

IDEAL GENERALIZATIONS OF EGOROFF'S THEOREM

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ABSTRACT. We investigate the classes of ideals for which the Egoroff's theorem or the generalized Egoroff's theorem holds between ideal versions of pointwise and uniform convergences. The paper is motivated by considerations of Michal Korch in [3].

1. INTRODUCTION

Analyzing the Pinciroli's approach to the generalized Egoroff's theorem [7], Michal Korch [3] isolated two properties between a pair of the convergences, denoted (H) and (\bar{H}) below, which ensure that all Pinciroli's arguments work for this pair of convergences. We recall some notation and terminology.

In the sets of functions of the form ${}^S\omega$ and ${}^T({}^S\omega)$ we consider the partial orderings, denoted in all cases by the same symbol as for the ordering of natural numbers \leq , given by $x \leq y$, if $x(s) \leq y(s)$ for all $x, y \in {}^S\omega$; and $\varphi \leq \psi$, if $\varphi(t) \leq \psi(t)$ for all $\varphi, \psi \in {}^T({}^S\omega)$, where $\varphi(t), \psi(t) \in {}^S\omega$. We consider also the eventual quasi-ordering \leq^* on ${}^\omega\omega$ defined by $x \leq^* y$, if $x(n) \leq y(n)$ for all but finitely many $n \in \omega$.

We say that a function $o : X \rightarrow P$ from a set X into a partially ordered set P is cofinal if for every $p \in P$ there exists $x \in X$ such that $p \leq o(x)$.

By a sequence of real functions $f_n : [0, 1] \rightarrow [0, 1]$ for $n \in \omega$ we understand a mapping $f : \omega \rightarrow {}^{[0,1]}[0, 1]$ where we let $f(n) = f_n$ for $n \in \omega$. It will be useful to consider sequences $f : S \rightarrow {}^{[0,1]}[0, 1]$ for arbitrary infinite countable sets S . We say that f is measurable, if $f_s : [0, 1] \rightarrow [0, 1]$ is measurable for all $s \in S$.

A mapping $o : {}^S({}^{[0,1]}[0, 1]) \rightarrow {}^{[0,1]}X$, where X is a topological space, is said to be a measurability preserving mapping, if $o(f) : [0, 1] \rightarrow X$ is measurable for every measurable f .

Assume that we are given two notions of convergence of sequences of functions $f \rightsquigarrow 0$ and $f \heartsuit 0$ on sets $A \subseteq [0, 1]$. We assume that the convergence on a set implies the convergence on a subset. Usually we assume that $f \heartsuit 0$ on A implies $f \rightsquigarrow 0$ on A , i.e., \heartsuit is stronger than \rightsquigarrow (however, this is not necessary because we can replace \heartsuit by conjunction of \heartsuit and \rightsquigarrow). Especially we are not interested in the results of the convergences and term 0 in the formulas $f \rightsquigarrow 0$ and $f \heartsuit 0$ can represent anything or nothing to which the sequence f converges. Denote

$$\mathcal{F}_{\rightsquigarrow} = \{f \in {}^S({}^{[0,1]}[0, 1]) : f \rightsquigarrow 0\}$$

and consider the following hypotheses between \rightsquigarrow and \heartsuit :

2000 *Mathematics Subject Classification*. Primary 28A20; Secondary 03E15, 03E17, 03E20, 26A03, 40A30, 40A35.

Key words and phrases. Egoroff's theorem, ideal on ω , ideal convergence, sequence of functions. The author was supported by grants VEGA 1/0097/16.

- (M) There exists a measurability preserving mapping $o : {}^S([0,1][0,1]) \rightarrow [0,1]({}^T\omega)$ with $|T| = \omega$ such that for every measurable $f \rightsquigarrow 0$ and measurable $A \subseteq [0,1]$, if $o(f)[A]$ is bounded in $({}^T\omega, \leq)$, then $f \heartsuit 0$ on A .
- (H) There exists $o : {}^S([0,1][0,1]) \rightarrow [0,1]({}^T\omega)$ with $|T| = \omega$ such that for every $f \rightsquigarrow 0$ and $A \subseteq [0,1]$, if $o(f)[A]$ is bounded in $({}^T\omega, \leq)$, then $f \heartsuit 0$ on A .
- (\bar{H}) There exists cofinal $o : \mathcal{F}_{\rightsquigarrow} \rightarrow ([0,1]({}^\omega\omega), \leq)$ such that for every $f \in \mathcal{F}_{\rightsquigarrow}$ and $A \subseteq [0,1]$, if $f \heartsuit 0$ on A , then $o(f)[A]$ is bounded in $({}^\omega\omega, \leq^*)$.

To verify (M) and (H) it is sufficient to define $o(f)$ only for those $f \in {}^S([0,1][0,1])$ such that $f \rightsquigarrow 0$ because for the other f , $o(f)$ can be defined arbitrarily. Obviously we can put $T = \omega$ in the conditions but sometimes definitions of functions o are easier to read if they do not contain enumerations of countable sets. In condition (\bar{H}) unlike Korch [3] we use the structure $({}^\omega\omega, \leq^*)$ instead of $({}^\omega\omega, \leq)$. On one hand this harms a bit the symmetry between the original Korch's conditions but on the other hand it seems that to look for a symmetry in the applications of these conditions is not important (see Lemma 2.1 below) in spite of the fact that in several cases (H) and (\bar{H}) can be witnessed by the same function o (see [3] and proof of Lemma 1.1 below).

Lemma 1.1 (Korch). *Conditions (M), (H), (\bar{H}) hold between the pointwise convergence \rightarrow and the uniform convergence \rightrightarrows .*

Proof. Take the measure preserving mapping $o : {}^\omega([0,1][0,1]) \rightarrow [0,1]({}^\omega\omega)$ defined so that $o(f)(x)(n) = \min\{m \in \omega : (\forall k \geq m) f_k(x) \leq 2^{-n}\}$, if $f \rightarrow 0$, $x \in [0,1]$ and $n \in \omega$. \square

Recall that \mathcal{N} is the σ -ideal of sets of reals of measure zero, $\text{non}(\mathcal{N})$ is the least cardinality of a set of reals not of measure zero, and \mathfrak{b} is the least cardinality of a \leq^* -unbounded subset of ${}^\omega\omega$.

The main motivation for [3] and also for the present paper relies on Korch's extraction of Theorem 1.2 and Theorem 1.3 from Pinciroli's arguments applied with the pair of the pointwise and the uniform convergences.

Theorem 1.2 (Korch). *Let $\Phi : [0,1] \rightarrow {}^\omega\omega$.*

- (1) *If Φ is measurable, then for every $\varepsilon > 0$ there exists $A \subseteq [0,1]$ such that $\mu(A) \geq 1 - \varepsilon$ and $\Phi[A]$ is bounded in $({}^\omega\omega, \leq)$.*
- (2) *If $\text{non}(\mathcal{N}) < \mathfrak{b}$, then for every $\varepsilon > 0$ there exists $A \subseteq [0,1]$ such that $\mu^*(A) \geq 1 - \varepsilon$ and $\Phi[A]$ is bounded in $({}^\omega\omega, \leq)$.*

Proof. (1) Since Φ is measurable we have $[0,1] = \bigcap_{n \in \omega} \bigcup_{k \in \omega} A_{n,k}$ with measurable sets $A_{n,k} = \{x \in [0,1] : \Phi(x)(n) = k\}$. For every $n \in \omega$ let $\varphi(n) \in \omega$ be minimal such that $\mu(\bigcup_{k < \varphi(n)} A_{n,k}) \geq 1 - \varepsilon 2^{-(n+1)}$ and let $A = \bigcap_{n \in \omega} \bigcup_{k < \varphi(n)} A_{n,k}$. Then $\Phi[A]$ is bounded by φ and $\mu(A) \geq 1 - \varepsilon \sum_{n \in \omega} 2^{-(n+1)} = 1 - \varepsilon$.

