN}, J)$-space.*
(5) *There is a nonempty (I,J)-space.*
(6) *There is a nonempty $(I\mathrm{QN}, J\mathrm{QN})$-space.*
(7) *There is a nonempty $(I\mathrm{QN}, J)$-space.*

Proof. Obviously, (1) $\Rightarrow$ (2) $\Rightarrow$ (5) $\Rightarrow$ (7), (1) $\Rightarrow$ (3) $\Rightarrow$ (6) $\Rightarrow$ (7), (1) $\Rightarrow$ (4) $\Rightarrow$ (7).

(7) $\Rightarrow$ (1) Let $X \neq \emptyset$. Given $a \in I$ assign a sequence $f^a \in {}^{\omega}C(X)$ defined by $f^a_n = \check{1}$, if $n \in a$, and $f^a_n = \check{0}$, if $n \in \omega \setminus a$. For every $a \in I$, $f^a \xrightarrow{I\mathrm{QN}} 0$ and, if $f^a \xrightarrow{J} 0$, then $a \in J$. □

Lemma 3.2. *For any ideals I and J on ω there is no nonempty space which is either $(\leq_{\mathrm{K}} I, J\mathrm{QN})$, or $(\leq_{\mathrm{K}} I, J)$, or $(\leq_{\mathrm{K}} I\mathrm{QN}, J\mathrm{QN})$, or $(\leq_{\mathrm{K}} I\mathrm{QN}, J)$.*

Proof. It is enough to prove that there is no nonempty $(\leq_{\mathrm{K}} I\mathrm{QN}, J)$-space because the other properties imply this property. Fix $a \in J^+$ and let $I' = \{x \subseteq \omega : x \setminus a \in \mathrm{Fin}\}$. Since I' is not tall, $I' \leq_{\mathrm{K}} \mathrm{Fin} \leq_{\mathrm{K}} I$. Therefore $(\leq_{\mathrm{K}} I\mathrm{QN}, J)$ implies $(I'\mathrm{QN}, J)$. But by Lemma 3.1 there is no $(I'\mathrm{QN}, J)$-space because $I' \nsubseteq J$. □

Lemma 3.3. *Let I and J be ideals on ω. The following conditions hold for arbitrary convergences β and γ:*

(a) $\mathrm{w}(\beta, J) \Leftrightarrow \mathrm{w}(\beta, \leq_{\mathrm{K}} J) \Leftrightarrow (\beta, \leq_{\mathrm{K}} J)$.
(b) $\mathrm{w}(\beta, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(\beta, \leq_{\mathrm{K}} J\mathrm{QN})$.
(c) $\mathrm{w}(I, \gamma) \Leftrightarrow \mathrm{w}(\leq_{\mathrm{K}} I, \gamma)$.
(d) $\mathrm{w}(I\mathrm{QN}, \gamma) \Leftrightarrow \mathrm{w}(\leq_{\mathrm{K}} I\mathrm{QN}, \gamma)$.

Proof. Note that the following implications are trivial: (a) $\mathrm{w}(\beta, J) \Rightarrow \mathrm{w}(\beta, \leq_{\mathrm{K}} J)$ and $(\beta, \leq_{\mathrm{K}} J) \Rightarrow \mathrm{w}(\beta, \leq_{\mathrm{K}} J)$; (b) $\mathrm{w}(\beta, J\mathrm{QN}) \Rightarrow \mathrm{w}(\beta, \leq_{\mathrm{K}} J\mathrm{QN})$; (c) $\mathrm{w}(\leq_{\mathrm{K}} I, \gamma) \Rightarrow \mathrm{w}(I, \gamma)$; (d) $\mathrm{w}(\leq_{\mathrm{K}} I\mathrm{QN}, \gamma) \Rightarrow \mathrm{w}(I\mathrm{QN}, \gamma)$. We prove the inverse implications. Let X be arbitrary space and let $f \in {}^{\omega}C(X)$.

(a) $\mathrm{w}(\beta, \leq_{\mathrm{K}} J) \Rightarrow \mathrm{w}(\beta, J)$ and $\mathrm{w}(\beta, \leq_{\mathrm{K}} J) \Rightarrow (\beta, \leq_{\mathrm{K}} J)$. Assume that there are $\varphi \in {}^{\omega}\omega$ and $K \leq_{\mathrm{K}} J$ such that $\varphi \circ f \xrightarrow{K} 0$. Let $\psi \in {}^{\omega}\omega$ be such that $K \subseteq \psi^{\rightarrow}(J)$. Then $\varphi \circ f \xrightarrow{\psi^{\rightarrow}(J)} 0$ and by Lemma 2.2 (a) (in both directions), $(\psi \circ \varphi) \circ f \xrightarrow{J} 0$ and $f \xrightarrow{(\psi\circ\varphi)^{\rightarrow}(J)} 0$.

(b) $\mathrm{w}(\beta, \leq_{\mathrm{K}} J\mathrm{QN}) \Rightarrow \mathrm{w}(\beta, J\mathrm{QN})$. Assume that there are $\varphi \in {}^{\omega}\omega$ and $K \leq_{\mathrm{K}} J$ such that $\varphi \circ f \xrightarrow{K\mathrm{QN}} 0$. Let $\psi \in {}^{\omega}\omega$ be such that $K \subseteq \psi^{\rightarrow}(J)$. Then by Lemma 2.2 (b), $(\psi \circ \varphi) \circ f \xrightarrow{J\mathrm{QN}} 0$.

(c)–(d) $\mathrm{w}(I, \gamma) \Rightarrow \mathrm{w}(\leq_{\mathrm{K}} I, \gamma)$ and $\mathrm{w}(I\mathrm{QN}, \gamma) \Rightarrow \mathrm{w}(\leq_{\mathrm{K}} I\mathrm{QN}, \gamma)$. Assume that $f \xrightarrow{\leq_{\mathrm{K}} I} 0$ (or $f \xrightarrow{\leq_{\mathrm{K}} I\mathrm{QN}} 0$). Hence there is an ideal K such that for some $\varphi \in {}^{\omega}\omega$, $K \subseteq \varphi^{\rightarrow}(I)$ and $f \xrightarrow{K} 0$ (or $f \xrightarrow{K\mathrm{QN}} 0$). Then by Lemma 2.2 (a) (or Lemma 2.2 (b)),

$\varphi \circ f \xrightarrow{I} 0$ (or $\varphi \circ f \xrightarrow{I\mathrm{QN}} 0$). Now, applying $\mathrm{w}(I, \gamma)$ (or $\mathrm{w}(I\mathrm{QN}, \gamma)$) we find $\psi \in {}^\omega\omega$ such that $(\psi \circ \varphi) \circ f \xrightarrow{\gamma} 0$. $\square$

Concerning to the four mentioned ideal convergences there are together 2×16 properties of the form (β, γ) and $\mathrm{w}(\beta, \gamma)$ (see the table below).

By Lemma 3.1 and Lemma 3.2 the following 7 properties are trivial:

- $(I\mathrm{QN}, J\mathrm{QN})$, $(I\mathrm{QN}, J)$, (I, J);
- $(\le_{\mathrm{K}} I\mathrm{QN}, J\mathrm{QN})$, $(\le_{\mathrm{K}} I, J\mathrm{QN})$, $(\le_{\mathrm{K}} I\mathrm{QN}, J)$, $(\le_{\mathrm{K}} I, J)$.

Due to Lemma 3.3 the remaining 25 properties are divided into the following 9 equivalence classes:

(1) $(I, J\mathrm{QN})$
(2) $(I\mathrm{QN}, \le_{\mathrm{K}} J\mathrm{QN})$
(3) $(I, \le_{\mathrm{K}} J\mathrm{QN})$
(4) $(\le_{\mathrm{K}} I\mathrm{QN}, \le_{\mathrm{K}} J\mathrm{QN})$
(5) $(\le_{\mathrm{K}} I, \le_{\mathrm{K}} J\mathrm{QN})$
(6) $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(I\mathrm{QN}, \le_{\mathrm{K}} J\mathrm{QN})$
$\Leftrightarrow \mathrm{w}(\le_K I\mathrm{QN}, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(\le_K I\mathrm{QN}, \le_{\mathrm{K}} J\mathrm{QN})$
(7) $\mathrm{w}(I, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(I, \le_{\mathrm{K}} J\mathrm{QN}) \Leftrightarrow \mathrm{w}(\le_K I, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(\le_K I, \le_{\mathrm{K}} J\mathrm{QN})$
(8) $\mathrm{w}(I\mathrm{QN}, J) \Leftrightarrow \mathrm{w}(I\mathrm{QN}, \le_{\mathrm{K}} J) \Leftrightarrow (I\mathrm{QN}, \le_{\mathrm{K}} J)$
$\Leftrightarrow \mathrm{w}(\le_K I\mathrm{QN}, J) \Leftrightarrow \mathrm{w}(\le_K I\mathrm{QN}, \le_{\mathrm{K}} J) \Leftrightarrow (\le_K I\mathrm{QN}, \le_{\mathrm{K}} J)$
(9) $\mathrm{w}(I, J) \Leftrightarrow \mathrm{w}(I, \le_{\mathrm{K}} J) \Leftrightarrow (I, \le_{\mathrm{K}} J)$
$\Leftrightarrow \mathrm{w}(\le_K I, J) \Leftrightarrow \mathrm{w}(\le_K I, \le_{\mathrm{K}} J) \Leftrightarrow (\le_K I, \le_{\mathrm{K}} J)$

– $(I\mathrm{QN}, J\mathrm{QN})$	– $(I\mathrm{QN}, J)$	**2** $(I\mathrm{QN}, \le_{\mathrm{K}} J\mathrm{QN})$	8 $(I\mathrm{QN}, \le_{\mathrm{K}} J)$
1 $(I, J\mathrm{QN})$	– (I, J)	**3** $(I, \le_{\mathrm{K}} J\mathrm{QN})$	9 $(I, \le_{\mathrm{K}} J)$
– $(\le_{\mathrm{K}} I\mathrm{QN}, J\mathrm{QN})$	– $(\le_{\mathrm{K}} I\mathrm{QN}, J)$	**4** $(\le_{\mathrm{K}} I\mathrm{QN}, \le_{\mathrm{K}} J\mathrm{QN})$	8 $(\le_{\mathrm{K}} I\mathrm{QN}, \le_{\mathrm{K}} J)$
– $(\le_{\mathrm{K}} I, J\mathrm{QN})$	– $(\le_{\mathrm{K}} I, J)$	**5** $(\le_{\mathrm{K}} I, \le_{\mathrm{K}} J\mathrm{QN})$	9 $(\le_{\mathrm{K}} I, \le_{\mathrm{K}} J)$

6 $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN})$	**8** $\mathrm{w}(I\mathrm{QN}, J)$	6 $\mathrm{w}(I\mathrm{QN}, \le_{\mathrm{K}} J\mathrm{QN})$	8 $\mathrm{w}(I\mathrm{QN}, \le_{\mathrm{K}} J)$
7 $\mathrm{w}(I, J\mathrm{QN})$	**9** $\mathrm{w}(I, J)$	7 $\mathrm{w}(I, \le_{\mathrm{K}} J\mathrm{QN})$	9 $\mathrm{w}(I, \le_{\mathrm{K}} J)$
6 $\mathrm{w}(\le_{\mathrm{K}} I\mathrm{QN}, J\mathrm{QN})$	8 $\mathrm{w}(\le_{\mathrm{K}} I\mathrm{QN}, J)$	6 $\mathrm{w}(\le_{\mathrm{K}} I\mathrm{QN}, \le_{\mathrm{K}} J\mathrm{QN})$	8 $\mathrm{w}(\le_{\mathrm{K}} I\mathrm{QN}, \le_{\mathrm{K}} J)$
7 $\mathrm{w}(\le_{\mathrm{K}} I, J\mathrm{QN})$	9 $\mathrm{w}(\le_{\mathrm{K}} I, J)$	7 $\mathrm{w}(\le_{\mathrm{K}} I, \le_{\mathrm{K}} J\mathrm{QN})$	9 $\mathrm{w}(\le_{\mathrm{K}} I, \le_{\mathrm{K}} J)$

As a consequence of the proved equivalences we get the following implications between the properties:

Lemma 3.4. *For any ideals I, J, and $K \subseteq J$ on ω the following implications hold:*

$$\begin{array}{ccccc}
 & & \mathrm{w}(I, J) & \Rightarrow & \mathrm{w}(I\mathrm{QN}, J) \\
 & & \Uparrow & & \Uparrow \\
\mathrm{w}(I, JK\mathrm{QN}) & \Rightarrow & \mathrm{w}(I, J\mathrm{QN}) & \Rightarrow & \mathrm{w}(I\mathrm{QN}, J\mathrm{QN}) \\
\Uparrow & & \Uparrow & & \Uparrow \\
(I, JK\mathrm{QN}) \Rightarrow (I, J\mathrm{QN}) & \Rightarrow & (I, \le_{\mathrm{K}} J\mathrm{QN}) & \Rightarrow & (I\mathrm{QN}, \le_{\mathrm{K}} J\mathrm{QN}) \\
 & & \Uparrow & & \Uparrow \\
 & & (\le_{\mathrm{K}} I, \le_{\mathrm{K}} J\mathrm{QN}) & \Rightarrow & (\le_{\mathrm{K}} I\mathrm{QN}, \le_{\mathrm{K}} J\mathrm{QN})
\end{array}$$

$\square$

Lemma 3.5. *There is a nonempty $(I, J\mathrm{QN})$-space if and only if $I \subseteq J$.*

Proof. A singleton is an $(I, J\mathrm{QN})$-space, if $I \subseteq J$. Conversely, $(I, J\mathrm{QN})$-space is an $(I\mathrm{QN}, J)$-space and, by Lemma 3.1, it exists only if $I \subseteq J$. $\square$

By next lemma for all pairs of ideals I and J and for every property from Lemma 3.4 to the right of $(I, J\mathrm{QN})$ there always exist nonempty spaces with this property.

Lemma 3.6. *For arbitrary ideals I and J on ω the following conditions hold:*

(1) $(\forall\varphi \in F(I))(\exists\psi \in F(\mathrm{Fin}))\ \psi\circ\varphi \in F(\mathrm{Fin})$ *and* $\varphi \in F(\psi^{\rightarrow}(\mathrm{Fin}))$.

(2) *There is a nonempty* $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})$*-space.*

Proof. (1) Let $\varphi \in F(I)$. For every $n \in \omega$ define $\psi(n) = \min(\varphi^{-1}(\{k\}))$ where k is the nth member of $\mathrm{rng}(\varphi)$, and hence $(\psi\circ\varphi)(n) = \varphi(\psi(n)) = k$ if and only if k is the nth member of $\mathrm{rng}(\varphi)$. Therefore the functions $\psi\circ\varphi$ and ψ are one-to-one.