(2) Since $\text{non}(\mathcal{N}) < \mathfrak{b}$ there is a set $Y \subseteq [0,1]$ such that $|Y| < \mathfrak{b}$ and $\mu^*(Y) = 1$. Since $|\Phi[Y]| < \mathfrak{b}$ there is a sequence of compact sets $B_n \subseteq {}^\omega\omega$ for $n \in \omega$ such that $\Phi[Y] \subseteq \bigcup_{n \in \omega} B_n$. Denote $A_n = \Phi^{-1}(\bigcup_{i < n} B_i)$ and find $n \in \omega$ such that $\mu^*(A_n) \geq 1 - \varepsilon$. Let $A = A_n$. Then $\Phi[A]$ is bounded because it is included in a compact set and $\mu^*(A) \geq 1 - \varepsilon$. \square

Theorem 1.3 (Korch). *Let \rightsquigarrow and \heartsuit be arbitrary convergences.*

- (1) Assume that (M) holds between \rightsquigarrow and \rightsquigarrow_+ . Then for every measurable $f \rightsquigarrow 0$ on $[0, 1]$ and $\varepsilon > 0$ there exists a measurable set $A \subseteq [0, 1]$ such that $\mu(A) \geq 1 - \varepsilon$ and $f \rightsquigarrow_+ 0$ on A .
- (2) Assume that (H) holds between \rightsquigarrow and \rightsquigarrow_+ . If $\text{non}(\mathcal{N}) < \mathfrak{b}$, then for every $f \rightsquigarrow 0$ on $[0, 1]$ and $\varepsilon > 0$ there exists $A \subseteq [0, 1]$ such that $\mu^*(A) \geq 1 - \varepsilon$ and $f \rightsquigarrow_+ 0$ on A .
- (3) Assume that (\bar{H}) holds between \rightsquigarrow and \rightsquigarrow_+ , $\text{non}(\mathcal{N}) = \mathfrak{c}$ and there exists a \mathfrak{c} -Lusin set. Then there exists $f \rightsquigarrow 0$ on $[0, 1]$ such that for all sets $A \subseteq [0, 1]$ with $\mu^*(A) > 0$, $f \not\rightsquigarrow_+ 0$ on A .

Proof. (1)–(2) Take $\Phi = o(f)$ in Theorem 1.2.

(3) Let $Z \subseteq [0, 1]$ be a \mathfrak{c} -Lusin set, i.e., $|B \cap Z| < \mathfrak{c}$ for every meager set $B \subseteq {}^\omega\omega$. Choose a bijection $\varphi : [0, 1] \rightarrow Z$. Applying (\bar{H}) let $f \in \mathcal{F}_{\rightsquigarrow}$ be such that $o(f) \geq \varphi$. Let $A \subseteq [0, 1]$ be arbitrary such that $f \rightsquigarrow_+ 0$ on A . By (\bar{H}) , $o(f)[A]$ is \leq^* -bounded and then also $\varphi[A]$ is \leq^* -bounded because $o(f) \geq \varphi$. Therefore $\varphi[A]$ is meager and $|A| = |\varphi[A]| < \mathfrak{c}$ because $\varphi[A] \subseteq Z$. Then $\mu(A) = 0$ because $\text{non}(\mathcal{N}) = \mathfrak{c}$. \square

Recall that the classical Egoroff's theorem is the conclusion of Theorem 1.3 (1) between the pointwise and the uniform convergences and the generalized Egoroff's theorem is the conclusion of Theorem 1.3 (2) for the pair of the pointwise and uniform convergences (see [1, 3, 5, 7]).

In Section 2 we show that (\bar{H}) holds for many pairs of ideal convergences. In Section 3 we prove closure properties of the systems of ideals such that (M) or (H) holds between the pointwise ideal convergence and the uniform ideal convergence (Theorems 3.2–3.5). In Section 4 we give several examples of ideals such that that the Egoroff's theorem and the generalized Egoroff's theorem (hence also (M) and (H)) do not hold between the pointwise ideal convergence and the uniform ideal convergence.

2. PAIRS OF IDEAL CONVERGENCES SATISFYING (\bar{H})

By an ideal I on an infinite set S we mean a family of subsets of S such $\emptyset \in I$, $S \notin I$, $[S]^{<\omega} \subseteq I$, and I is closed under subsets and finite unions. Hence $\text{Fin} = [S]^{<\omega}$ is the smallest ideal on S . For $f \in S^{([0,1][0,1])}$ we denote $f(s)(x)$ by $f_s(x)$; hence if $S = \omega$, $f = \langle f_m : m \in \omega \rangle$ is a sequence of real functions. We recall definitions of *pointwise I -convergence*, *quasinormal I -convergence*, and *uniform I -convergence* of a sequence of real functions on a set $A \subseteq [0, 1]$:

- (i) $f \rightarrow_I 0$ on A if $(\forall x \in A)(\forall \varepsilon > 0) \{s \in S : f_s(x) \geq \varepsilon\} \in I$;
- (ii) $f \xrightarrow{\text{QN}}_I 0$ on A if $(\exists \varepsilon \in S[0, 1]) [\varepsilon \rightarrow_I 0$ and $(\forall x \in A) \{s \in S : f_s(x) \geq \varepsilon_s\} \in I]$;
- (iii) $f \rightrightarrows_I 0$ on A if $(\forall \varepsilon > 0) \{s \in S : \sup_{x \in A} f_s(x) \geq \varepsilon\} \in I$.

We recall another three kinds of convergences called *I^* -pointwise convergence*, *I^* -quasinormal convergence*, and *I^* -uniform convergence*, respectively:

- (i) $f \rightarrow_{I^*} 0$ on A if $(\forall x \in A)(\exists M \in I)(\forall \varepsilon > 0) |\{s \in S : f_s(x) \geq \varepsilon\} \setminus M| < \omega$;
- (ii) $f \xrightarrow{\text{QN}}_{I^*} 0$ on A if $(\exists M \in I)(\exists \varepsilon \in S[0, 1]) [\varepsilon \rightarrow_{\text{Fin}} 0$ and $(\forall x \in A) |\{s \in S : f_s(x) \geq \varepsilon_s\} \setminus M| < \omega]$;
- (iii) $f \rightrightarrows_{I^*} 0$ on A if $(\exists M \in I)(\forall \varepsilon > 0) |\{s \in S : \sup_{x \in A} f_s(x) \geq \varepsilon\} \setminus M| < \omega$.

The above convergences were all studied in [1] but the notation used here coincides with [3]. Let $K \subseteq I$ be two ideals on S . The above convergences are the limiting cases of the following two-ideal convergences (with $K = I$ and $K = \text{Fin}$):

- (i) $f \rightarrow_{K,I} 0$ on A if $(\forall x \in A)(\exists M \in I)(\forall \varepsilon > 0) \{s \in S : f_s(x) \geq \varepsilon\} \setminus M \in K$;
- (ii) $f \xrightarrow{\text{QN}}_{K,I} 0$ on A if $(\exists M \in I)(\exists \varepsilon \in {}^S[0,1]) [\varepsilon \rightarrow_K 0 \text{ and } (\forall x \in A) \{s \in S : f_s(x) \geq \varepsilon_s\} \setminus M \in K]$;
- (iii) $f \rightrightarrows_{K,I} 0$ on A if $(\exists M \in I)(\forall \varepsilon > 0) \{s \in S : \sup_{x \in A} f_s(x) \geq \varepsilon\} \setminus M \in K$.

Recall that an ideal I on ω is a P -ideal, if for every sequence C_n for $n \in \omega$ of sets from I there is a set $C \in I$ such that $C_n \setminus C$ is finite for all $n \in \omega$. If I is a P -ideal, then $\rightarrow_{I^*} = \rightarrow_I$ and $\rightrightarrows_{I^*} = \rightrightarrows_I$. Moreover, if $K \subseteq I$ is a P -ideal (or contained in a P -ideal that is a subideal of I), then $\rightarrow_{I^*} = \rightarrow_{K,I}$ and $\rightrightarrows_{I^*} = \rightrightarrows_{K,I}$.

The implications between the convergences are summarized in the following diagram where “stronger” implies “weaker”:

$$\begin{array}{ccccccc}
\rightarrow_{\text{Fin}} & \longrightarrow & \rightarrow_{I^*} & \longrightarrow & \rightarrow_{K,I} & \longrightarrow & \rightarrow_I \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\text{QN} \rightarrow_{\text{Fin}} & \longrightarrow & \text{QN} \rightarrow_{I^*} & \longrightarrow & \text{QN} \rightarrow_{K,I} & \longrightarrow & \text{QN} \rightarrow_I \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\rightrightarrows_{\text{Fin}} & \longrightarrow & \rightrightarrows_{I^*} & \longrightarrow & \rightrightarrows_{K,I} & \longrightarrow & \rightrightarrows_I
\end{array}$$

If we consider that convergences are relations, the set operation $\rightsquigarrow \cup \leftrightsquigarrow$ takes sense. By next lemma many pairs of ideal convergences satisfy (\bar{H}) :

Lemma 2.1. *Let \rightsquigarrow and \leftrightsquigarrow be any convergences such that \rightsquigarrow is weaker than \rightarrow_{Fin} and \leftrightsquigarrow is stronger than $\rightrightarrows_I \cup \xrightarrow{\text{QN}}_{I^*}$ for an ideal I on ω . Then (\bar{H}) holds between \rightsquigarrow and \leftrightsquigarrow . In particular, (\bar{H}) holds between \rightarrow_I and \rightrightarrows_I and (\bar{H}) holds between \rightarrow_{I^*} and \rightrightarrows_{I^*} for every ideal I on ω .*

Proof. For $f \in \mathcal{F}_{\rightsquigarrow}$, $x \in [0,1]$, and $n \in \omega$ let $C_{f,x,n} = \{m \in \omega : f_m(x) < 2^{-n}\}$ and define $o : \mathcal{F}_{\rightsquigarrow} \rightarrow {}^{[0,1]}(\omega\omega)$ by $o(f)(x)(n) = \min(C_{f,x,n})$, if $C_{f,x,n} \neq \emptyset$, and $o(f)(x)(n) = 0$, otherwise.

We prove that o is cofinal. Given $\varphi : [0,1] \rightarrow {}^\omega\omega$ define $f^\varphi \in \mathcal{F}_{\text{Fin}} \subseteq \mathcal{F}_{\rightsquigarrow}$ by $f_m^\varphi(x) = \max\{2^{-n} : m \leq \varphi(x)(n) + n\}$. For all $m \leq \varphi(x)(n)$, $f_m^\varphi(x) \geq 2^{-n}$ and therefore $o(f^\varphi)(x)(n) > \varphi(x)(n)$ because $C_{f^\varphi,x,n} \neq \emptyset$.

Let $f \in \mathcal{F}_{\rightsquigarrow}$ and $f \leftrightsquigarrow 0$ on $A \subseteq [0,1]$. Then $f \rightrightarrows_I 0$ on A or $f \xrightarrow{\text{QN}}_{I^*} 0$ on A .

If $f \rightrightarrows_I 0$ on A , then the set $C_{f,n} = \{m \in \omega : \sup_{x \in A} f_m(x) < 2^{-n}\}$ is in the dual filter for every $n \in \omega$. Since $\emptyset \neq C_{f,n} \subseteq C_{f,x,n}$ for all $x \in A$, the function $\varphi(n) = \min(C_{f,n})$ is an upper bound of $o(f)[A]$ in (ω, \leq) . Therefore (\bar{H}) holds between \rightsquigarrow and \leftrightsquigarrow .

Assume that $f \xrightarrow{\text{QN}}_{I^*} 0$ on A . Then $A = \bigcup_{k \in \omega} A_k$ with $f \rightrightarrows_{I^*} 0$ on A_k for all $k \in \omega$ (this is an observation of Remark 3.2 in [1]). Since \rightrightarrows_{I^*} is stronger than \rightrightarrows_I , by previous case there is an upper bound $\varphi_k \in {}^\omega\omega$ of $o(f)[A_k]$ in $({}^\omega\omega, \leq)$ for all $k \in \omega$. Then the function $\varphi \in {}^\omega\omega$ defined by $\varphi(n) = \max\{\varphi_k(n) : k \leq n\}$ is an upper bound of $o(f)[A]$ in $({}^\omega\omega, \leq^*)$. \square

3. PAIRS OF IDEAL CONVERGENCES SATISFYING (M) AND (H)

Let I be an ideal on a set S (support of I), i.e., $S = \bigcup I$ and $[S]^{<\omega} \subseteq I$. The restriction of the ideal I onto a set $T \in \mathcal{P}(S) \setminus I$ is the ideal $I|_T = I \cap \mathcal{P}(T)$. We denote $\text{Fin} = \text{Fin}_S = [S]^{<\omega}$ and $\langle B \rangle = \langle B \rangle_S = \{E \subseteq S : |E \setminus B| < \omega\}$ for $B \subseteq S$; we omit the subscript S if the support S of the ideals is known from the context.

If \mathcal{I} is a family of ideals, the least ideal containing $\bigcup \mathcal{I}$ is denoted by $\bigvee \mathcal{I}$. The intersection $\bigcap \mathcal{I}$ is an ideal, provided that the intersection of supports of ideals in \mathcal{I} is not a member of $\bigcap \mathcal{I}$.

Let T be a finite or infinite set and for every $t \in T$ let I_t be an ideal on S_t . The set $S = \{(t, s) : t \in T \text{ and } s \in S_t\}$ is the disjoint sum of the family of sets $\{S_t : t \in T\}$. The ideal $\sum_{t \in T} I_t = \{A \subseteq S : (\forall t \in T) A_t \in I_t\}$ is the direct sum of the system of ideals $\{I_t : t \in T\}$, where $A_t = \{s \in S_t : (t, s) \in A\}$; $I_0 \oplus I_1 = \sum_{i \in \{0,1\}} I_i$. If T is infinite and J is an ideal on T , then the ideal $\sum_{t \in T}^J I_t = \{A \subseteq S : \{t \in T : A_t \notin I_t\} \in J\}$ is the sum of the system of ideals $\{I_t : t \in T\}$ with respect to the ideal J . The ideal $J \times I = \sum_{t \in T}^J I$ is the product of J and I . These definitions correspond with definitions of sums and products of filters in [2].

Sums of Borel ideals on ω are Borel and sums of analytic ideals on ω are analytic.

Let J be an ideal on T and for every $t \in T$ let I_t be an ideal on S_t . Define $\underline{\lim}_{t \in T}^J S_t = \{s : \{t \in T : s \notin S_t\} \in J\}$ and $\underline{\lim}_{t \in T}^J I_t = \{E : \{t \in T : E \notin I_t\} \in J\}$. Since $T \notin J$, $\underline{\lim}_{t \in T}^J S_t \subseteq \bigcup_{t \in T} S_t$ and $\underline{\lim}_{t \in T}^J I_t$ is a family of subsets of $\underline{\lim}_{t \in T}^J S_t$. Moreover, if $\underline{\lim}_{t \in T}^J S_t \notin \underline{\lim}_{t \in T}^J I_t$, then $\underline{\lim}_{t \in T}^J I_t$ is an ideal on $\underline{\lim}_{t \in T}^J S_t$. For $J = \text{Fin}$ this definition gives $\underline{\lim}_{n \in \omega} I_n = \bigcup_{m \in \omega} \bigcap_{n > m} I_n$.

We consider the following partial orderings of ideals on ω :

- (1) Rudin-Keisler partial ordering: $I \leq_{\text{RK}} J$ if there is $g : \omega \rightarrow \omega$ such that $I = \{E \subseteq \omega : g^{-1}(E) \in J\}$.
- (2) Rudin-Blass partial ordering: $I \leq_{\text{RB}} J$ if there is a finite-to-one $g : \omega \rightarrow \omega$ such that $I = \{E \subseteq \omega : g^{-1}(E) \in J\}$.

Remark 3.1. The partial orderings \leq_{RK} and \leq_{RB} are used also for ideals with different supports. Let I, J and I_n for $n \in \omega$ be ideals on ω .

(1) $J \leq_{\text{RK}} \sum_{n \in \omega}^J I_n$ is witnessed by the function $g : \omega \times \omega \rightarrow \omega$ defined by $g(m, n) = m$. In particular, $J \leq_{\text{RK}} J \times I$.

(2) $\bigcap_{n \in \omega} I_n \leq_{\text{RB}} \sum_{n \in \omega} I_n$ and $\underline{\lim}_{n \in \omega}^J I_n \leq_{\text{RB}} \sum_{n \in \omega}^J I_n$. The relation \leq_{RK} is witnessed by the function $g : \omega \times \omega \rightarrow \omega$ defined by $g(m, n) = n$ which is finite-to-one on the set $\{(m, n) \in \omega \times \omega : m \leq n\}$ from the dual filter. Therefore the relation \leq_{RB} holds between the ideals. In particular, $I \leq_{\text{RB}} J \times I$.