For $k \in \omega$, $\varphi^{-1}(\{k\})) \in \psi^{\rightarrow}(\mathrm{Fin})$ because $(\psi\circ\varphi)(\{k\}) = \psi^{-1}(\varphi^{-1}(\{k\})) \in \mathrm{Fin}$.

(2) We prove that the singleton space $X = 1$ is $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})$. Let $f \in {}^{\omega}\mathbb{R} \simeq {}^{\omega}C(X)$ be arbitrary such that $f \xrightarrow{I'} 0$ for some $I' \leq_{\mathrm{K}} I$. We apply the reductions τ and σ from Lemma 2.4. Hence $\tau(f) \in F(I')$. By (1) there is a one-to-one function $\psi \in {}^{\omega}\omega$ such that $\tau(f) \in F(\psi^{\rightarrow}(\mathrm{Fin}))$ and hence $f \xrightarrow{\psi^{\rightarrow}(\mathrm{Fin})} 0$ where $\psi^{\rightarrow}(\mathrm{Fin}) \leq_{\mathrm{K}} J$. Then $f \xrightarrow{\psi^{\rightarrow}(\mathrm{Fin})\mathrm{QN}} 0$ because $|X| = 1$. □

4. Cardinal invariants

Let I, J, K be ideals on ω. We consider the following weakenings of the P-property for ideals and for pairs of ideals:

(i) J is a $P(I)$ ideal $\Leftrightarrow (\forall\alpha \in F(J))(\exists c \in J^*)(\forall n \in \omega)\ \|\alpha = n\| \cap c \in I$.

(ii) (K, J) is weak $P(I) \Leftrightarrow (\forall\alpha \in F(K))(\exists c \in J^+)(\forall n \in \omega)\ \|\alpha = n\| \cap c \in I$.

(iii) (K, J) is $W(I) \Leftrightarrow (\forall\alpha \in F(K))(\forall\varphi \in F(J))(\exists c \in J^+)(\forall n \in \omega)\ \varphi(\|\alpha = n\| \cap c) \in I$.

(iv) J is a weak $P(I)$-ideal $\Leftrightarrow (J, J)$ is weak $P(I)$.

(v) J is a $W(I)$-ideal $\Leftrightarrow (J, J)$ is $W(I)$.

A P-ideal is a $P(\mathrm{Fin})$-ideal. $W(I)$ implies weak $P(I)$: (K, J) is $W(I)$ if and only if (K, J) is weak $P(\varphi^{\leftarrow}(I))$ for every $\varphi \in F(J)$ with $\mathrm{rng}(\varphi) \notin I$. The property $W(I, J, K)$ from [12] for $J \subseteq I$ means (J, I) is weak $P(K)$.

Note that Theorem 4.1 (a) below is similar to [8, Theorem 1.2]; Theorem 4.2 (a) is similar to [8, Theorem 5.1]; and Theorem 4.2 (b) is similar to [9, Proposition 2] because $\|\beta < \alpha\| = \bigcup_{n\in\omega}(\|\alpha = n\| \cap \bigcup_{k<n} \|\beta = k\|)$.

Theorem 4.1. *Let I, J, K be ideals on ω and let $K \subseteq J$.*

(a) *The following conditions are equivalent:*

(1) (K, J) *is not weak* $P(I)$.

(2) $(\exists\alpha \in F(K))(\forall\beta \in F(I))\ \|\beta \leq \alpha\| \in J$.

(3) $(\exists\alpha \in F(K))(\forall\beta \in F(I))\ \|\beta < \alpha\| \in J$.

(4) *Every space is* $(I, JK\mathrm{QN})$.

(5) $F(I)$ *as a subset of the Baire space is* $(I, JK\mathrm{QN})$.

(b) *The following conditions are equivalent:*

(1) (K, J) *is not* $W(I)$.

(2) $(\exists\alpha \in F(K))(\exists\varphi \in F(J))(\forall\beta \in F(I))\ \|\varphi\circ\beta \leq \alpha\| \in J$.

(3) $(\exists\alpha \in F(K))(\exists\varphi \in F(J))(\forall\beta \in F(I))\ \|\varphi\circ\beta < \alpha\| \in J$.

(4) *Every space is* $\mathrm{w}(I, JK\mathrm{QN})$.

(5) $F(I)$ *as a subset of the Baire space is* $\mathrm{w}(I, J\mathrm{QN})$.

(c) *The following conditions are equivalent:*

(1) $(\exists J' \leq_{\mathrm{K}} J$ *on* $\omega)$ J' *is not a weak* $P(I)$*-ideal.*
(2) $(\exists J' \leq_{\mathrm{K}} J$ *on* $\omega)(\exists \alpha \in F(J'))(\forall \beta \in F(I))$ $\|\beta \leq \alpha\| \in J'$.
(3) $(\exists J' \leq_{\mathrm{K}} J$ *on* $\omega)(\exists \alpha \in F(J'))(\forall \beta \in F(I))$ $\|\beta < \alpha\| \in J'$.
(4) *Every space is* $(I, \leq_{\mathrm{K}} J\mathrm{QN})$.
(5) $F(I)$ *as a subset of the Baire space is* $(I, \leq_{\mathrm{K}} J\mathrm{QN})$.

(d) *The following conditions are equivalent:*

(1) $(\forall I' \leq_{\mathrm{K}} I$ *on* $\omega)(\exists J' \leq_{\mathrm{K}} J$ *on* $\omega)$ J' *is not a weak* $P(I')$*-ideal.*
(2) $(\forall I' \leq_{\mathrm{K}} I$ *on* $\omega)(\exists J' \leq_{\mathrm{K}} J$ *on* $\omega)(\exists \alpha \in F(J'))(\forall \beta \in F(I'))$ $\|\beta \leq \alpha\| \in J'$.
(3) $(\forall I' \leq_{\mathrm{K}} I$ *on* $\omega)(\exists J' \leq_{\mathrm{K}} J$ *on* $\omega)(\exists \alpha \in F(J'))(\forall \beta \in F(I'))$ $\|\beta < \alpha\| \in J'$.
(4) *Every space is* $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})$.
(5) *The discrete space of cardinality* $\mathfrak{c}$ *is* $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})$.

(In quantifiers of (c) and (d) the relation $\leq_{\mathrm{K}}$ can be equivalently replaced by $\leq_{\mathrm{RK}}$ because $\leq_{\mathrm{K}} \;=\; \subseteq \circ \leq_{\mathrm{RK}}$. This fact will be applied in Theorem 4.2 and in the translation of the lemma into the the language of relations in Theorem 6.2 below.)

Proof. (a) (1) $\Rightarrow$ (2) If $\alpha \in F(K)$ witnesses that (K, J) is not weak $P(I)$, then for every $\beta \in F(I)$, $\|\beta \leq \alpha\| \in J$ because $\|\alpha = n\| \cap \|\beta \leq \alpha\| \subseteq \|\beta \leq n\| \in I$ for all $n \in \omega$.

(2) $\Rightarrow$ (1) Assume that $\alpha \in F(K)$ witnesses (2) and let $c \subseteq \omega$ be arbitrary such that $\|\alpha = n\| \cap c \in I$ for all $n \in \omega$. Define $\beta \in {}^\omega\omega$ by $\beta(k) = \alpha(k)$, if $k \in c$, and $\beta(k) = k$, otherwise. Then $\beta \in F(I)$ because $\|\beta = n\| \subseteq (\|\alpha = n\| \cap c) \cup \{n\} \in I$ for all $n \in \omega$ and $c \in J$ because $c \subseteq \|\beta \leq \alpha\| \in J$. Therefore (K, J) is not weak $P(I)$.

(2) $\Rightarrow$ (3) holds because $\|\beta < \alpha\| \subseteq \|\beta \leq \alpha\|$ and (3) $\Rightarrow$ (2) holds because $K \subseteq J$ and $\|\beta \leq \alpha \dot{-} 1\| \subseteq \|\beta < \alpha\| \cup \|\alpha = 0\|$ where $(\alpha \dot{-} 1)(n) = \max\{0, \alpha(n) - 1\}$.

(4) $\Rightarrow$ (5) is trivial.

(2) $\Rightarrow$ (4) Let $\alpha \in F(K)$ witness (2). Then by Lemma 2.4 (1) and (3), $\sigma(\alpha) \xrightarrow{K} 0$ and for every $\xi \in {}^\omega\mathbb{R}$ with $\xi \xrightarrow{I} 0$, $\tau(\xi) \in F(I)$ and $\{k \in \omega : |\xi_k| \geq \sigma_k(\alpha)\} \subseteq \|\tau(\xi) \leq \alpha\| \in J$. Therefore every I-convergent sequence of reals JKQN-converges with the same control $\sigma(\alpha)$.

(5) $\Rightarrow$ (3) Define $f \in {}^\omega(C(F(I))$ by $f_k(\beta) = \sigma_k(\beta)$. Since $f \xrightarrow{I} 0$ there is an $\varepsilon \in {}^\omega[0, \infty)$ K-converging to 0 wintessing the JKQN-convergence of f. By Lemma 2.4 (1) and (4), $\tau(\varepsilon) \in F(K)$ and for every $\beta \in F(I)$, $\|\beta < \tau(\varepsilon)\| \subseteq \{k \in \omega : f_k(\beta) \geq \varepsilon_k\} \in J$.

(b) (1) $\Rightarrow$ (2) If $(\alpha, \varphi) \in F(K) \times F(J)$ witnesses that (K, J) is not $W(I)$, then for every $\beta \in F(I)$, $\|\varphi \circ \beta \leq \alpha\| = \bigcup_{n \in \omega}(\|\alpha = n\| \cap \varphi^{-1}(\|\beta \leq n\|)) \in J$ because $\varphi(\|\alpha = n\| \cap \varphi^{-1}(\|\beta \leq n\|)) \subseteq \|\beta \leq n\| \in I$ for all $n \in \omega$.

(2) $\Rightarrow$ (1) Let $\alpha \in F(K)$ and $\varphi \in F(J)$ witness (2) and let $c \subseteq \omega$ be arbitrary such that $\varphi(\|\alpha = n\| \cap c) \in I$ for all $n \in \omega$. Define $\beta \in {}^\omega\omega$ by $\beta(m) = \min\{n \in \omega : m \in \varphi(\|\alpha = n\| \cap c)\}$, if $m \in \varphi(c)$, and $\beta(m) = m$, otherwise. Then $\beta \in F(I)$ because $\|\beta = n\| \subseteq \varphi(\|\alpha = n\| \cap c) \cup \{n\} \in I$ for all $n \in \omega$. Then $c \in J$ because $c \subseteq \{k \in \omega : \beta(\varphi(k)) \leq \alpha(k)\} = \|\varphi \circ \beta \leq \alpha\| \in J$. Therefore (K, J) is not $W(I)$.

(2) $\Leftrightarrow$ (3) holds by same arguments like in (a) and (4) $\Rightarrow$ (5) is trivial.

(2) $\Rightarrow$ (4) Let $\alpha \in F(K)$ and $\varphi \in F(J)$ witness (2). Then by Lemma 2.4 (1) and (5), $\sigma(\alpha) \xrightarrow{K} 0$ and for every $\xi \in {}^\omega\mathbb{R}$ with $\xi \xrightarrow{I} 0$, $\{k \in \omega : |\xi_{\varphi(k)}| \geq \sigma_k(\alpha)\} \subseteq \|\varphi \circ \tau(\xi) \leq \alpha\| \in J$. Therefore $\sigma(\alpha)$ controls the JKQN-convergence of every φ-subsequence of an I-convergent sequence of reals.

(5) $\Rightarrow$ (3) Define $f \in {}^\omega(C(F(I))$ by $f_k(\beta) = \sigma_k(\beta)$. Since $f \xrightarrow{I} 0$, there are $\varphi \in F(J)$ and $\varepsilon \in {}^\omega[0,\infty)$ K-converging to 0 witnessing the JKQN-convergence of $\varphi \circ f$ to 0. Then by Lemma 2.4 (a) and (6), $\tau(\varepsilon) \in F(K)$ and for every $\beta \in F(I)$, $\|\varphi \circ \beta < \tau(\varepsilon)\| \subseteq \{k \in \omega : f_{\varphi(k)}(\beta) \geq \varepsilon_k\} \in J$.

(c) Follows from (a) with $K = J$.

(d) (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from (a) with $K = J$ and (4) $\Rightarrow$ (5) is trivial.

(2) $\Rightarrow$ (4) Let X be arbitrary space and let $f \in {}^\omega C(X)$ be such that $f \xrightarrow{I'} 0$ for some $I' \leq_{\mathrm{K}} I$. Let $J' \leq_{\mathrm{K}} J$ and $\alpha \in F(J')$ witness (2) and for every $x \in X$ let $\xi^x \in {}^\omega\mathbb{R}$ be defined by $\xi^x_k = f_k(x)$. Hence $\sigma(\alpha) \xrightarrow{J'} 0$ and for every $x \in X$, $\xi^x \xrightarrow{I'} 0$ and $\tau(\xi^x) \in F(I')$. Then for every $x \in X$, by Lemma 2.4 (3), $\{k \in \omega : |f_k(x)| \geq \sigma_k(\alpha)\} \subseteq \|\tau(\xi^x) \leq \alpha\| \in J'$. Therefore $f \xrightarrow{J'\mathrm{QN}} 0$ and $J' \leq_{\mathrm{K}} J$.

(5) $\Rightarrow$ (3) We prove that if the discrete space $X = \bigcup\{F(I') : I' \leq_{\mathrm{K}} I$ is an ideal on $\omega\}$ is $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})$, then (3) holds. Let $I' \leq_{\mathrm{K}} I$ be an ideal on ω. Let $f \in {}^\omega({}^X\mathbb{R})$ be defined by $f_k(\beta) = \sigma_k(\beta)$, if $\beta \in F(I')$, and $f_k(\beta) = 0$, otherwise. Since $f \xrightarrow{I'} 0$, there is $J_0 \leq_{\mathrm{K}} J$ such that f J_0QN-converges to 0. Let $J' \supseteq J_0$ be such that $J' \leq_{\mathrm{KB}} J$. Then f J'QN-converges to 0 and let $\varepsilon \in {}^\omega[0,\infty)$ be a sequence J'-converging to 0 witnessing the J'QN-convergence of f. By Lemma 2.4 (1) and (4), $\tau(\varepsilon) \in F(J')$ and for every $\beta \in F(I')$, $\|\beta < \tau(\varepsilon)\| \subseteq \{k \in \omega : f_k(\beta) \geq \varepsilon_k\} \in J'$. $\square$

By Theorem 4.1 the following holds (a possible interpretation for the symbol ∞ is the cardinal $\mathfrak{c}^+$):

(a) If (K,J) is not weak $P(I)$, then $\mathrm{non}((I, JK\mathrm{QN})\text{-space}) = \infty$.
(b) If (K,J) is not $W(I)$, then $\mathrm{non}(\mathrm{w}(I, JK\mathrm{QN})\text{-space}) = \infty$.
(c) If there is an ideal $J' \leq_{\mathrm{K}} J$ on ω that is not a weak $P(I)$-ideal, then $\mathrm{non}((I, \leq_{\mathrm{K}} J\mathrm{QN})\text{-space}) = \infty$.
(d) If for every ideal $I' \leq_{\mathrm{K}} I$ on ω there is an ideal $J' \leq_{\mathrm{K}} J$ on ω that is not a weak $P(I')$-ideal, then $\mathrm{non}((\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})\text{-space}) = \infty$.