Theorem 3.2. *Let \mathcal{I} be the class of ideals I on countable sets such that (M) holds between \rightarrow_I and \Rightarrow_I .*

- (1) $\text{Fin} \in \mathcal{I}$ and $\langle B \rangle \in \mathcal{I}$ for coinfinite $B \subseteq \omega$.
- (2) \mathcal{I} is closed under restrictions of ideals.
- (3) \mathcal{I} is downward \leq_{RK} -closed.
- (4) \mathcal{I} is closed under direct sums $\sum_{n \in \omega} I_n$, where $I_n \in \mathcal{I}$ for all $n \in \omega$.
- (5) \mathcal{I} is closed under ideals that are intersections of countable subfamilies of \mathcal{I} .
- (6) \mathcal{I} is closed under increasing countable unions of measurable ideals from \mathcal{I} .
- (7) \mathcal{I} is closed under sums $\sum_{n \in \omega}^J I_n$, where $J, I_n \in \mathcal{I}$ and I_n are analytic for all $n \in \omega$.

- (8) \mathcal{I} is closed under ideals of the form $\underline{\lim}_{n \in \omega}^J I_n$, where $J, I_n \in \mathcal{I}$ and I_n are analytic for all $n \in \omega$.

Theorem 3.3. Let \mathcal{I} be the class of ideals I on countable sets such that (H) holds between \rightarrow_I and \rightrightarrows_I .

- (1) $\text{Fin} \in \mathcal{I}$ and $\langle B \rangle \in \mathcal{I}$ for coinfinite $B \subseteq \omega$.
- (2) \mathcal{I} is closed under restrictions of ideals.
- (3) \mathcal{I} is downward \leq_{RK} -closed.
- (4) \mathcal{I} is closed under direct sums $\sum_{n \in \omega} I_n$, where $I_n \in \mathcal{I}$ for all $n \in \omega$.
- (5) \mathcal{I} is closed under ideals that are intersections of countable subfamilies of \mathcal{I} .
- (6) (a) If $I \in \mathcal{I}$ and $K \in \mathcal{I}$ is a P -ideal, then $I \vee K \in \mathcal{I}$.
(b) \mathcal{I} is closed under increasing countable unions of ideals from \mathcal{I} .
- (7) \mathcal{I} is closed under sums $\sum_{n \in \omega}^J I_n$ where $J, I_n \in \mathcal{I}$ for all $n \in \omega$.
- (8) \mathcal{I} is closed under ideals of the form $\underline{\lim}_{n \in \omega}^J I_n$, where $J, I_n \in \mathcal{I}$ for all $n \in \omega$.

Theorem 3.4. Let \mathcal{I} be the class of ideals I on countable sets such that (M) holds between \rightarrow_{I^*} and \rightrightarrows_{I^*} .

- (1) $\text{Fin} \in \mathcal{I}$ and $\langle B \rangle \in \mathcal{I}$ for coinfinite $B \subseteq \omega$.
- (2) \mathcal{I} is closed under restrictions of ideals.
- (3) \mathcal{I} is downward \leq_{RB} -closed.
- (4) \mathcal{I} is closed under direct sums $\sum_{n \in \omega} I_n$, where $I_n \in \mathcal{I}$ for all $n \in \omega$.
- (5) \mathcal{I} is closed under ideals that are intersections of countable subfamilies of \mathcal{I} .
- (6) \mathcal{I} is closed under increasing countable unions of analytic ideals from \mathcal{I} .
- (7) \mathcal{I} is closed under sums $\sum_{n \in \omega}^J I_n$ where $J, I_n \in \mathcal{I}$ and I_n are analytic for all $n \in \omega$.
- (8) \mathcal{I} is closed under ideals of the form $\underline{\lim}_{n \in \omega}^J I_n$, where $J, I_n \in \mathcal{I}$ and I_n are analytic for all $n \in \omega$.

Theorem 3.5. Let \mathcal{I} be the class of ideals I on countable sets such that (H) holds between \rightarrow_{I^*} and \rightrightarrows_{I^*} .

- (1) $\text{Fin} \in \mathcal{I}$ and $\langle B \rangle \in \mathcal{I}$ for coinfinite $B \subseteq \omega$.
- (2) \mathcal{I} is closed under restrictions of ideals.
- (3) \mathcal{I} is downward \leq_{RB} -closed.
- (4) \mathcal{I} is closed under direct sums $\sum_{n \in \omega} I_n$, where $I_n \in \mathcal{I}$ for all $n \in \omega$.
- (5) \mathcal{I} is closed under ideals that are intersections of countable subfamilies of \mathcal{I} .
- (6) \mathcal{I} is closed under ideals that are joins of countable subfamilies of \mathcal{I} .
- (7) \mathcal{I} is closed under sums $\sum_{n \in \omega}^J I_n$ where $J, I_n \in \mathcal{I}$ for all $n \in \omega$.
- (8) \mathcal{I} is closed under ideals of the form $\underline{\lim}_{n \in \omega}^J I_n$, where $J, I_n \in \mathcal{I}$ for all $n \in \omega$.

Proof of the theorems. The assertions of theorems follow by the following lemmas:

- (1) Lemma 3.6; (2) Lemma 3.7; (3) Lemma 3.9; (4) Lemma 3.10; (5) Lemma 3.11; (6) Lemma 3.12 and Lemma 3.13; (7) Lemma 3.15; (8) Lemma 3.16. \square

To simplify speaking we use the following phrases:

- “ I satisfies (M') ”, if “ (M) holds between \rightarrow_I and \rightrightarrows_I ”,
- “ I satisfies (H') ”, if “ (H) holds between \rightarrow_I and \rightrightarrows_I ”,
- “ I satisfies (M'') ”, if “ (M) holds between \rightarrow_{I^*} and \rightrightarrows_{I^*} ”,
- “ I satisfies (H'') ”, if “ (H) holds between \rightarrow_{I^*} and \rightrightarrows_{I^*} ”.

The following lemma consists of special cases of paper [3] results.

Lemma 3.6. *The ideals Fin and $\langle B \rangle$ on ω satisfy (M') , (H') , (M'') , (H'') for all coinfinite sets $B \subseteq \omega$.*

Proof. Since the ideal $I = \langle B \rangle$ is a P -ideal, it is enough to verify (M') and (H') for I because $\rightarrow_I = \rightarrow_{I^*}$ and $\rightrightarrows_I = \rightrightarrows_{I^*}$. Define $o : {}^\omega([0,1]^{[0,1]}) \rightarrow [0,1]^{({}^\omega\omega)}$ for $f \rightarrow_I 0$ by $o(f)(x)(n) = \min\{k \in \omega : \{m \geq k : f_m(x) \geq 2^{-n}\} \subseteq B\}$.

Obviously, o is measurability preserving. If $\varphi \in {}^\omega\omega$ is a bound of $o(f)[A]$ where $f \rightarrow_I 0$ and $A \subseteq [0,1]$, then $\{m \geq \varphi(n) : \sup_{x \in A} f_m(x) \geq 2^{-n}\} \subseteq B$ for every $n \in \omega$, i.e., $f \rightrightarrows_I 0$ on A . \square

Lemma 3.7. *Let I be an ideal on ω and $S \in \mathcal{P}(\omega) \setminus I$. The restriction $I \upharpoonright S$ has any of the properties (M') , (H') , (M'') , (H'') , whenever I has the same property.*

Proof. Let z denote the constant real function with value 0. For $f \in S^{([0,1]^{[0,1]})}$ define $\bar{f} \in {}^\omega([0,1]^{[0,1]})$ by $\bar{f}_m = f_m$ for $m \in S$ and $\bar{f}_m = z$ for $m \in \omega \setminus S$. For both ideal convergences, \bar{f} converges with respect to I if and only if f converges with respect to $I \upharpoonright S$. It is easy to verify that whenever $o_I : {}^\omega([0,1]^{[0,1]}) \rightarrow [0,1]^{({}^\omega\omega)}$ is a witness for a particular property concerning to I required by the lemma, then $o : S^{([0,1]^{[0,1]})} \rightarrow [0,1]^{({}^\omega\omega)}$ defined by $o(f) = o_I(\bar{f})$ is a witness for the same property concerning to $I \upharpoonright S$. \square

Remark 3.8. Let $I \leq_{\text{RK}} J$ be ideals on ω and let $I = \{E \subseteq \omega : g^{-1}(E) \in J\}$ for some $g : \omega \rightarrow \omega$. If $\text{Fin} \subsetneq J$, then there is a surjection $h : \omega \rightarrow \omega$ such that $I = \{E \subseteq \omega : h^{-1}(E) \in J\}$ and h is finite-to-one, if g is finite-to-one; h is injective, if g is injective. To see this, find $B \in J$ either finite or infinite such that $|B| = |\omega \setminus g(\omega \setminus B)|$ and choose a bijection $\pi : B \rightarrow \omega \setminus g(\omega \setminus B)$. The function $h = \pi \cup g \upharpoonright (\omega \setminus B)$ has the required properties.

Lemma 3.9. *Let I and J be ideals on ω .*

- (1) *If $I \leq_{\text{RK}} J$, then*
 - (a) *I satisfies (M') , whenever J satisfies (M') ;*
 - (b) *I satisfies (H') , whenever J satisfies (H') .*
- (2) *If $I \leq_{\text{RB}} J$, then*
 - (a) *I satisfies (M'') , whenever J satisfies (M'') ;*
 - (b) *I satisfies (H'') , whenever J satisfies (H'') .*

Proof. Let $I = \{E \subseteq \omega : g^{-1}(E) \in J\}$ for a function $g : \omega \rightarrow \omega$ that is finite-to-one in case (2). If $J = \text{Fin}$, then $I = \langle \omega \setminus \text{rng}(g) \rangle$ and both assertions of the lemma follow by Lemma 3.6. Therefore we can assume that $J \neq \text{Fin}$ and by Remark 3.8 we can moreover assume that g is a surjection. For $f \in {}^\omega([0,1]^{[0,1]})$ define $\bar{f} \in {}^\omega([0,1]^{[0,1]})$ by $\bar{f}_m = f_{g(m)}$ for $m \in \omega$.

If $f \rightarrow_I 0$, then $\bar{f} \rightarrow_J 0$ because for $x \in [0,1]$ and $\varepsilon > 0$, $\{m \in \omega : \bar{f}_m(x) \geq \varepsilon\} = g^{-1}(\{k \in \omega : f_k(x) \geq \varepsilon\}) \in J$.

If $f \rightarrow_{I^*} 0$ and g is finite-to-one, then $\bar{f} \rightarrow_{J^*} 0$. To see this assume that for every $x \in [0,1]$ there is $M_x \in I$ be such that $\{k \in \omega : f_k(x) \geq \varepsilon\} \setminus M_x \in \text{Fin}$ for all $\varepsilon > 0$. Then $\{m \in \omega : \bar{f}_m(x) \geq \varepsilon\} \setminus g^{-1}(M_x) = g^{-1}(\{k \in \omega : f_k(x) \geq \varepsilon\} \setminus M_x) \in \text{Fin}$ for all $\varepsilon > 0$ because g is finite-to-one.

Assume that $o_J : {}^\omega([0,1]^{[0,1]}) \rightarrow [0,1]^{({}^\omega\omega)}$ witnesses any of the properties (M') , (H') , (M'') , (H'') for J and define $o : {}^\omega([0,1]^{[0,1]}) \rightarrow [0,1]^{({}^\omega\omega)}$ by $o(f) = o_J(\bar{f})$; o is measurability preserving, whenever o_J is such.

(1) If $f \rightarrow_I 0$ and $A \subseteq [0,1]$ are such that $o(f)[A] = o_J(\bar{f})[A]$ is bounded in ${}^\omega\omega$, then (in cases of (M') and (H') for J) $\bar{f} \rightrightarrows_J 0$ on A and hence for $\varepsilon > 0$,

$g^{-1}(\{k \in \omega : \sup_{x \in A} f_k(x) \geq \varepsilon\}) = \{m \in \omega : \sup_{x \in A} \bar{f}_m(x) \geq \varepsilon\} \in J$. Therefore $f \rightrightarrows_I 0$ on A .

(2) Assume that $f \rightarrow_{I^*} 0$ and $A \subseteq [0, 1]$ are such that $o_I(f)[A] = o_J(\bar{f})[A]$ is bounded in ${}^\omega\omega$. Let $M_\varepsilon = \{m \in \omega : \sup_{x \in A} \bar{f}_m(x) \geq \varepsilon\}$ and $N_\varepsilon = \{k \in \omega : \sup_{x \in A} f_k(x) \geq \varepsilon\}$ for $\varepsilon > 0$. Since (in cases of (M'') and (H'') for J) $\bar{f} \rightrightarrows_{J^*} 0$ on A , there is $N \in J$ such that $N_\varepsilon \setminus N \in \text{Fin}$ for all $\varepsilon > 0$. Let $M = \{k \in \omega : g^{-1}(\{k\}) \subseteq N\}$. Then $M \in I$ and $g^{-1}(\{k\}) \cap (N_\varepsilon \setminus N) \neq \emptyset$ for all $k \in M_\varepsilon \setminus M$. Therefore $M_\varepsilon \setminus M \in \text{Fin}$ for all $\varepsilon > 0$ and $f \rightrightarrows_{I^*} 0$ on A . \square

Lemma 3.10. *The direct sum $\sum_{t \in T} I_t$ of a countable family of ideals on countable sets satisfies any of the properties (M') , (H') , (M'') , (H'') , whenever all ideals I_t for $t \in T$ satisfy the same property.*

Proof. Let $I = \sum_{t \in T} I_t$; I is an ideal on the set $S = \bigcup_{t \in T} \{t\} \times S_t$. For $f : S \rightarrow [0, 1]$ define $\hat{f}^t : S_t \rightarrow [0, 1]$ by $(\hat{f}^t)_s(x) = f_{t,s}(x)$. For every $t \in T$, if $f \rightarrow_I 0$, then $\hat{f}^t \rightarrow_{I_t} 0$; if $f \rightarrow_{I^*}$, then $\hat{f}^t \rightarrow_{I_t^*}$; if f is measurable, then \hat{f}^t is measurable. For any of the required properties it is easy to verify that whenever $o_t : S_t \rightarrow [0, 1] \rightarrow [0, 1]^{S_t}$ witnesses this property for I_t for all $t \in T$, then the function $o : S \rightarrow [0, 1] \rightarrow [0, 1]^{S\omega}$ defined by $o(f)(x)(t, s) = o_t(\hat{f}^t)(x)(s)$ witnesses the same property for I . \square

Lemma 3.11. *If an intersection $\bigcap_{n \in \omega} I_n$ of a sequence of ideals on countable sets is an ideal, then it has any of the properties (M') , (H') , (M'') , (H'') , whenever all ideals I_n have the same property.*

Proof. Without loss of generality we can assume that all ideals I_n are ideals on the same set (otherwise apply Lemma 3.7 for I_n). Then the lemma is a consequence of Lemma 3.10 and Lemma 3.9 because $\bigcap_{n \in \omega} I_n \leq_{\text{RB}} \sum_{n \in \omega} I_n$ by Remark 3.1. \square

For example, the P -ideal of pseudo-intersections of a countable sequence of infinite subsets of ω satisfies (M') , (H') , (M'') , (H'') .

Let I and J be ideals on ω and $g : [0, 1] \rightarrow I \vee J$ be measurable. In general we can say nothing about the existence of a measurable function $h : [0, 1] \rightarrow J$ such that $g(x) \setminus h(x) \in I$ and $h(x) \subseteq g(x)$. This is one of the reasons why measurable variants are omitted in next lemma (they hold for analytic ideals due to Σ_2^1 -uniformization provided that Σ_2^1 sets are Lebesgue measurable).

Lemma 3.12. *Let I and J be ideals on countable sets.*

- (1) $I \vee J$ satisfies (H') , whenever I and J satisfy (H') and J is a P -ideal.
- (2) $I \vee J$ satisfies (H'') , whenever I and J satisfy (H'') .

Proof. Let I, J , and $K = I \vee J$ be ideals on sets S, T , and $U = S \cup T$, respectively. For functions $f : U \rightarrow [0, 1]$ and $\varepsilon > 0$ denote $U_{f,x,\varepsilon} = \{m \in U : f_m(x) \geq \varepsilon\}$.