Observe that the equivalences (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) in assertions (a), (b), (c) of Theorem 4.1 hold by same proofs when reduced to spaces and subsets of $F(I)$ of a given infinite cardinality. Therefore we get the following ((d) goes directly):

Theorem 4.2. *Let I, J, and $K \subseteq J$ be ideals on ω.*

(a) *If (K,J) is weak $P(I)$, then*

$$\mathrm{non}((I, JK\mathrm{QN})\textit{-space}) = \min\{|X| : X \subseteq F(I) \textit{ and } (\forall \alpha \in F(K))(\exists \beta \in X)\ \|\beta < \alpha\| \notin J\} \leq \mathfrak{c}.$$

(b) *If (K,J) is $W(I)$, then*

$$\mathrm{non}(\mathrm{w}(I, JK\mathrm{QN})\textit{-space}) = \min\{|X| : X \subseteq F(I) \textit{ and } (\forall \alpha \in F(K))(\forall \varphi \in F(J))(\exists \beta \in X)\ \|\varphi \circ \beta < \alpha\| \notin J\} \leq \mathfrak{c}.$$

(c) *If every ideal $\leq_{\mathrm{K}}$-below J is a weak $P(I)$ ideal, then*

$$\mathrm{non}((I, \leq_{\mathrm{K}} J\mathrm{QN})\textit{-space}) = \min\{|X| : X \subseteq F(I) \textit{ and } (\forall J' \leq_{\mathrm{RK}} J \textit{ on } \omega) (\forall \alpha \in F(J'))(\exists \beta \in X)\ \|\beta < \alpha\| \notin J'\} \leq \mathfrak{c}.$$

(d) *If there is an ideal $I' \leq_{\mathrm{K}} I$ on ω such that every ideal $J' \leq_{\mathrm{K}} J$ on ω is a weak $P(I')$-ideal, then*

$$\mathrm{non}((\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})\textit{-space}) = \min\{\mathrm{non}((I', \leq_{\mathrm{K}} J\mathrm{QN})\textit{-space}) : I' \leq_{\mathrm{K}} I\} \leq \mathfrak{c}. \qquad \square$$

In the next theorem we identify $\mathcal{P}(\omega)$ with the Cantor space.

Theorem 4.3. *For any ideals I and J on ω the following conditions are equivalent:*

(1) $I \leq_{\mathrm{K}} J$.
(2) *Every space is* $\mathrm{w}(I, J)$.
(3) *Every space is* $(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$.
(4) *Every space is* $(I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$.
(5) *Every space is* $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN})$.
(6) *Every space is* $\mathrm{w}(I\mathrm{QN}, J)$.
($\bar{2}$) *I as a subset of the Cantor space is* $\mathrm{w}(I, J)$.
($\bar{3}$) *I as a subset of the Cantor space is* $(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$.
($\bar{4}$) *I as a subset of the Cantor space is* $(I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$.
($\bar{5}$) *I as a subset of the Cantor space is* $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN})$.
($\bar{6}$) *I as a subset of the Cantor space is* $\mathrm{w}(I\mathrm{QN}, J)$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ ($\bar{2}$) Assume that $I \leq_{\mathrm{K}} J$, i.e., there $\varphi \in {}^{\omega}\omega$ such that $I \subseteq \varphi^{\rightarrow}(J)$. Let X be arbitrary space and let $f \in {}^{\omega}({}^{X}\mathbb{R})$. If $f \xrightarrow{I} 0$, then $f \xrightarrow{\varphi^{\rightarrow}(J)} 0$ and by Lemma 2.2 (a), $\varphi \circ f \xrightarrow{J} 0$.

(1) $\Rightarrow$ (3) $\Rightarrow$ ($\bar{3}$) If $I \leq_{\mathrm{K}} J$ and $I' \leq_{\mathrm{K}} I$, then $I' \leq_{\mathrm{K}} J$ and by Lemma 3.1, every space is $(I'\mathrm{QN}, I'\mathrm{QN})$. Therefore every space is $(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$.

(2) $\Rightarrow$ (6), ($\bar{2}$) $\Rightarrow$ ($\bar{6}$) and (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6), ($\bar{3}$) $\Rightarrow$ ($\bar{4}$) $\Rightarrow$ ($\bar{5}$) $\Rightarrow$ ($\bar{6}$) hold by Lemma 3.4; (6) $\Rightarrow$ ($\bar{6}$) is trivial.

($\bar{6}$) $\Rightarrow$ (1) Let $f \in {}^{\omega}C(I)$ be defined by

$$f_n(x) = \begin{cases} 1, & \text{if } n \in x, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f \xrightarrow{I\mathrm{QN}} 0$ because $\{n \in \omega : |f_n(x)| \geq 2^{-n}\} = x \in I$ for all $x \in I$. Let $\varphi \in {}^{\omega}\omega$ be such that $\varphi \circ f \xrightarrow{J} 0$. Then for every $x \in I$, $\varphi^{-1}(x) = \{k \in \omega : f_{\varphi(k)}(x) \geq 1\} \in J$. Therefore $I \subseteq \varphi^{\rightarrow}(J)$ and hence $I \leq_{\mathrm{K}} J$. $\square$

For ideals I and J on ω we define

$$\mathfrak{k}_{I,J} = \min\{|X| : X \subseteq I \text{ and } (\forall \varphi \in F(J)\ X \setminus \varphi^{\rightarrow}(J) \neq \emptyset\},$$
$$\mathfrak{k}^*_{I,J} = \min\{|X| : X \subseteq I \text{ and } (\forall \varphi \in F(\mathrm{Fin}))\ X \setminus \varphi^{\rightarrow}(J) \neq \emptyset\}.$$

If $I \leq_{\mathrm{K}} J$, then we let $\mathfrak{k}_{I,J} = \infty$. If $I \leq_{\mathrm{KB}} J$, then we let $\mathfrak{k}^*_{I,J} = \infty$. Hence, $\mathfrak{k}_{I,J} < \infty \Leftrightarrow \mathfrak{k}_{I,J} \leq \mathfrak{c} \Leftrightarrow I \not\leq_{\mathrm{K}} J$ and $\mathfrak{k}^*_{I,J} < \infty \Leftrightarrow \mathfrak{k}^*_{I,J} \leq \mathfrak{c} \Leftrightarrow I \not\leq_{\mathrm{K}} J$.

Lemma 4.4. *Let I and J be ideals on ω.*

(a) *$I \leq^{\kappa}_{\mathrm{KB}} J$ if and only if either $I \leq_{\mathrm{KB}} J$ or else $\kappa \leq \mathfrak{k}^*_{I,J}$.*
(b) *$I \leq^{\kappa}_{\mathrm{K}} J$ if and only if either $I \leq_{\mathrm{K}} J$ or else $\kappa \leq \mathfrak{k}_{I,J}$.*
(c) $\mathfrak{p} \leq \mathfrak{k}^*_{I,J} \leq \mathfrak{k}_{I,J}$.
(d) *If J is a P-ideal, then $\mathfrak{k}^*_{I,J} = \mathfrak{k}_{I,J}$.*
(e) *For every $\kappa \leq \mathfrak{p}$, $\leq^{\kappa}_{\mathrm{KB}} = \leq^{\kappa}_{\mathrm{K}} = \{(I_1, I_2) : I_1$ and I_2 are ideals on $\omega\}$.*

Proof. (c) If $X \subseteq I$ does not generate a tall ideal, then there is a one-to-one function $\varphi \in {}^\omega\omega$ such that $X \subseteq \varphi^\to(\mathrm{Fin}) \subseteq \varphi^\to(J)$. The minimal size of a set $X \subseteq I$ generating a tall ideal is $\geq \mathfrak{p}$ $\square$

Note that the cardinal number $\mathrm{cov}^*(I) = \min\{X \subseteq I : X \text{ generates a tall ideal}\}$ used by Kwela in a similar context, equals to $\mathfrak{k}_{I,\mathrm{Fin}}$ and to $\mathfrak{k}^*_{I,\mathrm{Fin}}$.

Theorem 4.5. *Let I and J be ideals on ω such that $I \not\leq_{\mathrm{K}} J$ and let X be a discrete space. The following conditions are equivalent:*

(1) $|X| < \mathfrak{k}_{I,J}$.
(2) X *is* $\mathrm{w}(I,J)$.
(3) X *is* $(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$.
(4) X *is* $(I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$.
(5) X *is* $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN})$.
(6) X *is* $\mathrm{w}(I\mathrm{QN}, J)$.

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). Let $|X| < \mathfrak{k}_{I,J}$ and let $f \in {}^\omega C(X)$ be arbitrary such that either $f \xrightarrow{I} 0$ or $f \xrightarrow{\leq_{\mathrm{K}} I\mathrm{QN}} 0$.

The convergence $f \xrightarrow{I} 0$ is witnessed by a family $I_0 \subseteq I$ of cardinality $\leq |X| \cdot \omega < \mathfrak{k}_{I,J}$. Let $\psi \in {}^\omega\omega$ be such that $I_0 \subseteq \psi^\to(J)$ and denote $I' = I \cap \psi^\to(J)$. Then $I' \leq_{\mathrm{K}} J$ and $f \xrightarrow{I'} 0$. By Theorem 4.3, X is a $\mathrm{w}(I', J)$-space and therefore there is $\varphi \in {}^\omega\omega$ such that $\varphi \circ f \xrightarrow{J} 0$.

If $f \xrightarrow{\leq_{\mathrm{K}} I\mathrm{QN}} 0$, then $f \xrightarrow{\eta^\to(I)\mathrm{QN}} 0$ for for some $\eta \in F(I)$ and this convergence is witnessed by a family $I_0 \subseteq \eta^\to(I)$ of cardinality $< \mathfrak{k}_{I,J}$. Let $\psi \in {}^\omega\omega$ be such that $\{\eta^{-1}(a) : a \in I_0\} \subseteq \psi^\to(J)$ and denote $I' = I \cap \psi^\to(J)$. Then $I' \leq_{\mathrm{K}} J$ and $f \xrightarrow{\leq_{\mathrm{K}} I'\mathrm{QN}} 0$ because $I_0 \subseteq \eta^\to(I')$. By Theorem 4.3, X is a $(\leq_{\mathrm{K}} I'\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$-space and therefore there is $J' \leq_{\mathrm{K}} J$ such that $f \xrightarrow{J'\mathrm{QN}} 0$.

(2) $\Rightarrow$ (6) and (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) hold by Lemma 3.4.

(6) $\Rightarrow$ (1) Assume that X is $\mathrm{w}(I\mathrm{QN}, J)$. Then $|X| < \mathfrak{c}$. Otherwise, X is not $\mathrm{w}(I\mathrm{QN}, J)$ because I is a continuous image of X and by Theorem 4.3, I is not $\mathrm{w}(I\mathrm{QN}, J)$. Since $I \neq \mathrm{Fin} \leq_{\mathrm{K}} J$, $|X| < |I| = \mathfrak{c}$ and every set $Y \subseteq I$ of cardinality $|Y| = |X|$ is $\mathrm{w}(I\mathrm{QN}, J)$. Replacing I by any such Y in the proof of $(\bar{6}) \Rightarrow (1)$ of Theorem 4.3 we find φ such that $Y \subseteq \varphi^\to(J)$. This proves that $|X| < \mathfrak{k}_{I,J}$. $\square$

Corollary 4.6. *For any ideals I and J on ω,*

$$\begin{aligned}\mathrm{non}(\mathrm{w}(I,J)\text{-}space) &= \mathrm{non}(\mathrm{w}(I\mathrm{QN},J)\text{-}space) = \mathrm{non}(\mathrm{w}(I\mathrm{QN},J\mathrm{QN})\text{-}space)\\ &= \mathrm{non}((I\mathrm{QN},\leq_{\mathrm{K}} J\mathrm{QN})\text{-}space) = \mathrm{non}((\leq_{\mathrm{K}} I\mathrm{QN},\leq_{\mathrm{K}} J\mathrm{QN})\text{-}space) = \mathfrak{k}_{I,J}. \quad \square\end{aligned}$$

Proof. Theorem 4.3 and Theorem 4.5 (recall that $\mathfrak{k}_{I,J} < \infty$ if and only if $I \not\leq_{\mathrm{K}} J$). $\square$

Theorem 4.7. *Let $I \subseteq J$ be ideals on ω. Then*

$$\mathrm{add}((J, J\mathrm{QN})\text{-}space) = \mathrm{non}((J, J\mathrm{QN})\text{-}space) \leq \mathrm{add}((I, J\mathrm{QN})\text{-}space).$$

Proof. Assume that $\kappa < \mathrm{non}(J, J\mathrm{QN})$, $X = \bigcup_{\xi<\kappa} X_\xi$, $f \in {}^\omega C(X)$, and sequences $\varepsilon^\xi \in {}^\omega[0,\infty)$ that J-converge to 0 control the $J\mathrm{QN}$-convergence of f on X_ξ. Define $g_n : \kappa \to X$ by $g_n(\xi) = \varepsilon^\xi_n$. Since $g \xrightarrow{J} 0$ on κ, it follows that $g \xrightarrow{J\mathrm{QN}} 0$ on κ and some

$\delta \in {}^{\omega}[0,\infty)$ that J-converges to 0 controls this convergence. Every $x \in X$ belongs to some X_ξ and for any such x and ξ we have

$$\{n \in \omega : |f_n(x)| \geq \delta_n\} \subseteq \{n \in \omega : |f_n(x)| \geq \varepsilon_n^\xi\} \cup \{n \in \omega : g_n(\xi) \geq \delta_n\} \in J.$$

Therefore non($(J, J\mathrm{QN})$-space) $\leq$ add($(I, J\mathrm{QN})$-space). For $I = J$ we get the equality because add($(I, J\mathrm{QN})$-space) $\leq$ non($(I, J\mathrm{QN})$-space). $\square$

5. Reductions of the properties with respect to $\leq_{\mathrm{K}}$

In this section we present some results on the influence of the Katětov partial ordering of ideals on the investigated properties of spaces. Most of these results can be expressed also via morphisms between binary relations investigated in next section.