(1) If $f \rightarrow_K 0$, then $U_{f,x,\varepsilon} \in K$. Since J is a P -ideal, for every $x \in [0, 1]$ we can find $M_{f,x} \in J$ (hence $M_{f,x} \subseteq T$) such that $U_{f,x,\varepsilon} \setminus M_{f,x} \in I \vee [T \setminus S]^{<\omega}$ for all $\varepsilon > 0$. Let us define $f^I : S \rightarrow [0, 1]$ and $f^J : T \rightarrow [0, 1]$ by

$$f_m^I(x) = \begin{cases} f_m(x), & \text{if } m \in S \setminus M_{f,x}, \\ 0, & \text{if } m \in S \cap M_{f,x}, \end{cases}$$

$$f_m^J(x) = \begin{cases} f_m(x), & \text{if } m \in T \setminus (S \setminus M_{f,x}), \\ 0, & \text{if } m \in T \cap (S \setminus M_{f,x}). \end{cases}$$

Then $f^I \rightarrow_I 0$ and $f^J \rightarrow_J 0$ because for all $\varepsilon > 0$,

$$\begin{aligned} \{m \in S : f_m^I(x) \geq \varepsilon\} &= U_{f,x,\varepsilon} \cap (S \setminus M_{f,x}) = (U_{f,x,\varepsilon} \setminus M_{f,x}) \cap S \in I, \\ \{m \in T : f_m^J(x) \geq \varepsilon\} &\subseteq U_{f,x,\varepsilon} \setminus (S \setminus M_{f,x}) \subseteq (U_{f,x,\varepsilon} \setminus S) \cup M_{f,x} \in J. \end{aligned}$$

Let $o_I : S^{([0,1][0,1])} \rightarrow [0,1]^{(\omega\omega)}$ and $o_J : T^{([0,1][0,1])} \rightarrow [0,1]^{(\omega\omega)}$ witness (M') or (H') for I and J . Define $o : U^{([0,1][0,1])} \rightarrow [0,1]^{(\omega\omega)}$ for $f \rightarrow_K 0$ by

$$o(f)(x)(n) = \max\{o_I(f^I)(x)(n), o_J(f^J)(x)(n)\}.$$

Assume that $f \rightarrow_K 0$ and $A \subseteq [0,1]$ are such that $o(f)[A]$ is bounded in $\omega\omega$. Then $o_I(f^I)[A]$ and $o_J(f^J)[A]$ are bounded and applying (M') or (H') we have $f^I \rightarrow_I 0$ on A and $f^J \rightarrow_J 0$ on A . Denote $E_\varepsilon = \{m \in S : \sup_{x \in A} f_m^I(x) \geq \varepsilon\}$ and $F_\varepsilon = \{m \in T : \sup_{x \in A} f_m^J(x) \geq \varepsilon\}$ for $\varepsilon > 0$; hence $E_\varepsilon \in I$ and $F_\varepsilon \in J$. Then for every $\varepsilon > 0$, $\{m \in U : \sup_{x \in A} f_m(x) \geq \varepsilon\} = E_\varepsilon \cup F_\varepsilon \in K$ because $\sup_{x \in A} f_m(x) = \max\{\sup_{x \in A} f_m^I(x), \sup_{x \in A} f_m^J(x)\}$, where every missing term $f_m^I(x)$ for $m \in T \setminus S$ and $f_m^J(x)$ for $m \in S \setminus T$ is replaced by 0. Hence $f \rightarrow_K 0$.

(2) If $f \rightarrow_{K^*} 0$, then for every $x \in [0,1]$ fix $N_{f,x} \in I$ and $M_{f,x} \in J$ such that $N_{f,x} \cap M_{f,x} = \emptyset$ and $U_{f,x,\varepsilon} \setminus (N_{f,x} \cup M_{f,x}) \in \text{Fin}$ for all $\varepsilon > 0$. Let us define $f^I : S \rightarrow [0,1]$ and $f^J : T \rightarrow [0,1]$ by the same formula as in case (1). Then $f^I \rightarrow_{I^*} 0$ and $f^J \rightarrow_{J^*} 0$ because for all $\varepsilon > 0$ the following sets are finite:

$$\begin{aligned} \{m \in S : f_m^I(x) \geq \varepsilon\} \setminus N_{f,x} &\subseteq U_{f,x,\varepsilon} \setminus (M_{f,x} \cup N_{f,x}), \\ \{m \in T : f_m^J(x) \geq \varepsilon\} \setminus M_{f,x} &\subseteq (U_{f,x,\varepsilon} \setminus (S \setminus M_{f,x})) \setminus M_{f,x} = U_{f,x,\varepsilon} \setminus (S \cup M_{f,x}). \end{aligned}$$

Let $o_I : S^{([0,1][0,1])} \rightarrow [0,1]^{(\omega\omega)}$ and $o_J : T^{([0,1][0,1])} \rightarrow [0,1]^{(\omega\omega)}$ witness (M'') or (H'') for I and J . Define $o : U^{([0,1][0,1])} \rightarrow [0,1]^{(\omega\omega)}$ by

$$o(f)(x)(n) = \max\{o_I(f^I)(x)(n), o_J(f^J)(x)(n)\}.$$

Assume that $f \rightarrow_{K^*} 0$ and $A \subseteq [0,1]$ are such that $o(f)[A]$ is bounded in $\omega\omega$. Then $o_I(f^I)[A]$ and $o_J(f^J)[A]$ are bounded and applying (M'') or (H'') we get $f^I \rightarrow_{I^*} 0$ on A and $f^J \rightarrow_{J^*} 0$ on A . Hence, if we denote $E_\varepsilon = \{m \in S : \sup_{x \in A} f_m^I(x) \geq \varepsilon\}$ and $F_\varepsilon = \{m \in T : \sup_{x \in A} f_m^J(x) \geq \varepsilon\}$, then there are $N \in I$ and $M \in J$ such that $E_\varepsilon \setminus N$ and $F_\varepsilon \setminus M$ are finite for all $\varepsilon > 0$. Then $\{m \in U : \sup_{x \in A} f_m(x) \geq \varepsilon\} \setminus (N \cup M) = (E_\varepsilon \cup F_\varepsilon) \setminus (N \cup M)$ is finite for all $\varepsilon > 0$ because $\sup_{x \in A} f_m(x) = \max\{\sup_{x \in A} f_m^I(x), \sup_{x \in A} f_m^J(x)\}$, where every missing term $f_m^I(x)$ for $m \in T \setminus S$ and $f_m^J(x)$ for $m \in S \setminus T$ is replaced by 0. \square

Korch [3] proved that countably generated ideals satisfy (H') and (H'') . The following lemma together with Lemma 3.6 and Lemma 3.12 gives much more. A proof of assertion (1a) for analytic ideals and proof of assertions (1b), (2a), (2b) can be obtained also by Lemma 3.16 below.

Lemma 3.13. *Let the ideal $I = \bigcup_{k \in \omega} I_k$ be the union of an increasing sequence of ideals on countable sets.*

- (1) (a) I satisfies (M') , whenever the ideals I_k for $k \in \omega$ are measurable and satisfy (M') .
- (b) I satisfies (H') , whenever the ideals I_k for $k \in \omega$ satisfy (H') .
- (2) (a) I satisfies (M'') , whenever the ideals I_k for $k \in \omega$ are analytic and satisfy (M'') .
- (b) I satisfies (H'') , whenever the ideals I_k for $k \in \omega$ satisfy (H'') .

Proof. Applying direct sums $J \oplus \text{Fin}$, restrictions $J \upharpoonright E$, and isomorphisms (i.e., Lemma 3.7, Lemma 3.9, Lemma 3.10) for the ideals $J = I_k$, without loss of generality we can assume that all ideals I_k are ideals on ω . Moreover we can assume that $I_k \neq I_{k+1}$ for all $k \in \omega$.

(1) If $f \rightarrow_I 0$, denote $B_{f,x,n} = \{m \in \omega : f_m(x) > 2^{-n}\}$ for $x \in [0, 1]$ and $n \in \omega$, and define $\psi(f) : [0, 1] \rightarrow {}^\omega\omega$ and $\bar{f}^k : \omega \rightarrow [0, 1]$ for $k \in \omega$ by

$$\begin{aligned} \psi(f)(x)(n) &= \min\{k \in \omega : B_{f,x,n} \in I_k\}, \\ (\bar{f}^k)_m(x) &= \begin{cases} f_m(x), & \text{if } (\exists n \in \omega) 2^{-n} < f_m(x) \leq 2^{-(n-1)} \text{ and } \psi(f)(x)(n) \leq k, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Then $\bar{f}^k \rightarrow_{I_k} 0$ because for every $x \in [0, 1]$ and $n \in \omega$, $\{m \in \omega : (\bar{f}^k)_m(x) > 2^{-n}\} = \{m \in \omega : (\exists i \leq n) 2^{-i} < f_m(x) \leq 2^{-(i-1)} \text{ and } \psi(f)(x)(i) \leq k\} \subseteq \bigcup\{B_{f,x,i} : i \leq n \text{ and } B_{f,x,i} \in I_k\} \in I_k$.