Lemma 5.1. *Let I and J be ideals on ω and $\varphi \in {}^{\omega}\omega$.*

1. *If $\varphi \in F(I)$ and $\mathrm{rng}(\varphi) \in J^*$, then $(I, \varphi^{\leftarrow}(J)\mathrm{QN}) \Rightarrow (\varphi^{\rightarrow}(I), J\mathrm{QN})$.*
2. *If $\varphi \in F(\mathrm{Fin})$, then $(I, \varphi^{\rightarrow}(J)\mathrm{QN}) \Rightarrow (\varphi^{\leftarrow}(I), J\mathrm{QN})$.*
3. *If φ is injective, then $(\varphi^{\rightarrow}(I), \varphi^{\rightarrow}(J)\mathrm{QN}) \Rightarrow (I, J\mathrm{QN})$.*
4. *If $\mathrm{rng}(\varphi) \in I^*$, then $(\varphi^{\leftarrow}(I), \varphi^{\leftarrow}(J)\mathrm{QN}) \Rightarrow (I, J\mathrm{QN})$.*
5. *If $\varphi \in F(\mathrm{Fin})$ and $\mathrm{rng}(\varphi) \in I^*$, then $(\varphi^{\leftarrow}(I), \varphi^{\leftarrow}(J)\mathrm{QN}) \Leftrightarrow (I, J\mathrm{QN})$.*

Proof. (1) Let X be an $(I, \varphi^{\leftarrow}(J)\mathrm{QN})$-space. Let $f \in {}^{\omega}C(X)$ be arbitrary such that $f \xrightarrow{\varphi^{\rightarrow}(I)} 0$. Then by Lemma 2.2 (a), $\varphi \circ f \xrightarrow{I} 0$, and then $\varphi \circ f \xrightarrow{\varphi^{\leftarrow}(J)\mathrm{QN}} 0$. Since $\mathrm{rng}(\varphi) = \omega$, by Lemma 2.3 (d), $f \xrightarrow{J\mathrm{QN}} 0$.

(2) Let $X \neq \emptyset$ be an $(I, \varphi^{\rightarrow}(J)\mathrm{QN})$-space. Then $\mathrm{rng}(\varphi) \in I^+$ because, by Lemma 3.5, $I \subseteq \varphi^{\rightarrow}(J)$. Let $f \in {}^{\omega}C(X)$ be arbitrary such that $f \xrightarrow{\varphi^{\leftarrow}(I)} 0$. Then $\varphi * f \in {}^{\omega}C(X)$ because φ is finite-to-one. By Lemma 2.3 (a), $\varphi * f \xrightarrow{I} 0$, therefore $\varphi * f \xrightarrow{\varphi^{\rightarrow}(J)\mathrm{QN}} 0$, and then, by Lemma 2.2 (d), $f \xrightarrow{J\mathrm{QN}} 0$.

(3) This is a consequence of (2) because $\varphi^{\leftarrow}(\varphi^{\rightarrow}(I)) = I$, if φ is injective.

In (4) and (5) we can assume by Lemma 3.5 that $I \subseteq J$ and hence $I^* \subseteq J^*$.

(4) Let X be an $(\varphi^{\leftarrow}(I), \varphi^{\leftarrow}(J)\mathrm{QN})$-space. We apply Lemma 2.3 (c) and (d). If $f \xrightarrow{I} 0$, then by (c), $\varphi \circ f \xrightarrow{\varphi^{\leftarrow}(I)} 0$, then $\varphi \circ f \xrightarrow{\varphi^{\leftarrow}(J)\mathrm{QN}} 0$, and by (d), $f \xrightarrow{J\mathrm{QN}} 0$.

(5) The implication $\Rightarrow$ holds by (4) and the implication $\Leftarrow$ follows by (2) because $\varphi^{\rightarrow}(\varphi^{\leftarrow}(J)) = J$ due to $\mathrm{rng}(\varphi) \in J^*$. $\square$

Lemma 5.2. *Let I and J be ideals on ω.*

1. *$(I, J\mathrm{QN}) \Rightarrow (\forall I' \subseteq I)(\forall J' \supseteq J)\ (I', J'\mathrm{QN})$.*
2. *$(I, J\mathrm{QN}) \Rightarrow (I \vee \langle \omega \setminus b\rangle, J \vee \langle \omega \setminus b\rangle\mathrm{QN}) \Rightarrow (I{\restriction}b, J{\restriction}b\mathrm{QN})$ for $b \in J^+$.*
3. *$(I, J\mathrm{QN}) \Leftrightarrow (I \vee \langle a\rangle, J\mathrm{QN})$ for all $a \in J$.*
4. *$(\mathrm{Fin}, J\mathrm{QN}) \Rightarrow (\forall J' \geq_{\mathrm{KB}} J$ on $\omega)\ (\mathrm{Fin}, J'\mathrm{QN})$.*

Proof. In (2) and (3) we can assume by Lemma 3.5 that $I \subseteq J$.

(2) The first implication is easy to prove and the second one is a consequence of Lemma 5.1 (3) because for $K = I, J$ and for the identity function $\varphi : b \to \omega$ we have $K \vee \langle \omega \setminus b\rangle = \varphi^{\rightarrow}(K{\restriction}b)$.

(3) The implication $\Leftarrow$ holds by (1) and the implication $\Rightarrow$ holds by (2) for $b = \omega \setminus a$.

(4) Let J' be an ideal on ω such that $J \leq_{\mathrm{KB}} J'$, i.e., $J \subseteq \varphi^{\rightarrow}(J')$ for some finite-to-one $\varphi \in {}^{\omega}\omega$. If X be a $(\mathrm{Fin}, J\mathrm{QN})$-space, then X is a $(\mathrm{Fin}, \varphi^{\rightarrow}(J')\mathrm{QN})$-space, and by Lemma 5.1 (2), X is a a $(\mathrm{Fin}, J'\mathrm{QN})$-space because $\varphi^{\leftarrow}(\mathrm{Fin}) = \mathrm{Fin}$. $\square$

The analogues of Lemma 5.2 (4) for the other properties have a bit stronger form.

Lemma 5.3. *Let I and J be ideals on ω.*

(1) $\mathrm{w}(I, J\mathrm{QN}) \Rightarrow (\forall I' \leq_{\mathrm{K}} I)(\forall J' \geq_{\mathrm{K}} J)\ \mathrm{w}(I', J'\mathrm{QN})$.
(2) $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN}) \Rightarrow (\forall I' \leq_{\mathrm{K}} I)(\forall J' \geq_{\mathrm{K}} J)\ \mathrm{w}(I'\mathrm{QN}, J'\mathrm{QN})$.
(3) $\mathrm{w}(I, J) \Rightarrow (\forall I' \leq_{\mathrm{K}} I)(\forall J' \geq_{\mathrm{K}} J)\ \mathrm{w}(I', J')$.
(4) $\mathrm{w}(I\mathrm{QN}, J) \Rightarrow (\forall I' \leq_{\mathrm{K}} I)(\forall J' \geq_{\mathrm{K}} J)\ \mathrm{w}(I'\mathrm{QN}, J')$.

Similar implications hold for $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})$*,* $(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$*,* $(I, \leq_{\mathrm{K}} J\mathrm{QN})$*, and* $(I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$*.*

Proof. (1)–(4) The implications follow by Lemma 3.3. For example, by (c) and (b) of Lemma 3.3 we have $\mathrm{w}(I, J\mathrm{QN}) \Rightarrow (\forall I' \leq_{\mathrm{K}} I)\ \mathrm{w}(I', J\mathrm{QN})$ because $\mathrm{w}(I, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(\leq_{\mathrm{K}} I, J\mathrm{QN}) \Rightarrow \mathrm{w}(I', J\mathrm{QN})$; and $\mathrm{w}(I, J\mathrm{QN}) \Rightarrow (\forall J' \geq_{\mathrm{K}} J)\ \mathrm{w}(I, J'\mathrm{QN})$ because $\mathrm{w}(I, J\mathrm{QN}) \Rightarrow \mathrm{w}(I, \leq_{\mathrm{K}} J'\mathrm{QN}) \Leftrightarrow \mathrm{w}(I, J'\mathrm{QN})$. □

Corollary 5.4. *Let I and J be ideals on ω and let $a \in I^+$ and $b \in J^+$.*

$$\mathrm{w}(I, J\mathrm{QN}) \Rightarrow \mathrm{w}(I, J \vee \langle \omega \setminus b \rangle \mathrm{QN}) \Rightarrow \mathrm{w}(I, J{\restriction}b\mathrm{QN}),$$
$$\mathrm{w}(I{\restriction}a, J\mathrm{QN}) \Rightarrow \mathrm{w}(I \vee \langle \omega \setminus a \rangle, J\mathrm{QN}) \Rightarrow \mathrm{w}(I, J\mathrm{QN}).$$

Similar implications hold for $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN})$*,* $\mathrm{w}(I, J)$*,* $\mathrm{w}(I\mathrm{QN}, J)$*, and also for* $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN})$*,* $(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$*,* $(I, \leq_{\mathrm{K}} J\mathrm{QN})$*, and* $(I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$*.*

Proof. This is a consequence of Lemma 5.3 because $I \leq_{\mathrm{KB}} I \vee \langle \omega \setminus a \rangle \leq_{\mathrm{KB}} I{\restriction}a$. In fact, $I \leq_{\mathrm{KB}} I \vee \langle \omega \setminus a \rangle$ holds due to the inclusion $I \subseteq I \vee \langle \omega \setminus a \rangle$ and $I \vee \langle \omega \setminus a \rangle \leq_{\mathrm{KB}} I{\restriction}a$ is witnessed by the identity function $i : a \to \omega$. □

Statements (a1) and (b1) of the next corollary are proved in [3, Theorem 3.3].

Corollary 5.5. *Let I and J be ideals on ω.*

(a) (1) *If J is not tall, then* $(\mathrm{Fin}, J\mathrm{QN}) \Leftrightarrow \mathrm{QN}$*.*
(2) *If J is not tall, then* $(I, \leq_{\mathrm{K}} J\mathrm{QN}) \Leftrightarrow (I, \leq_{\mathrm{K}} \mathrm{FinQN})$*.*
(3) *If J is not tall, then* $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN}) \Leftrightarrow (\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} \mathrm{FinQN})$*;*
if I is not tall, then $(\leq_{\mathrm{K}} I, \leq_{\mathrm{K}} J\mathrm{QN}) \Leftrightarrow (\leq_{\mathrm{K}} \mathrm{Fin}, \leq_{\mathrm{K}} J\mathrm{QN})$*.*
(4) *If J is not tall, then* $(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN}) \Leftrightarrow (\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} \mathrm{FinQN})$*;*
if I is not tall, then $(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN}) \Leftrightarrow (\leq_{\mathrm{K}} \mathrm{FinQN}, \leq_{\mathrm{K}} J\mathrm{QN})$*.*
(5) *If J is not tall, then* $(I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN}) \Leftrightarrow (I\mathrm{QN}, \leq_{\mathrm{K}} \mathrm{FinQN})$*.*

(b) (1) *If J is not tall, then* $\mathrm{w}(I, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(I, \mathrm{FinQN})$*;*
if I is not tall, then $\mathrm{w}(I, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(\mathrm{Fin}, J\mathrm{QN})$*.*
(2) *If J is not tall, then* $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(I\mathrm{QN}, \mathrm{FinQN})$*;*
if I is not tall, then $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN}) \Leftrightarrow \mathrm{w}(\mathrm{FinQN}, J\mathrm{QN})$*.*
(3) *If J is not tall, then* $\mathrm{w}(I, J) \Leftrightarrow \mathrm{w}(I, \mathrm{Fin})$*;*
if I is not tall, then $\mathrm{w}(I, J) \Leftrightarrow \mathrm{w}(\mathrm{Fin}, J)$*.*
(4) *If J is not tall, then* $\mathrm{w}(I\mathrm{QN}, J) \Leftrightarrow \mathrm{w}(I\mathrm{QN}, \mathrm{Fin})$*;*
If I is not tall, then $\mathrm{w}(I\mathrm{QN}, J) \Leftrightarrow \mathrm{w}(\mathrm{FinQN}, J)$*.*

(c) *If I is not tall, then every space is* $(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$*,* $(I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$*,* $\mathrm{w}(I, J)$*,* $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN})$*, and* $\mathrm{w}(I\mathrm{QN}, J)$*.*

Proof. (a)–(b) We apply Lemma 5.2 (4) and Lemma 5.3. If an ideal K is not tall, then $K =_{\mathrm{KB}} \mathrm{Fin}$ because there is $a \in [\omega]^\omega$ such that $K{\restriction}a = [a]^{<\omega}$ and hence $K \leq_{\mathrm{KB}} K{\restriction}a =_{\mathrm{KB}} \mathrm{Fin} \leq_{\mathrm{KB}} K$.

(c) This is a consequence of Theorem 4.3 and the equivalences proved in (a)–(b) because $\mathrm{Fin} \leq_{\mathrm{K}} J$ for every ideal J. $\square$

By Lemma 4.4 for $\kappa \leq \mathfrak{p}$ all ideals are $\leq^{\kappa}_{\mathrm{K}}$-equivalent and $\leq^{\kappa}_{\mathrm{KB}}$-equivalent. Lemma 5.2 (5) and Lemma 5.3 can be generalized for spaces of small cardinalities as follows:

Lemma 5.6. *Let I and J be ideals on ω and let $\omega_1 \leq \kappa \leq \mathfrak{c}$. For a set X of cardinality $< \kappa$ the following conditions hold:*

1. *If X is $(\mathrm{Fin}, J\mathrm{QN})$, then $(\forall J' \geq^{\kappa}_{\mathrm{KB}} J$ on $\omega)$ X is $(\mathrm{Fin}, J'\mathrm{QN})$.*
2. *If X is $\mathrm{w}(I, J\mathrm{QN})$, then $(\forall I' \leq^{\kappa}_{\mathrm{K}} I)(\forall J' \geq^{\kappa}_{\mathrm{K}} J)$ X is $\mathrm{w}(I', J'\mathrm{QN})$.*
3. *If X is $\mathrm{w}(I\mathrm{QN}, J\mathrm{QN})$, then $(\forall I' \leq^{\kappa}_{\mathrm{K}} I)(\forall J' \geq^{\kappa}_{\mathrm{K}} J)$ X is $\mathrm{w}(I'\mathrm{QN}, J'\mathrm{QN})$.*
4. *If X is $\mathrm{w}(I, J)$, then $(\forall I' \leq^{\kappa}_{\mathrm{K}} I)(\forall J' \geq^{\kappa}_{\mathrm{K}} J)$ X is $\mathrm{w}(I', J')$.*
5. *If X is $\mathrm{w}(I\mathrm{QN}, J)$, then $(\forall I' \leq^{\kappa}_{\mathrm{K}} I)(\forall J' \geq^{\kappa}_{\mathrm{K}} J)$ X is $\mathrm{w}(I'\mathrm{QN}, J')$.*

Proof. Apply Lemma 5.2 and Lemma 5.3 using the following easy observations for sequences of functions $f \in {}^{\omega}({}^{X}\mathbb{R})$.

If $J' \geq^{\kappa}_{\mathrm{K}} J$ and $f \xrightarrow{J\mathrm{QN}} 0$, then there is an ideal $J'' \subseteq J$ generated by $\leq \omega \cdot |X| < \kappa$ sets such that $f \xrightarrow{J''\mathrm{QN}} 0$, and then $J' \geq_{\mathrm{K}} J''$ (the same relation holds between $\leq^{\kappa}_{\mathrm{KB}}$ and $\leq_{\mathrm{KB}}$). This should be applied in the proof of (1)–(3).