Assume that for every $k \in \omega$, $o_k : {}^\omega([0, 1][0, 1]) \rightarrow [0, 1]({}^\omega\omega)$ witnesses (M') or (H') for I_k and define $o : {}^\omega([0, 1][0, 1]) \rightarrow [0, 1]({}^\omega \times {}^\omega\omega)$ for $f \rightarrow_I 0$ by

$$o(f)(x)(k, n) = \max\{o_k(\bar{f}^k)(x)(n), \psi(f)(x)(n)\}.$$

If f is measurable and I_k are measurable ideals for all $k \in \omega$, then all \bar{f}^k are measurable. Hence o is measurability preserving, whenever all o_k are such.

Let $f \rightarrow_I 0$ and $A \subseteq [0, 1]$ be such that $o(f)[A]$ is bounded by a function $\varphi : \omega \times \omega \rightarrow \omega$. Then $o_k(\bar{f}^k)[A]$ is bounded by $\varphi_k \in {}^\omega\omega$ defined by $\varphi_k(n) = \varphi(k, n)$ and therefore $\bar{f}^k \rightarrow_{I_k} 0$ on A for all $k \in \omega$. We apply this fact for all k of the form $k = \varphi_0(n)$ for $n \in \omega$. For every $x \in A$ and $n \in \omega$, $\{m \in \omega : f_m(x) > 2^{-n}\} = \{m \in \omega : (\exists i \leq n) 2^{-i} < f_m(x) \leq 2^{-(i-1)}\} \subseteq \{m \in \omega : (\bar{f}^{\varphi_0(n)})_m(x) > 2^{-n}\}$ because $\psi(f)(x)(i) \leq \psi(f)(x)(n) \leq \varphi_0(n)$ for all $i \leq n$. Therefore $\{m \in \omega : \sup_{x \in A} f_m(x) > 2^{-n}\} \subseteq \{m \in \omega : \sup_{x \in A} (\bar{f}^{\varphi_0(n)})_m(x) > 2^{-n}\} \in I_{\varphi_0(n)} \subseteq I$ for all $n \in \omega$.

(2) If $f \rightarrow_{I^*} 0$, denote $B_{f,x,\varepsilon} = \{m \in \omega : f_m(x) \geq \varepsilon\}$ for $x \in [0, 1]$ and $\varepsilon > 0$, and define $\psi(f) : [0, 1] \rightarrow {}^\omega\omega$ and $\bar{f}^k : \omega \rightarrow [0, 1]$ for $k \in \omega$ by

$$\begin{aligned} \psi(f)(x) &= \min\{k \in \omega : (\exists M \in I_k)(\forall \varepsilon > 0) |B_{f,x,\varepsilon} \setminus M| < \omega\}, \\ (\bar{f}^k)_m(x) &= \begin{cases} f_m(x), & \text{if } \psi(f)(x) \leq k, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We show that if $f \rightarrow_{I^*} 0$, then $\bar{f}^k \rightarrow_{I_k^*} 0$. For every $x \in [0, 1]$ choose $M_{f,x} \in I_{\psi(f)(x)}$ such that $|B_{f,x,\varepsilon} \setminus M_{f,x}| < \omega$ for all $\varepsilon > 0$. Either $\psi(f)(x) \leq k$ and then $M_{f,x} \in I_k$ and $\{m \in \omega : (\bar{f}^k)_m(x) \geq \varepsilon\} \setminus M_{f,x} = B_{f,x,\varepsilon} \setminus M_{f,x} \in \text{Fin}$, or $\psi(f)(x) > k$ and $\{m \in \omega : (\bar{f}^k)_m(x) \geq \varepsilon\} = \emptyset$.

Assume that for every $k \in \omega$, $o_k : {}^\omega([0, 1][0, 1]) \rightarrow [0, 1]({}^\omega\omega)$ witnesses (M'') or (H'') for I_k and define $o : {}^\omega([0, 1][0, 1]) \rightarrow [0, 1]({}^\omega \times {}^\omega\omega)$ for $f \rightarrow_{I^*} 0$ by

$$o(f)(x)(k, n) = \max\{o_k(\bar{f}^k)(x)(n), \psi(f)(x)\}.$$

If f is measurable and I_k are analytic ideals for all $k \in \omega$, then all \bar{f}^k are measurable ($f_m \upharpoonright B$ are Borel for some Borel set $B \subseteq [0, 1]$ of full measure, therefore $(\bar{f}^k)_m \upharpoonright B$ are analytic, and then \bar{f}^k is measurable). Hence o is measurability preserving, whenever all o_k are such.

Let $f \rightarrow_{I^*} 0$ and $A \subseteq [0, 1]$ be such that $o(f)[A]$ is bounded by a function $\varphi : \omega \times \omega \rightarrow \omega$. Then $o_k(\bar{f}^k)[A]$ is bounded by $\varphi_k \in {}^\omega\omega$ defined by $\varphi_k(n) = \varphi(k, n)$ and therefore $\bar{f}^k \rightarrow_{I_k^*} 0$ on A . Let $k = \varphi(0, 0)$ and let $M \in I_k$ be such that

$|\{m \in \omega : \sup_{x \in A} (\bar{f}^k)_m(x) > \varepsilon\} \setminus M| < \omega$ for all $\varepsilon > 0$. For $x \in A$ and $\varepsilon > 0$, $\psi(f)(x) \leq k$ and hence, $\{m \in \omega : f_m(x) > \varepsilon\} = \{m \in \omega : (\bar{f}^k)_m(x) > \varepsilon\}$. Therefore $\{m \in \omega : \sup_{x \in A} f_m(x) > \varepsilon\} \setminus M = \{m \in \omega : \sup_{x \in A} (\bar{f}^k)_m(x) > \varepsilon\} \setminus M \in \text{Fin}$ for all $\varepsilon > 0$. \square

Lemma 3.14. *Let J be an ideal on ω , let I_t be ideals on S_t for $t \in \omega$, let $I = \sum_{t \in \omega}^J I_t$ be the ideal on $S = \bigcup_{t \in \omega} \{t\} \times S_t$, and let $a \in {}^S[0, 1]$, $f \in {}^S([0, 1]^{[0, 1]})$, $A \subseteq [0, 1]$.*

- (1) $a \rightarrow_{I^*} 0$ if and only if $\{t \in \omega : \langle a_{t,s} \rangle_{s \in S_t} \not\rightarrow_{I_t^*} 0\} \in J$.
- (2) $f \rightarrow_{I^*} 0$ on A if and only if $(\forall x \in A) \{t \in \omega : \langle f_{t,s}(x) \rangle_{s \in S_t} \not\rightarrow_{I_t^*} 0\} \in J$.
- (3) $f \rightrightarrows_{I^*} 0$ on A if and only if $\{t \in \omega : \langle f_{t,s} \rangle_{s \in S_t} \not\rightarrow_{I_t^*} 0 \text{ on } A\} \in J$.

Proof. (1) Denote $E = \{t \in \omega : \langle a_{t,s} \rangle_{s \in S_t} \not\rightarrow_{I_t^*} 0\}$.

Assume $a \rightarrow_{I^*} 0$. Let $M \in I$ be such that $|\{(t, s) \in S : a_{t,s} \geq \varepsilon\} \setminus M| < \omega$ for all $\varepsilon > 0$. Denote $M_t = \{s \in S_t : (t, s) \in M\}$ and $N = \{t \in \omega : M_t \notin I_t\}$. Then $N \in J$. If $t \in \omega \setminus N$, then $M_t \in I_t$ and for every $\varepsilon > 0$, $|\{s \in S_t : a_{t,s} \geq \varepsilon\} \setminus M_t| \leq |\{(u, s) \in S : a_{u,s} \geq \varepsilon\} \setminus M| < \omega$ and hence $t \notin E$. Therefore $E \subseteq N \in J$.

Assume $E \in J$. For each $t \in \omega \setminus E$ let $M_t \in I_t$ be such that $F_{t,n} = \{s \in S_t : a_{t,s} \geq 2^{-n}\} \setminus M_t$ is finite for all $n \in \omega$. The set $M = \{(t, s) : t \in \omega \setminus E \text{ and } s \in M_t \cup F_{t,t}\}$ belongs to I and $\{(t, s) \in S : a_{t,s} \geq 2^{-n}\} \setminus M \subseteq \{(t, s) \in S : t < n \text{ and } s \in F_{t,n}\}$ are finite sets for all $n \in \omega$. Therefore $a \rightarrow_{I^*} 0$.