If $I' \leq^{\kappa}_{\mathrm{K}} I$ and $f \xrightarrow{I'} 0$, then there is an ideal $I'' \subseteq I'$ generated by $< \kappa$ sets such that $f \xrightarrow{I''} 0$, and then $I'' \leq_{\mathrm{K}} I$. This should be applied in the proof of (2). $\square$

Note that Lemma 5.6 can be extended also for the other four properties from Lemma 3.4. For example, for a set X of cardinality $< \kappa$ the following holds:

- If X is $(\leq_{\mathrm{K}} I\mathrm{QN}, \leq_{\mathrm{K}} J\mathrm{QN})$, then $(\forall I' \leq^{\kappa}_{\mathrm{K}} I)(\forall J' \geq^{\kappa}_{\mathrm{K}} J)$ X is $(\leq_{\mathrm{K}} I'\mathrm{QN}, \leq_{\mathrm{K}} J'\mathrm{QN})$.

6. Unbounding and dominating cardinal numbers

Following [1], a binary relation is a triple (R_-, R_+, R) with $R \subseteq R_- \times R_+$. The dual relation is $(R_+, R_-, R^{\perp})$ where $x\ R^{\perp}\ y$ if and only if $\neg(y\ R\ x)$. A morphism $(\Phi, \Psi) : (R_-, R_+, R) \to (S_-, S_+, S)$ is a pair of functions $\Phi : S_- \to R_-$ and $\Psi : R_+ \to S_+$ such that $\Phi(x)\ R\ y \Rightarrow x\ S\ \Psi(y)$; then $(\Psi, \Phi) : (S_-, S_+, S)^{\perp} \to (R_-, R_+, R)^{\perp}$ is a morphism, too. We denote

$$\mathfrak{b}(R) = \min\{|B| : B \subseteq R_- \text{ and } (\forall x \in R_+))(\exists y \in B)\ \neg(y\ R\ x)\},$$
$$\mathfrak{d}(R) = \min\{|D| : D \subseteq R_+ \text{ and } (\forall x \in R_-))(\exists y \in D)\ x\ R\ y\}$$

provided that these cardinals exist, otherwise the value is ∞; then $\mathfrak{b}(R^{\perp}) = \mathfrak{d}(R)$. We write $R \preccurlyeq S$, if there is a morphism from R to S; then $\mathfrak{b}(R) \leq \mathfrak{b}(S)$ and $\mathfrak{d}(S) \leq \mathfrak{d}(R)$. We write $R \approx S$, if $R \preccurlyeq S$ and $S \preccurlyeq R$. For example, if R is a partial (complete) ordering without maximal elements, then $R \preccurlyeq R^{\perp}$ ($R \approx R^{\perp}$).

Below we express any of the invariants

$$\mathrm{non}((I, J\mathrm{QN})\text{-space}), \quad \mathrm{non}((I, \leq_{\mathrm{K}} J\mathrm{QN})\text{-space}), \quad \mathrm{non}(\mathrm{w}(I, J\mathrm{QN})\text{-space})$$

in the form $\mathfrak{b}(R)$ for a suitable binary relation R.

Recall that for $\alpha, \beta, \varphi \in {}^{\omega}\omega$, $\|\beta \leq \alpha\| = \{k \in \omega : \beta(k) \leq \alpha(k)\}$ and $\varphi \circ \alpha \in {}^{\omega}\omega$ is defined by $(\varphi \circ \alpha)(k) = \alpha(\varphi(k))$ for $k \in \omega$. Then $\|\varphi \circ \alpha \leq \varphi \circ \beta\| = \varphi^{-1}(\|\alpha \leq \beta\|)$.

Let I, J, K be ideals on ω. Denote

$$\begin{aligned} F_2(J) &= \{(\varphi, \alpha) \in {}^\omega\omega \times {}^\omega\omega : \varphi \circ \alpha \in F(J)\} \\ &= \{(\varphi, \alpha) \in F(J) \times {}^\omega\omega : \alpha \in F(\varphi^{\rightarrow}(J))\} \end{aligned}$$

and consider the following binary relations:

$$\begin{array}{ll} \mathrm{Z}^K_{I,J} \subseteq F(I) \times F(K), & \beta\ \mathrm{Z}^K_{I,J}\ \alpha \Leftrightarrow \|\beta < \alpha\| \in J; \\ \mathrm{A}_{I,J} \subseteq F(I) \times F(I), & \beta\ \mathrm{A}_{I,J}\ \alpha \Leftrightarrow \|\beta < \alpha\| \in J; \\ \mathrm{B}_{I,J} \subseteq F(I) \times F(J), & \beta\ \mathrm{B}_{I,J}\ \alpha \Leftrightarrow \|\beta < \alpha\| \in J; \\ \mathrm{C}_{I,J} \subseteq F(I) \times F_2(J), & \beta\ \mathrm{C}_{I,J}\ (\varphi, \alpha) \Leftrightarrow \varphi^{-1}(\|\beta < \alpha\|) \in J; \\ \mathrm{D}_{I,J} \subseteq F(I) \times (F(J) \times F(J)), & \beta\ \mathrm{D}_{I,J}\ (\varphi, \alpha) \Leftrightarrow \|\varphi \circ \beta < \alpha\| \in J; \\ \mathrm{E}_{I,J} \subseteq I \times F(J), & a\ \mathrm{E}_{I,J}\ \varphi \Leftrightarrow \varphi^{-1}(a) \in J. \end{array}$$

Hence, $\mathrm{A}_{I,J} = \mathrm{Z}^I_{I,J}$, $\mathrm{B}_{I,J} = \mathrm{Z}^J_{I,J}$, and $\mathrm{A}_{J,J} = \mathrm{B}_{J,J} = \mathrm{Z}^J_{J,J}$. The inclusions $I \subseteq K$ and $K \subseteq J$ imply that the following pairs of identities are morphisms:

$$\mathrm{A}_{I,J} \xrightarrow{(\mathrm{id}_{F(I)}, \mathrm{id}_{F(I)})} \mathrm{Z}^K_{I,J} \xrightarrow{(\mathrm{id}_{F(I)}, \mathrm{id}_{F(K)})} \mathrm{B}_{I,J}.$$

Denote

$$\begin{array}{ll} \Psi_1 : F(J) \to F_2(J), & \Psi_1(\alpha) = (\mathrm{id}_\omega, \alpha), \\ \Psi_2 : F_2(J) \to F(J) \times F(J), & \Psi_2(\varphi, \alpha) = (\varphi, \varphi \circ \alpha), \\ \multicolumn{2}{l}{\Phi_3 : I \to F(I), \quad \Phi_3(a)(k) = 0, \text{ if } k \in a, \text{ and } \Phi_3(a)(k) = k, \text{ if } k \in \omega \setminus a,} \\ \Psi_3 : F(J) \times F(J) \to F(J), & \Psi_3(\varphi, \alpha) = \varphi. \end{array}$$

Lemma 6.1. *The following pairs of functions are morphisms:*

$$\mathrm{B}_{I,J} \xrightarrow{(\mathrm{id}_{F(I)}, \Psi_1)} \mathrm{C}_{I,J} \xrightarrow{(\mathrm{id}_{F(I)}, \Psi_2)} \mathrm{D}_{I,J} \xrightarrow{(\Phi_3, \Psi_3)} \mathrm{E}_{I,J}.$$

Consequently, $\mathfrak{b}(\mathrm{B}_{I,J}) \le \mathfrak{b}(\mathrm{C}_{I,J}) \le \mathfrak{b}(\mathrm{D}_{I,J}) \le \mathfrak{b}(\mathrm{E}_{I,J})$.

Proof. For example, the implication $\Phi_3(a)\ \mathrm{D}_{I,J}\ (\varphi, \alpha) \Rightarrow a\ \mathrm{E}_{I,J}\ \Psi_3(\varphi, \alpha)$ holds because $\|\varphi \circ \Phi_3(a) < \alpha\| \in J$ implies $\varphi^{-1}(a) \subseteq \|\varphi \circ \Phi_3(a) < \alpha\| \cup \|\alpha = 0\| \in J$. $\square$

Theorem 6.2. *Let I, J, K be ideals on ω.*

(a) *If $K \subseteq J$, then* $\mathfrak{b}(\mathrm{Z}^K_{I,J}) = \mathrm{non}((I, JK\mathrm{QN})$*-space) and* $\mathfrak{b}(\mathrm{Z}^K_{I,J}) < \infty$ *if and only if (K, J) is weak $P(I)$.*

(b) *If $I \subseteq J$, then* $\mathfrak{b}(\mathrm{A}_{I,J}) = \mathfrak{b}(\mathrm{Z}^I_{I,J}) = \mathrm{non}((I, JI\mathrm{QN})$*-space*$) \le \mathfrak{c}$.

(c) $\mathfrak{b}(\mathrm{B}_{I,J}) = \mathfrak{b}(\mathrm{Z}^J_{I,J}) = \mathrm{non}((I, J\mathrm{QN})$*-space) and* $\mathfrak{b}(\mathrm{B}_{I,J}) < \infty$ *if and only if J is a weak $P(I)$-ideal.*

(d) $\mathfrak{b}(\mathrm{C}_{I,J}) = \mathrm{non}((I, \le_{\mathrm{K}} J\mathrm{QN})$*-space) and* $\mathfrak{b}(\mathrm{C}_{I,J}) < \infty$ *if and only if every ideal $\le_{\mathrm{K}}$-below J is a weak $P(I)$-ideal.*

(e) $\mathfrak{b}(\mathrm{D}_{I,J}) = \mathrm{non}(\mathrm{w}(I, J\mathrm{QN})$*-space) and* $\mathfrak{b}(\mathrm{D}_{I,J}) < \infty$ *if and only if J is a $W(I)$-ideal.*

(f) $\mathfrak{b}(\mathrm{E}_{I,J}) = \mathfrak{k}_{I,J}$ *and* $\mathfrak{b}(\mathrm{E}_{I,J}) < \infty$ *if and only if* $I \not\le_{\mathrm{K}} J$.

Proof. (a)–(e) Use Theorem 4.2 with $J' = \varphi^{\rightarrow}(J)$ for the property $(I, \le_{\mathrm{K}} J\mathrm{QN})$. (f) is a consequence of Theorem 4.3 and Theorem 4.5. $\square$

By Lemma 6.1 and Theorem 6.2 it follows that

$$I \not\leq_{\mathrm{K}} J \ \Rightarrow\ J \text{ is } W(I) \ \Rightarrow\ (\forall J' \leq_{\mathrm{K}} J)\ J' \text{ is weak } P(I) \ \Rightarrow\ J \text{ is weak } P(I).$$

Let $\leq_J$ denote the binary relation defined for $\alpha, \beta \in {}^\omega\omega$ by

$$\alpha \leq_J \beta \Leftrightarrow \|\alpha < \beta\| \in J.$$

Lemma 6.3. *For every ideal J on ω the relation $\leq_J$ is a partial quasi-ordering of ${}^\omega\omega$. If J is a maximal ideal, then $\leq_J$ is a quasi-ordering of ${}^\omega\omega$.* $\square$

So, $\mathrm{A}_{J,J} = (F(J), \leq_J)$ and $\mathrm{A}_{I,J} = (F(I), \leq_J)$ are partial quasi-orderings. Let

$$\mathfrak{b}_{I,J} = \mathfrak{b}(\mathrm{A}_{I,J}), \quad \mathfrak{d}_{I,J} = \mathfrak{d}(\mathrm{A}_{I,J}), \quad \mathfrak{b}_J = \mathfrak{b}_{J,J}, \quad \mathfrak{d}_J = \mathfrak{d}_{J,J}.$$

We consider also some restrictions of two partial quasi-orderings of ${}^\omega\mathcal{P}(\omega)$:

$$f \leq^0_J g \Leftrightarrow (\exists x \in J)(\forall n \in \omega)\ f(n) \subseteq g(n) \cup x,$$
$$f \leq^1_J g \Leftrightarrow \{n \in \omega : f(n) \not\subseteq g(n)\} \in J.$$

The relations $\leq^0_J$ and $\leq^1_J$ are natural generalizations of the eventual partial quasi-ordering $\leq^*$ on ${}^\omega\omega$ and $({}^\omega\mathrm{Fin}, \leq^0_{\mathrm{Fin}}) \approx ({}^\omega\mathrm{Fin}, \leq^1_{\mathrm{Fin}}) \approx ({}^\omega\omega, \leq^*)$ because ${}^\omega\omega$ is cofinal in ${}^\omega\mathrm{Fin}$. Note that for $\alpha, \beta \in {}^\omega\omega$, $\alpha \leq_J \beta \Leftrightarrow \beta \leq^1_J \alpha$, i.e., $\leq_J$ is inverse to $\leq^1_J$.

Lemma 6.4. *Let I, J, K and I', J' be ideals on ω.*

(a) $\mathrm{Z}^K_{I,J} \approx ({}^\omega I \times {}^\omega K, \leq^0_J)$, $\mathrm{A}_{I,J} \approx ({}^\omega I \times {}^\omega I, \leq^0_J)$, *and* $\mathrm{B}_{I,J} \approx ({}^\omega I \times {}^\omega J, \leq^0_J)$.

(b) *There are morphisms according to the diagram*

$$\begin{array}{ccccccc}
 & & \mathrm{A}_{I,I} & & & & \\
 & & \uparrow & & & & \\
\mathrm{A}_{I,\mathrm{Fin}} & \to & ({}^\omega I, \leq^1_{\mathrm{Fin}}) & \to & ({}^\omega\omega, \leq^*) & \to & ({}^\omega\omega, \leq^0_J) \approx \mathrm{A}_{\mathrm{Fin},J} \\
 & & \downarrow & & \downarrow & & \\
 & & ({}^\omega I, \leq^1_J) & \to & ({}^\omega\omega, \leq^1_J) & &
\end{array}$$

(c) $\mathfrak{b}_{\mathrm{Fin}} = \mathfrak{b}$ *and* $\mathfrak{d}_{\mathrm{Fin}} = \mathfrak{d}$.

(d) *If $I' \subseteq I \subseteq J \subseteq J'$, then $\mathrm{A}_{I,J} \preccurlyeq \mathrm{A}_{I,J'}$, $\mathrm{A}_{I,I} \preccurlyeq \mathrm{A}_{I,J} \preccurlyeq \mathrm{B}_{I,J} \preccurlyeq \mathrm{B}_{I,J'}$, $\mathrm{A}_{J,J} \preccurlyeq \mathrm{B}_{I,J} \preccurlyeq \mathrm{B}_{I',J}$.*

(e) *If $J \supseteq I$ is a $P(I)$-ideal, then for all ideals K, L with $I \subseteq K \subseteq L \subseteq J$ and for all $J' \supseteq J$, $\mathrm{A}_{K,J'} \preccurlyeq \mathrm{A}_{L,J'} \preccurlyeq \mathrm{B}_{K,J'} \approx \mathrm{B}_{L,J'}$ and $\mathrm{A}_{K,J} \approx \mathrm{B}_{K,J}$. Consequently, $\mathrm{A}_{I,I} \preccurlyeq \mathrm{A}_{I,J} \approx \mathrm{A}_{K,J} \approx \mathrm{A}_{J,J} \approx \mathrm{B}_{K,J}$.*

(f) *If J is a P-ideal, then $\mathfrak{b} \leq \mathfrak{b}_J$ and $\mathfrak{b}_I \leq \mathfrak{b}_{I,J} = \mathfrak{b}_J$ for all ideals $I \subseteq J$ on ω. Dually, $\mathfrak{d} \geq \mathfrak{d}_J$ and $\mathfrak{d}_I \geq \mathfrak{d}_{I,J} = \mathfrak{d}_J$ for all ideals $I \subseteq J$.*

Proof. (a) Define $\Phi_I : F(I) \to {}^\omega I$ and $\Psi_I : {}^\omega I \to F(I)$ by $\Phi_I(\alpha)(n) = \|\alpha \leq n\|$ and $\Psi_I(f)(n) = \min\{m \in \omega : n \in m \cup f(m)\}$ for $\alpha \in F(I)$, $f \in {}^\omega I$, and $n \in \omega$. We show that the following pairs of functions are morphisms:

$$(\Phi_I, \Psi_K) : ({}^\omega I \times {}^\omega K, \leq^0_J) \to (F(I) \times F(K), \leq_J),$$
$$(\Psi_I, \Phi_K) : (F(I) \times F(K), \leq_J) \to ({}^\omega I \times {}^\omega K, \leq^0_J).$$

Assume that $\alpha \in F(I)$, $f \in {}^\omega K$, and $\Phi_I(\alpha) \leq^0_J f$. There is an $x \in J$ such that $\|\alpha = n\| \subseteq f(n) \cup x$ for all $n \in \omega$ and then, $\alpha(k) = n$ and $k \notin x$ implies $\Psi_K(f)(k) \leq n = \alpha(k)$. Therefore $\|\alpha < \Psi_K(f)\| \subseteq x \in J$, i.e., $\alpha \leq_J \Psi_K(f)$.

Assume that $\alpha \in F(K)$, $f \in {}^\omega I$, and $\Psi_I(f) \leq_J \alpha$, i.e., $\|\Psi_I(f) < \alpha\| \in J$. Then for every $n \in \omega$, $f(n) \subseteq \|\Psi_I(f) \leq n\| \subseteq \|\alpha \leq n\| \cup \|\Psi_I(f) < \alpha\| = \Phi_K(\alpha)(n) \cup \|\Psi_I(f) < \alpha\|$. Therefore $f \leq^0_J \Phi_K(\alpha)$.

(b) We apply $\mathrm{A}_{I,K} \approx ({}^\omega I, \leq^0_K)$ for $K = I$, Fin. The morphisms $({}^\omega I, \leq^1_{\mathrm{Fin}}) \to ({}^\omega I, \leq^0_I)$, $({}^\omega I, \leq^1_{\mathrm{Fin}}) \to ({}^\omega I, \leq^1_J)$, $({}^\omega \omega, \leq^*) \to ({}^\omega \omega, \leq^0_J)$, $({}^\omega \omega, \leq^*) \to ({}^\omega \omega, \leq^1_J)$ are created by pairs of indentities. Define $\Psi_1 : {}^\omega I \to {}^\omega I$ and $\Psi_2 : {}^\omega I \to {}^\omega \omega$ by $\Psi_1(f)(n) = f(n) \cup n$ and $\Psi_2(f)(k) = \max\{m \in \omega : m \subseteq f(k)\}$ for $f \in {}^\omega I$. Then $(\mathrm{id}_{({}^\omega I)}, \Psi_1) : ({}^\omega I, \leq^0_{\mathrm{Fin}}) \to ({}^\omega I, \leq^1_{\mathrm{Fin}})$ and $(\mathrm{id}_{({}^\omega \omega)}, \Psi_2) : ({}^\omega I, \leq^1_J) \to ({}^\omega \omega, \leq^1_J)$ are morphisms. Taking $J = \mathrm{Fin}$ we get also a morphism $({}^\omega I, \leq^1_{\mathrm{Fin}}) \to ({}^\omega \omega, \leq^*)$.

(c) Because by (a), $\mathrm{A}_{\mathrm{Fin},\mathrm{Fin}} \approx ({}^\omega \mathrm{Fin}, \leq^0_{\mathrm{Fin}}) \approx ({}^\omega \omega, \leq^*)$.

(d) All these morphism consists of identity functions.

(e) By (d), $\mathrm{A}_{L,J'} \preccurlyeq \mathrm{B}_{L,J'} \preccurlyeq \mathrm{B}_{K,J'}$ and $\mathrm{A}_{K,J} \preccurlyeq \mathrm{B}_{K,J}$. To get $\mathrm{A}_{K,J'} \preccurlyeq \mathrm{A}_{L,J'}$, $\mathrm{B}_{K,J'} \preccurlyeq \mathrm{B}_{L,J'}$, and $\mathrm{B}_{K,J} \preccurlyeq \mathrm{A}_{K,J}$ we find morphisms of the form

$$\begin{aligned}
(\Phi, \mathrm{id}) &: ({}^\omega K \times {}^\omega K, \leq^0_{J'}) \to ({}^\omega L \times {}^\omega L, \leq^0_{J'}),\\
(\Phi, \mathrm{id}) &: ({}^\omega K \times {}^\omega J', \leq^0_{J'}) \to ({}^\omega L \times {}^\omega J', \leq^0_{J'}),\\
(\mathrm{id}, \Phi) &: ({}^\omega K \times {}^\omega J, \leq^0_{J}) \to ({}^\omega K \times {}^\omega K, \leq^0_{J}),
\end{aligned}$$

where each Φ means a restriction of the same function $\Phi : {}^\omega J \to {}^\omega I$ (note that ${}^\omega L \subseteq {}^\omega J$ and ${}^\omega I \subseteq {}^\omega K$). Since J is a $P(I)$-ideal we can define $\Phi : {}^\omega J \to {}^\omega I$ as follows: For every $h \in {}^\omega J$ fix a set $x_h \in J$ such that $h(n) \setminus x_h \in I$ for all $n \in \omega$ and let $\Phi(h)(n) = h(n) \setminus x_h$.

Assume that $\Phi(g) \leq^0_{J'} f$ for some $g \in {}^\omega L$ and $f \in {}^\omega K$ or $f \in {}^\omega J'$ and let $x \in J'$ be such that $\Phi(g)(n) \subseteq f(n) \cup x$ for all $n \in \omega$. Then $g(n) \subseteq f(n) \cup (x \cup x_g)$ for all $n \in \omega$ and $x \cup x_g \in J'$. Therefore $g \leq^0_{J'} f$.

Assume that $g \leq^0_J f$ for some $g \in {}^\omega K$ and $f \in {}^\omega J$ and let $x \in J$ be such that $g(n) \subseteq f(n) \cup x$ for all $n \in \omega$. Then $g(n) \subseteq \Phi(f)(n) \cup (x_f \cup x)$ for all $n \in \omega$ and $x_f \cup x \in J$. Therefore $g \leq^0_J \Phi(f)$.

(f) Since J is a $P(I)$-ideal for every ideal $I \subseteq J$, then by (e), $\mathfrak{b}_I \leq \mathfrak{b}_{I,J} = \mathfrak{b}_J$. By (c), $\mathfrak{b}_{\mathrm{Fin}} = \mathfrak{b}$. The dual arguments are same. $\square$

If $I \subseteq J$, then $({}^\omega I, \leq^1_{\mathrm{Fin}}) \preccurlyeq \mathrm{A}_{I,I} \preccurlyeq \mathrm{A}_{I,J} \preccurlyeq {\mathrm{A}_{I,J}}^\perp \preccurlyeq {\mathrm{A}_{I,I}}^\perp \preccurlyeq ({}^\omega I, \leq^1_{\mathrm{Fin}})^\perp$ by Lemma 6.4 and $\mathrm{A}_{I,J} \preccurlyeq \mathrm{B}_{I,J} \preccurlyeq \mathrm{C}_{I,J} \preccurlyeq \mathrm{D}_{I,J}$.

Lemma 6.5. *Let I and J be ideals on ω.*

(a) $\mathrm{B}_{I,J} \preccurlyeq \mathrm{A}^\perp_{I,J}$, *if J is a weak $P(I)$-ideal.*
(b) $\mathrm{C}_{I,J} \preccurlyeq ({}^\omega I, \leq^1_{\mathrm{Fin}})^\perp$, *if every ideal $\leq_{\mathrm{K}}$-below J is a weak $P(I)$-ideal.*
(c) $\mathrm{D}_{I,J} \preccurlyeq ({}^\omega I, \leq^1_{\mathrm{Fin}})^\perp$, *if J is a $W(I)$-ideal.*

Proof. (a) If J is a weak $P(I)$-ideal, then by Theorem 4.1 for every $\alpha \in F(J)$ let $\Psi(\alpha) \in F(I)$ be such that $\Psi(\alpha) \not\leq_J \alpha$. We claim that $(\mathrm{id}_{F(I)}, \Psi) : \mathrm{B}_{I,J} \to \mathrm{A}^\perp_{I,J}$ is a morphism, i.e., $\beta \; \mathrm{B}_{I,J} \; \alpha \Rightarrow \beta \; \mathrm{A}^\perp_{I,J} \; \Psi_0(\alpha)$ for $\beta \in F(I)$ and $\alpha \in F(J)$.

Assume that $\beta \; \mathrm{B}_{I,J} \; \alpha$, i.e., $\beta \leq_J \alpha$. Then $\Psi_0(\alpha) \not\leq_J \beta$ by transitivity of $\leq_J$, i.e., $\beta \; \mathrm{A}^\perp_{I,J} \; \Psi_0(\alpha)$.

(b)–(c) Under the hypotheses we define $\Phi : {}^\omega I \to F(I)$, $\Psi_1 : F_2(J) \to {}^\omega I$, and $\Psi_2 : F(J) \times F(J) \to {}^\omega I$ such that

$$\begin{aligned}
&\Phi(f) \; \mathrm{C}_{I,J} \; (\varphi, \alpha) \Rightarrow f \not\geq^1_{\mathrm{Fin}} \Psi_1(\varphi, \alpha), && f \in {}^\omega I,\ (\varphi, \alpha) \in F_2(J),\\
&\Phi(f) \; \mathrm{D}_{I,J} \; (\varphi, \alpha) \Rightarrow f \not\geq^1_{\mathrm{Fin}} \Psi_2(\varphi, \alpha), && f \in {}^\omega I,\ (\varphi, \alpha) \in F(J) \times F(J).
\end{aligned}$$

Let $\Phi(f)(k) = \min\{m \in \omega : k \in m \cup f(m)\}$ for $f \in {}^\omega I$ and $n \in \omega$. Then $\|\Phi(f) \leq n\| = n \cup \bigcup_{m \leq n} f(m) \in I$.

If every ideal $\leq_{\mathrm{K}}$-belov J is a $P(I)$-ideal, then for every $(\varphi, \alpha) \in F_2(J)$ there is a set $c_{\varphi,\alpha} \in \varphi^{\to}(J)^+$ such that $\|\alpha = n\| \cap c_{\varphi,\alpha} \in I$ for all $n \in \omega$. Let $\Psi_1(\varphi, \alpha)(n) = \|\alpha \leq n+1\| \cap c_{\varphi,\alpha}$ for all $n \in \omega$.

Assume that $\Psi_1(\varphi, \alpha) \leq^1_{\mathrm{Fin}} f$. There is $n_0 \in \omega$ such that for every $n \geq n_0$, $\|\alpha = n+1\| \cap c_{\varphi,\alpha} \subseteq f(n) \subseteq \|\Phi(f) \leq n\|$ and hence $\|\alpha = n+1\| \cap c_{\varphi,\alpha} \subseteq \|\Phi(f) < \alpha\|$. Then $\|\Phi(f) < \alpha\| \supseteq c_{\varphi,\alpha} \setminus \|\alpha \leq n_0\| \in \varphi^{\to}(J)^+$ because $\|\alpha \leq n_0\| \in \varphi^{\to}(J)$. Therefore $\neg(\Phi(f) \; \mathrm{C}_{I,J} \; (\varphi, \alpha))$.

If J is a $W(I)$-ideal, then for every $\varphi, \alpha \in F(J)$ there is a set $d_{\varphi,\alpha} \in J^+$ such that $\varphi(\|\alpha = n\| \cap d_{\varphi,\alpha}) \in I$ for all $n \in \omega$. Let $\Psi_2(\varphi, \alpha)(n) = \varphi(\|\alpha \leq n+1\| \cap d_{\varphi,\alpha})$ for all $n \in \omega$.

Assume that $\Psi_2(\varphi, \alpha) \leq^1_{\mathrm{Fin}} f$. Let $n_0 \in \omega$ be such that for every $n \geq n_0$, $\varphi(\|\alpha = n+1\| \cap d_{\varphi,\alpha}) \subseteq f(n) \subseteq \|\Phi(f) \leq n\|$ and hence $\|\alpha = n+1\| \cap d_{\varphi,\alpha} \subseteq \{k \in \omega : \Phi(f)(\varphi(k)) < \alpha(k)\} = \|\varphi \circ \Phi(f) < \alpha\|$. Then $\neg(\Phi(f) \; \mathrm{D}_{I,J} \; (\varphi, \alpha))$ because $\|\varphi \circ \Phi(f) < \alpha\| \supseteq d_{\varphi,\alpha} \setminus \|\alpha \leq n_0\| \in J^+$. □

Recall that $\mu : \mathcal{P}(\omega) \to [0, \infty]$ is a submeasure on ω if $\mu(\emptyset) = 0$, $\mu(\{n\}) < \infty$ for all $n \in \omega$, and $\mu(a) \leq \mu(a \cup b) \leq \mu(a) + \mu(b)$ for all $a, b \subseteq \omega$. Denote $\mathrm{Fin}(\mu) = \{a \subseteq \omega : \mu(a) < \infty\}$ and $\mathrm{Exh}(\mu) = \{a \subseteq \omega : \lim_{n \in \omega} \mu(a \setminus n) = 0\}$. We say that μ is unbounded on $I \subseteq \mathrm{Fin}(\mu)$, if for every $n \in \omega$ there is $a \in I$ such that $\mu(a) \geq n$; μ is unbounded, if it is unbounded on $\mathrm{Fin}(\mu)$; μ is lower semi-continuous, if $\mu(a) = \lim_{n \in \omega} \mu(a \cap n)$ for all $a \subseteq \omega$.

Lemma 6.6. *For ideals I, J, I_n, J_n the following holds:*

(a) *If there is a submeasure μ on ω such that $I \subseteq J \subseteq \mathrm{Fin}(\mu)$ and μ is unbounded on I, then $\mathrm{A}_{I,J} \preccurlyeq ({}^\omega\omega, \leq^*)$.*

(b) *If $J = \bigcup_{n \in \omega} J_n$ with ideals $J_n \subsetneq J_{n+1}$, then $\mathrm{A}_{J,J} \preccurlyeq ({}^\omega\omega, \leq^*)$.*

(c) *If $I = \sum^J_{n \in \omega} I_n$, then $\mathrm{A}_{I,I} \preccurlyeq \mathrm{A}_{J,J}$.*

(d) *If $I_1 \vee I_2$ and $J_1 \vee J_2$ are proper ideals, then $\mathrm{A}_{I_1,J_1} \times \mathrm{A}_{I_2,J_2} \preccurlyeq \mathrm{A}_{I_1 \vee I_2, J_1 \vee J_2}$, $\min\{\mathfrak{b}_{I_1,J_1}, \mathfrak{b}_{I_2,J_2}\} \leq \mathfrak{b}_{I_1 \vee I_2, J_1 \vee J_2}$, and $\mathfrak{d}_{I_1 \vee I_2, J_1 \vee J_2} \leq \max\{\mathfrak{d}_{I_1,J_1}, \mathfrak{d}_{I_2,J_2}\}$.*

Proof. (a) For every $n \in \omega$ fix $a_n \in I$ such that $n \leq \mu(a_n) < \infty$. Define $\Phi : {}^\omega\omega \to {}^\omega I$ and $\Psi : {}^\omega I \to {}^\omega\omega$ as follows: For $f \in {}^\omega\omega$, $g \in {}^\omega I$, and $n \in \omega$ let $\Phi(f)(n) = a_{f(n)+n}$ and $\Psi(g)(n) = \lceil \mu(g(n)) \rceil$. Let $\Phi(f) \leq^0_J g$ and let $x \in J$ be such that for every $n \in \omega$, $\Phi(f)(n) \subseteq g(n) \cup x$ and hence $f(n) + n \leq \mu(g(n) \cup x) \leq \Psi(g(n)) + \mu(x)$. Then $f \leq^* \Psi(g)$ because $f(n) \leq \Psi(g(n))$ for all $n \geq \mu(x)$.

(b) This is a consequence of (a) because the submeasure μ on ω defined by $\mu(a) = \inf\{n \in \omega : a \in J_n\}$ for $a \subseteq \omega$ is unbounded and $J = \mathrm{Fin}(\mu)$.

(c) Let $I = \sum^J_{n \in \omega} I_n = \{a \subseteq \omega \times \omega : \{n \in \omega : a_{(n)} \notin I_n\} \in J\}$ where $a_{(n)} = \{k \in \omega : (n, k) \in a\}$ for $a \subseteq \omega \times \omega$. Define $\Phi : {}^\omega J \to {}^\omega I$ and $\Psi : {}^\omega I \to {}^\omega J$ by $\Phi(f)(k) = f(k) \times \omega$ and $\Psi(g)(k) = \{n \in \omega : g(k)_{(n)} \notin I_n\}$. Let $\Phi(f) \leq^0_I g$ and $x \in I$ be such that $f(k) \times \omega \subseteq g(k) \cup x$ for all $k \in \omega$. Denote $y = \{n \in \omega : x_{(n)} \notin I_n\}$. Then $y \in J$ and for every $k \in \omega$, $f(k) \subseteq \{n \in \omega : (g(k) \cup x)_{(n)} \notin I_n\} = \Psi(g)(k) \cup y$. Therefore $f \leq^0_J \Psi(g)$.

(d) Define $\Phi = (\Phi_1, \Phi_2) : {}^\omega(I_1 \vee I_2) \to {}^\omega I_1 \times {}^\omega I_2$ and $\Psi : {}^\omega I_1 \times {}^\omega I_2 \to {}^\omega(I_1 \vee I_2)$ so that for $f \in {}^\omega(I_1 \vee I_2)$ and $(h, g) \in {}^\omega I_1 \times {}^\omega I_2$, $\Phi_1(f)(n) \cup \Phi_2(f)(n) \supseteq f(n)$ and $\Psi(h, g)(n) = h(n) \cup g(n)$ for all $n \in \omega$. One can verify that, if $\Phi_1(f) \leq^0_{J_1} h$ and $\Phi_2(f) \leq^0_{J_2} g$, then $f \leq^0_{J_1 \vee J_2} \Psi(h, g)$. □

Mazur [10] proved that an ideal J on ω is an F_σ ideal if and only if $J = \mathrm{Fin}(\mu)$ for some lower semi-countinous submeasure μ on ω such that $\mu(\omega) = \infty$. Solecki [11] proved that an ideal J on ω is an analytic P-ideal if and only if $J = \mathrm{Exh}(\mu)$ for a bounded lower semi-countinous submeasure μ on ω. Kwela [9, Theorem 5] proved a result paraphrasing $\mathrm{D}_{\mathrm{Fin},J} \preccurlyeq ({}^\omega\omega, \leq^*)$ for all subideals J of F_σ-ideals (see also weaker Lemma 6.6 (a)). Obviously, $\mathrm{Exh}(\mu) \subseteq \mathrm{Fin}(\mu)$ but $\mathrm{Fin}(\mu)$ is a proper F_σ ideal only if $\mu(\omega) = \infty$ and it remained open whether this result includes also all analytic P-ideals. This leads to the following notion of capacity on ω familiar to the notion of capacity in topological spaces in sense of [2].

A capacity on ω is a function $\nu : \mathcal{P}(\omega) \to [0, \infty]$ with the following properties:

(i) $\nu(\emptyset) = 0$ and $a \subseteq b \subseteq \omega$ implies $\nu(a) \leq \nu(b)$.
(ii) $\nu(n) < \infty$ and $\nu(\omega \setminus n) = \infty$ for every $n \in \omega$.
(iii) $\lim_{n\in\omega} \nu(a \cap n) = \nu(a)$ for every $a \subseteq \omega$.

We say that an ideal J on ω is capacitous, if there is a capacity ν on ω such that $J \subseteq \mathrm{Fin}^*(\nu) = \{a \subseteq \omega : \nu(a) < \infty \text{ and } \nu(\omega \setminus a) = \infty\}$. The ceiling of a capacity is a capacity and thus it is enough to consider capacities with values in $\omega \cup \{\infty\}$. By next lemma subideals of F_σ ideals and of analytic P-ideals are capacitous and therefore Theorem 6.8 below generalizes Kwela's result (with the same proof).

Lemma 6.7. *Let μ be a lower semi-continuous submeasure on ω.*

(1) *If $\omega \notin \mathrm{Fin}(\mu)$, then $\mathrm{Fin}(\mu)$ is a capacitous ideal on ω.*
(2) *If $\omega \notin \mathrm{Exh}(\mu)$, then $\mathrm{Exh}(\mu)$ is a capacitous ideal on ω.*
(3) *If $J \leq_{\mathrm{K}} I$ are ideals on ω and I is capacitous, then also J is capacitous.*
(4) *The direct sum of a pair of capacitous ideals is a capacitous ideal.*

Proof. (1) If $\mu(\omega) = \infty$, then μ is a capacity and $\mathrm{Fin}(\mu) = \mathrm{Fin}^*(\mu)$.

(2) By (1) we can assume that $\mu(\omega) < \infty$ because $\mathrm{Exh}(\mu) \subseteq \mathrm{Fin}(\mu)$. Let $\varepsilon = \inf_{n\in\omega} \mu(\omega \setminus n)$; $\varepsilon > 0$ because $\omega \notin \mathrm{Exh}(\mu)$. Then $\mathrm{Exh}(\mu) \subseteq \mathrm{Fin}^*(\nu)$ where ν is the capacity defined by $\nu(a) = \min\{n \leq \omega : \mu(a \setminus n) \leq \varepsilon/2\}$. To see (iii) note that if $\nu(a) \geq k+1$, then $\mu(a \setminus k) > \varepsilon/2$ and there is $n \in \omega$ such that $\mu((a \setminus k) \cap n) > \varepsilon/2$, and then $\nu(a \cap n) \geq k + 1$.

(3) Let $\varphi \in {}^\omega\omega$ be such that $J \subseteq \varphi^{\rightarrow}(I)$ and assume that I is capacitous by a capacity ν. Let $\nu^\varphi : \mathcal{P}(\omega) \to [0, \infty]$ be the capacity on ω defined by $\nu^\varphi(a) = \nu(\varphi^{-1}(a))$ for $a \in \mathcal{P}(\omega)$. Then $J \subseteq \varphi^{\rightarrow}(\mathrm{Fin}^*(\nu)) = \mathrm{Fin}^*(\nu^\varphi)$.

(4) If ν_0 and ν_1 are capacities on disjoint sets a_0 and a_1, then the function ν defined on $\mathcal{P}(a_0 \cup a_1)$ by $\nu(x) = \min\{\nu_0(x \cap a_0), \nu_1(x \cap a_1)\}$ is a capacity on $a_0 \cup a_1$ and $\mathrm{Fin}^*(\nu)$ is the direct sum of $\mathrm{Fin}^*(\nu_0)$ and $\mathrm{Fin}^*(\nu_1)$. ☐

Theorem 6.8 (Kwela). $\mathrm{A}_{\mathrm{Fin},J} \approx \mathrm{B}_{\mathrm{Fin},J} \approx \mathrm{C}_{\mathrm{Fin},J} \approx \mathrm{D}_{\mathrm{Fin},J} \approx ({}^\omega\omega, \leq^*)$ *for every capacitous ideal J on ω.*

Proof. By Lemma 6.4 and Lemma 6.1, $({}^\omega\omega, \leq^*) \approx \mathrm{A}_{\mathrm{Fin},\mathrm{Fin}} \preccurlyeq \mathrm{A}_{\mathrm{Fin},J} \preccurlyeq \mathrm{B}_{\mathrm{Fin},J} \preccurlyeq \mathrm{C}_{\mathrm{Fin},J} \preccurlyeq \mathrm{D}_{\mathrm{Fin},J}$. We prove $\mathrm{D}_{\mathrm{Fin},J} \preccurlyeq ({}^\omega\omega, \leq^*)$. Let ν be a capacity on ω such that $J \subseteq \mathrm{Fin}^*(\nu)$. We find $\Phi : {}^\omega\omega \to F(\mathrm{Fin})$ and $\Psi : F(J) \times F(J) \to {}^\omega\omega$ such that $\Phi(f)\ \mathrm{D}_{\mathrm{Fin},J}\ (\varphi, \alpha) \Rightarrow f \leq^* \Psi(\varphi, \alpha)$.

Define $\Phi(f)(m) = \min\{n \in \omega : m < f(n) + n\}$ for $f \in {}^\omega\omega$ and $m \in \omega$.

We can define $\Psi(\varphi, \alpha)(n) = \max\{m \in \omega : \nu(\|\varphi < m\| \setminus \|\alpha \leq n\|) \leq n\}$ for $(\varphi, \alpha) \in F(J) \times F(J)$ and $n \in \omega$ because $\lim_{k\in\omega} \nu(k \setminus \|\alpha \leq n\|) = \infty$.

Assume that $\Phi(f)\ \mathrm{D}_{\mathrm{Fin},J}\ (\varphi, \alpha)$. Then $\nu(\|\varphi \circ \Phi(f) < \alpha\|) < \infty$. For every $n \geq \nu(\|\varphi \circ \Phi(f) < \alpha\|)$ then $\nu(\|\varphi < f(n)+n\| \setminus \|\alpha \leq n\|) = \nu(\{k \in \omega : \Phi(f)(\varphi(k)) \leq n < \alpha(k)\}) \leq \nu(\|\varphi \circ \Phi(f) < \alpha\|) \leq n$ and hence $f(n) + n \leq \Psi(\varphi, \alpha)(n)$. Therefore $f \leq^* \Psi(\varphi, \alpha)$. ☐

Corollary 6.9. *Let J be a capacitous ideal on ω.*

(a) *J is a $W(\mathrm{Fin})$-ideal.*
(b) *$\mathfrak{b}_J \le \mathfrak{b}$ and $\mathfrak{b}_{I,J} \le \mathfrak{b}$ for every ideal $I \subseteq J$.*
(c) *If J is a P-ideal, then $\mathfrak{b}_{I,J} = \mathfrak{b}_J = \mathfrak{b}$ for every ideal $I \subseteq J$.*

Proof. (a) $\mathfrak{b}(\mathrm{D}_{I,J}) \le \mathfrak{c}$ by Theorem 6.8 and therefore by Theorem 6.2, J is a $W(I)$-ideal for every ideal I on ω.

(b) $\mathrm{A}_{J,J} = \mathrm{B}_{J,J} \preccurlyeq \mathrm{B}_{I,J}$ and $\mathrm{A}_{I,J} \preccurlyeq \mathrm{B}_{I,J} \preccurlyeq \mathrm{B}_{\mathrm{Fin},J} \approx ({}^\omega\omega, \le^*)$ by Lemma 6.4 and Theorem 6.8.

(c) If J is a P-ideal, then $\mathfrak{b} \le \mathfrak{b}_{I,J} = \mathfrak{b}_J$ by Lemma 6.4 and $\mathfrak{b}_J \le \mathfrak{b}$ by (b). $\square$

Kwela used the following result to show that the existence of an ideal J for which $\mathfrak{c} \ge \mathrm{non}(\mathrm{w}(\mathrm{Fin}, J\mathrm{QN})\text{-space}) > \mathfrak{b}$ is consistent with ZFC. The same arguments prove that $\mathfrak{c} \ge \mathrm{non}((\mathrm{Fin}, \le_{\mathrm{K}} J\mathrm{QN})\text{-space}) > \mathfrak{b}$ for some J is consistent with ZFC. This ideal cannot be capacitous.

Lemma 6.10 ([9, Lemma 1])**.** *For every ideal I on ω there is a weak P-ideal J such that $({}^\omega\omega, \le_I^1) \preccurlyeq \mathrm{C}_{\mathrm{Fin},J}$.*

(Note that if J is a weak P-ideal, then every ideal $J' \le_{\mathrm{K}} J$ is a weak P-ideal and by Theorem 6.2, $\mathfrak{b}(\mathrm{C}_{\mathrm{Fin},J}) \le \mathfrak{c}$.)

Proof. Let $\alpha \in {}^\omega\omega$ be such that $\|\alpha = n\|$ is infinite for every $n \in \omega$ and denote

$$J = \{x \in \alpha^{\leftarrow}(I) : (\forall^\infty n \in \omega)\ \|\alpha = n\| \cap x \in \mathrm{Fin}\}.$$

(i) J is a weak P-ideal. Let $\beta \in F(J)$. Denote $a_n = \{k \in \omega : \|\alpha = n\| \cap \|\beta = k\|$ is infinite$\}$ and by induction define $m_n \in \omega$ and $k_n = \beta(m_n)$ as follows:

$$m_n = \begin{cases} \min(\|\alpha = n\| \setminus \bigcup\{\|\beta = k_i\| : i < n\}), & \text{if } \|\alpha = n\| \nsubseteq \bigcup\{\|\beta = k_i\| : i < n\}, \\ \min(\|\alpha = n\| \cap \bigcup\{\|\beta = k\| : k \in a_n\}), & \text{otherwise.} \end{cases}$$

Denote $x = \{m_n : n \in \omega\}$. Then $x \notin J$ because $\alpha(x) = \omega \notin I$. For every $k \in \omega$, $\|\beta = k\| \cap x \in \mathrm{Fin}$ because $\|\alpha = n\| \cap \|\beta = k\|$ may be infinite only for finitely many n.

(ii) We find a morphism $(\Phi, \Psi) : ({}^\omega\omega, \le_I^1) \to \mathrm{C}_{\mathrm{Fin},J}$. For $\beta \in F(\mathrm{Fin})$ define $\Phi(\beta) \in {}^\omega\omega$ by $\Phi(\beta)(n) = \min\{m \in \omega : \|\beta \le n\| \subseteq m\}$. For $g \in {}^\omega\omega$ find a one-to-one function $\varphi \in {}^\omega\omega$ such that for every $n \in \omega$, φ maps $\|\alpha = n\|$ into $\|\alpha = n\| \setminus g(n)$. Then $\varphi \circ \alpha \in F(J)$ because $\alpha \in F(J)$ and $\|\varphi \circ \alpha = n\| = \varphi^{-1}(\|\alpha = n\|) = \|\alpha = n\|$. Define $\Psi(g) \in F_2(J)$ by $\Psi(g) = (\varphi, \alpha)$.

Let $\beta \in F(\mathrm{Fin})$ and $g \in {}^\omega\omega$ be such that $\Phi(\beta) \le_I^1 g$, i.e., $y = \{n \in \omega : \Phi(\beta)(n) > g(n)\} \in I$. Let $\Psi(g) = (\varphi, \alpha)$. Note that $\varphi^{-1}(\|\beta < \alpha\|) = \|\varphi \circ \beta < \varphi \circ \alpha\| = \|\varphi \circ \beta < \alpha\|$ because $\|\varphi \circ \alpha = n\| = \|\alpha = n\|$. We prove $\beta\ \mathrm{C}_{\mathrm{Fin},J}\ \Psi(g)$, i.e., $\|\varphi \circ \beta < \alpha\| = \varphi^{-1}(\|\beta < \alpha\|) \in J$. For every $n \in \omega \setminus y$, $\|\varphi \circ \beta < \alpha\| \cap \|\alpha = n\| = \emptyset$ because for every $k \in \|\alpha = n\|$, $\varphi(k) \ge g(n) \ge \Phi(\beta)(n)$ and hence, $\beta(\varphi(k)) > n$. Therefore $\alpha(\|\varphi \circ \beta < \alpha\|) \subseteq y \in I$. For every $n \in \omega$, $\|\varphi \circ \beta < \alpha\| \cap \|\alpha = n\| \subseteq \|\varphi \circ \beta < n\| = \varphi^{-1}(\|\beta < n\|) \in \mathrm{Fin}$. $\square$

Let I and J be ideals on ω. If $I \subseteq J$, denote $\nu_{I,J} = \sup\{\mathfrak{b}_{I',J} : I \subseteq I' \subseteq J\}$; then $\mathfrak{b}_I \le \mathfrak{b}_{I,J} \cdot \mathfrak{b}_J \le \nu_{I,J}$. If $I \le_{\mathrm{K}} J$, then let

$$\kappa_{I,J} = \sup\{\mathfrak{b}_{I',J'} : I \subseteq I' \subseteq J' \le_{\mathrm{K}} J\}, \quad \lambda_{I,J} = \sup\{\mathfrak{b}_{I',J'} : I \le_{\mathrm{K}} I' \subseteq J' \le_{\mathrm{K}} J\};$$

then $\mathfrak{b}_I \le \kappa_{I,J} \le \lambda_{I,J}$ and $\mathfrak{b}_I \cdot \mathfrak{b}_J \le \lambda_{I,J}$; and if $I \not\le_{\mathrm{K}} J$, then let

$$\kappa_{I,J} = \min\{\kappa_{I' \cap I,J} : I' \le_{\mathrm{K}} J\}, \qquad \lambda_{I,J} = \min\{\lambda_{I' \cap I,J} : I' \le_{\mathrm{K}} J\};$$

then $\kappa_{I,J} \le \lambda_{I,J}$ and $\mathfrak{b}_J \le \lambda_{I,J}$.

Due to Lemma 6.4 the next result is a generalization of [9, Theorem 4].

Theorem 6.11. *Let I and J be ideals on ω.*

(a) $\mathfrak{b}_I = \mathrm{non}((I, I\mathrm{QN})\text{-}space) \le \nu_{I,J} \le \mathrm{non}((I, J\mathrm{QN})\text{-}space)$*, if $I \subseteq J$.*
(b) $\mathrm{non}((I, J\mathrm{QN})\text{-}space) \le \mathfrak{d}_{I,J}$*, if $I \subseteq J$ and J is a weak $P(I)$-ideal.*
(c) $\mathrm{non}((I, \le_{\mathrm{K}} J\mathrm{QN})\text{-}space) \le \mathfrak{d}({}^{\omega}I, \le^1_{\mathrm{Fin}})$*, if every ideal $\le_{\mathrm{K}}$-below J is a weak $P(I)$-ideal.*
(d) $\mathrm{non}(\mathrm{w}(I, J\mathrm{QN})\text{-}space) \le \mathfrak{d}({}^{\omega}I, \le^1_{\mathrm{Fin}})$*, if J is a $W(I)$-ideal.*
(e) *If $I \le_{\mathrm{K}} J$, then $\mathfrak{b}_I \le \kappa_{I,J} \le \lambda_{I,J}$, $\mathfrak{b}_I \cdot \mathfrak{b}_J \le \lambda_{I,J}$ and*

$$\kappa_{I,J} \le \mathrm{non}((I, \le_{\mathrm{K}} J\mathrm{QN})\text{-}space),$$
$$\lambda_{I,J} \le \mathrm{non}(\mathrm{w}(I, J\mathrm{QN})\text{-}space).$$

(f) *If $I \not\le_{\mathrm{K}} J$, then $\kappa_{I,J} \le \lambda_{I,J}$, $\mathfrak{b}_J \le \lambda_{I,J}$, and*

$$\min\{\mathfrak{k}_{I,J}, \kappa_{I,J}\} \le \mathrm{non}((I, \le_{\mathrm{K}} J\mathrm{QN})\text{-}space) \le \min\{\mathfrak{k}_{I,J}, \mathfrak{d}({}^{\omega}I, \le^1_{\mathrm{Fin}})\},$$
$$\min\{\mathfrak{k}_{I,J}, \lambda_{I,J}\} \le \mathrm{non}(\mathrm{w}(I, J\mathrm{QN})\text{-}space) \le \min\{\mathfrak{k}_{I,J}, \mathfrak{d}({}^{\omega}I, \le^1_{\mathrm{Fin}})\}.$$

Proof. (a) holds by definitions and (b)–(d) hold by Lemma 6.5.

(e) The displayed inequalities are consequences of (a) because if $I \subseteq I' \subseteq J' \le_{\mathrm{K}} J$, then $(I', J'\mathrm{QN}) \Rightarrow (I, \le_{\mathrm{K}} J\mathrm{QN})$, and by Lemma 5.3, if $I \le_{\mathrm{K}} I' \subseteq J' \le_{\mathrm{K}} J$, then $(I', J'\mathrm{QN}) \Rightarrow \mathrm{w}(I, J\mathrm{QN})$.

(f) The inequalities $\mathrm{non}((I, \le_{\mathrm{K}} J\mathrm{QN})\text{-space}) \le \mathrm{non}(\mathrm{w}(I, J\mathrm{QN})\text{-space}) \le \mathfrak{k}_{I,J}$ hold by Lemma 6.1 and Theorem 6.2. Since $\mathfrak{k}_{I,J} \le \mathfrak{c}$, it follows by Theorem 6.2 that we can apply (c) and (d) above and therefore $\mathrm{non}((I, \le_{\mathrm{K}} J\mathrm{QN})\text{-space}) \le \mathrm{non}(\mathrm{w}(I, J\mathrm{QN})\text{-space}) \le \mathfrak{d}({}^{\omega}I, \le^1_{\mathrm{Fin}})$. This proves the upper estimations.

We prove the lower estimations. Let $X \subseteq F(I)$ and let $|X| < \min\{\mathfrak{k}_{I,J}, \kappa_{I,J}\}$. Since $|X| < \mathfrak{k}_{I,J}$, there is $\psi \in F(J)$ such that $\beta^{-1}(\{n\}) \in \psi^{\rightarrow}(J)$ for all $\beta \in X$ and $n \in \omega$. Denote $I' = \psi^{\rightarrow}(J)$. Then $X \subseteq F(I' \cap I)$ and $I' \le_{\mathrm{K}} J$. Since $|X| < \kappa_{I' \cap I,J}$, then by (e), $|X| < \mathrm{non}((I' \cap I, \le_{\mathrm{K}} J\mathrm{QN})\text{-space})$, and then, by Theorem 6.2 for the relation $\mathrm{C}_{I' \cap I,J}$, there is $(\varphi, \alpha) \in F_2(J)$ such that for all $\beta \in X$, $\varphi^{-1}(\|\beta < \alpha\|) \in J$. Therefore $\min\{\mathfrak{k}_{I,J}, \kappa_{I,J}\} \le \mathrm{non}((I, \le_{\mathrm{K}} J\mathrm{QN})\text{-space})$. In the same way we get $\min\{\mathfrak{k}_{I,J}, \lambda_{I,J}\} \le \mathrm{non}(\mathrm{w}(I, J\mathrm{QN})\text{-space})$. □

For a pair of ideals (I, J) on ω consider the following conditions:

$$H_1(I, J) \Leftrightarrow (\exists J' \text{ a } P\text{-ideal})(I \subseteq J' \subseteq J).$$
$$H_2(I, J) \Leftrightarrow (\forall I' \le_{\mathrm{K}} J)(\exists J' \text{ a } P\text{-ideal})(I' \cap I \subseteq J' \le_{\mathrm{K}} J).$$
$$H_3(I, J) \Leftrightarrow (\forall I' \le_{\mathrm{K}} J)(\exists J' \text{ a } P\text{-ideal})(I' \cap I \le_{\mathrm{K}} J' \le_{\mathrm{K}} J).$$

The quantifier $\forall I'$ can be reduced to $I' = I$ in $H_2(I, J)$ and $H_3(I, J)$, if $I \le_{\mathrm{K}} J$.

Lemma 6.12. *Let I and J be ideals on ω.*

(a) *If $H_1(I, J)$ holds, then* $\mathrm{non}((I, J\mathrm{QN})\text{-}space) \ge \nu_{I,J} \ge \mathfrak{b}$.
(b) *If $H_2(I, J)$ holds, then $\lambda_{I,J} \ge \kappa_{I,J} \ge \mathfrak{b}$ and* $\mathrm{non}(\mathrm{w}(I, J\mathrm{QN})\text{-}space) \ge \mathrm{non}((I, \le_{\mathrm{K}} J\mathrm{QN})\text{-}space) \ge \min\{\mathfrak{k}_{I,J}, \mathfrak{b}\}$.
(c) *If $H_3(I, J)$ holds, then $\lambda_{I,J} \ge \mathfrak{b}$ and* $\mathrm{non}(\mathrm{w}(I, J\mathrm{QN})\text{-}space) \ge \min\{\mathfrak{k}_{I,J}, \mathfrak{b}\}$.

Note that $\min\{\mathfrak{k}_{I,J},\mathfrak{b}\} \geq \mathfrak{p}$ because $\mathfrak{b} \geq \mathfrak{p}$ and by Lemma 4.4, $\mathfrak{k}_{I,J} = \infty$, if $I \leq_{\mathrm{K}} J$, and otherwise, $\mathfrak{c} \geq \mathfrak{k}_{I,J} \geq \mathfrak{p}$.

Proof. (a) Let J' be a P-ideal from $H_1(I,J)$. By Lemma 6.4 (d) and Theorem 6.11, $\mathfrak{b} \leq \mathfrak{b}_{J'} \leq \nu_{I,J} \leq \mathrm{non}((I,J\mathrm{QN})\text{-space})$.

(b)–(c) The proofs are same and we prove (b) only. We prove that $\kappa_{I,J} = \min\{\kappa_{I'\cap I,J} : I' \leq_{\mathrm{K}} J\} \geq \mathfrak{b}$; the second part follows by Theorem 6.11. Let $I' \leq_{\mathrm{K}} J$ be arbitrary and let J' be a P-ideal such that $I' \cap I \subseteq J' \leq_{\mathrm{K}} J$. Then by Lemma 6.4 (d) and by definition of $\kappa_{I'\cap I,J}$ we get $\mathfrak{b} \leq \mathfrak{b}_{J',J'} \leq \kappa_{I'\cap I,J}$. Since I' was arbitrary, it follows that $\mathfrak{b} \leq \kappa_{I,J}$. □

Corollary 6.13. *Let J be a capacitous ideal on ω.*

(a) $\kappa_{I,J} \leq \lambda_{I,J} \leq \mathfrak{b}$ *for every ideal I on ω.*
(b) *If $H_1(I,J)$ holds, then* $\mathrm{non}((I,J\mathrm{QN})\text{-}space) = \mathfrak{b}$.
(c) *If $H_2(I,J)$ holds, then $\kappa_{I,J} = \lambda_{I,J} = \mathfrak{b}$ and* $\mathrm{non}((I,\leq_{\mathrm{K}}J\mathrm{QN})\text{-}space) = \mathrm{non}(\mathrm{w}(I,J\mathrm{QN})\text{-}space) = \min\{\mathfrak{k}_{I,J},\mathfrak{b}\}$.
(d) *If $H_3(I,J)$ holds, then $\lambda_{I,J} = \mathfrak{b}$ and* $\mathrm{non}(\mathrm{w}(I,J\mathrm{QN})\text{-}space) = \min\{\mathfrak{k}_{I,J},\mathfrak{b}\}$.

Proof. (a) The inequality $\lambda_{I,J} \leq \mathfrak{b}$ follows by definitions and by Corollary 6.9 because every ideal $\leq_{\mathrm{K}}$-below J is capacitous.

(b) By Lemma 6.12 and Theorem 6.8.

(c)–(d) The equalities for $\kappa_{I,J}$ and $\lambda_{I,J}$ follow by (a) and by Lemma 6.12. The other equalities follow by Theorem 6.11 and Theorem 6.8. □

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Mathematical Institute of the Slovak Academy of Sciences, Grešákova 6, 040 01 Košice, Slovak Republic

E-mail address: repicky@saske.sk