- (2) Use (1) with $a_{t,s} = f_{t,s}(x)$ for $x \in A$.
- (3) Use (1) with $a_{t,s} = \sup_{x \in A} f_{t,s}(x)$; $a \rightarrow_{I^*} 0$ if and only if $f \rightrightarrows_{I^*} 0$ on A . \square

Mrozek [6] and Korch [3] showed that for iterations Fin^α , $\alpha < \omega_1$ of the ideal Fin obtained by repeating the operation \sum^{Fin} , the Egoroff's theorem and the generalized Egoroff's theorem hold between $\rightarrow_{\text{Fin}^\alpha}$ and $\rightrightarrows_{\text{Fin}^\alpha}$. The following lemma generalizes both these results.

Lemma 3.15. *Let J be an ideal on T , let I_t be ideals on S_t for $t \in T$, where T and all sets S_t are infinite countable, and let $I = \sum_{t \in T}^J I_t$ be the ideal on the set $S = \bigcup_{t \in T} \{t\} \times S_t$.*

- (1) (a) I satisfies (M') , whenever J and I_t for $t \in T$ satisfy (M') and the ideals I_t for $t \in T$ are analytic.
- (b) I satisfies (H') , whenever J and I_t for $t \in T$ satisfy (H') .
- (2) (a) I satisfies (M'') , whenever J and I_t for $t \in T$ satisfy (M'') and the ideals I_t for $t \in T$ are analytic.
- (b) I satisfies (H'') , whenever J and I_t for $t \in T$ satisfy (H'') .

Proof. (1) For $f : S \rightarrow [0, 1]$ we define $\bar{f} : T \rightarrow [0, 1]$ and $\hat{f}^t : S_t \rightarrow [0, 1]$ for $t \in T$ in the following way: Let $t \in T$ and $x \in [0, 1]$. If $\langle f_{t,s}(x) \rangle_{s \in S_t} \rightarrow_{I_t} 0$, then for all $s \in S_t$ let

$$\bar{f}_t(x) = 0 \quad \text{and} \quad (\hat{f}^t)_s(x) = f_{t,s}(x).$$

Otherwise, let $n(f, t, x)$ be the least $n \in \omega$ such that $\{s \in S_t : f_{t,s}(x) \geq 2^{-n}\} \notin I_t$ and for all $s \in S_t$ let

$$\bar{f}_t(x) = 2^{-n(f,t,x)},$$

$$(\hat{f}^t)_s(x) = \begin{cases} f_{t,s}(x), & \text{if } f_{t,s}(x) \geq 2^{-(n(f,t,x)-1)}, \\ 0, & \text{if } f_{t,s}(x) < 2^{-(n(f,t,x)-1)}. \end{cases}$$

If $f \rightarrow_I 0$, then $\bar{f} \rightarrow_J 0$ because for all $x \in [0, 1]$ and $n \in \omega$, $\{t \in T : \bar{f}_t(x) \geq 2^{-n}\} = \{t \in T : n \geq n(f, t, x)\} = \{t \in T : \{s \in S_t : f_{t,s}(x) \geq 2^{-n}\} \notin I_t\} \in J$; and $\hat{f}^t \rightarrow_{I_t} 0$ because either $\langle (\hat{f}^t)_s(x) \rangle_{s \in S_t} = \langle f_{t,s}(x) \rangle_{s \in S_t} \rightarrow_{I_t} 0$, or otherwise $\{s \in S_t : (\hat{f}^t)_s(x) > 0\} \in I_t$.

One can verify that if f is measurable, then \bar{f} and \hat{f}^t for $t \in T$ are measurable: If f is Borel, the functions \bar{f} and \hat{f}^t are obtained by changing f in a simple way on sets $X_t = \{x \in [0, 1] : \langle f_{t,s}(x) \rangle_{s \in S_t} \rightarrow_{I_t} 0\}$ and $Z_{t,n} = \{x \in [0, 1] \setminus X_t : n(f, t, x) = n\}$ that are measurable because they belong to the algebra generated by analytic sets since the ideals I_t are analytic. If f is measurable, then there is a Borel measure zero set $B \subseteq [0, 1]$ such that $f \upharpoonright ([0, 1] \setminus B)$ is Borel and then the sets X_t and $Z_{t,n}$ are measurable because $X_t \setminus B$ and $Z_{t,n} \setminus B$ are measurable by previous argument.

Assume that $o_J : T^{[0,1]}[0, 1] \rightarrow^{[0,1]}(T\omega)$ and $o_t : S_t^{[0,1]}[0, 1] \rightarrow^{[0,1]}(S_t\omega)$ are witnesses that the ideals J and I_t for $t \in T$ satisfy (M') or (H') . We define $o : S^{[0,1]}[0, 1] \rightarrow^{[0,1]}(S\omega)$ by

$$o(f)(x)(t, s) = \max\{o_J(\bar{f})(x)(t), o_t(\hat{f}^t)(x)(s)\}$$

for $f \rightarrow_I 0$, $x \in [0, 1]$, $t \in T$ and $s \in S_t$. If o_J and o_t are all measurability preserving, then also o is such.

Let $f \rightarrow_I 0$ and $A \subseteq [0, 1]$ be such that $o(f)[A]$ is bounded in $(S\omega, \leq)$ by a function $\nu : S \rightarrow \omega$. Then $o_J(\bar{f})[A]$ is bounded in $(T\omega, \leq)$ by the function $\bar{\nu} : T \rightarrow \omega$ defined by $\bar{\nu}(t) = \nu(t, 0)$ and $o_t(\hat{f}^t)[A]$ is bounded in $(S_t\omega, \leq)$ by the function $\hat{\nu}_t : S_t \rightarrow \omega$ defined by $\hat{\nu}_t(s) = \nu(t, s)$. It follows that $\bar{f} \rightrightarrows_J 0$ on A and $\hat{f}^t \rightrightarrows_{I_t} 0$ on A for all $t \in T$. We show that the set $X_n = \{(t, s) \in S : \sup_{x \in A} f_{t,s}(x) \geq 2^{-n}\}$ is in I for all $n \in \omega$ and hence $f \rightrightarrows_I 0$ on A .

For $n \in \omega$ and $t \in T$ let $T_n = \{v \in T : \sup_{x \in A} \bar{f}_v(x) \geq 2^{-n}\}$ and $S_{t,n} = \{s \in S_t : \sup_{x \in A} (\hat{f}^t)_s(x) \geq 2^{-n}\}$; hence $T_n \in J$ and $S_{t,n} \in I_t$. Fix $t \in T \setminus T_n$ and $s \in S_t \setminus S_{t,n}$. Then $\sup_{x \in A} \bar{f}_t(x) < 2^{-n}$ and $\sup_{x \in A} (\hat{f}^t)_s(x) < 2^{-n}$. Let $x \in A$ be arbitrary. If $\langle f_{t,u}(x) \rangle_{u \in S_t} \rightarrow_{I_t} 0$, then $f_{t,s}(x) = (\hat{f}^t)_s(x) < 2^{-n}$. Otherwise, $2^{-n(f,t,x)} = \bar{f}_t(x) < 2^{-n}$. Hence $n(f, t, x) > n$ and, either $f_{t,s}(x) \geq 2^{-(n(f,t,x)-1)}$ and then $f_{t,s}(x) = (\hat{f}^t)_s(x) < 2^{-n}$, or $f_{t,s}(x) < 2^{-(n(f,t,x)-1)} \leq 2^{-n}$. Therefore $(t, s) \notin X_n$ for all $t \in T \setminus T_n$ and $s \in S_t \setminus S_{t,n}$. It follows that $X_n \in I$.

(2) For $f : S \rightarrow^{[0,1]}[0, 1]$ we define $\bar{f} : T \rightarrow^{[0,1]}[0, 1]$ and $\hat{f}^t : S_t \rightarrow^{[0,1]}[0, 1]$ for $t \in T$ by the formulas

$$\begin{aligned} \bar{f}_t(x) &= 0 & \text{and} & & (\hat{f}^t)_s(x) &= f_{t,s}(x), & \text{if } \langle f_{t,u}(x) \rangle_{u \in S_t} \rightarrow_{I_t^*} 0, \\ \bar{f}_t(x) &= 1 & \text{and} & & (\hat{f}^t)_s(x) &= 0, & \text{otherwise.} \end{aligned}$$

Clearly $\hat{f}^t \rightarrow_{I_t^*} 0$ for all $t \in T$. By Lemma 3.14 (2), $\bar{f} \rightarrow_{J^*} 0$. If f is measurable, then also \bar{f} and \hat{f}^t are measurable because the ideals I_t are analytic.

Assume that $o_J : T^{[0,1]}[0, 1] \rightarrow^{[0,1]}(T\omega)$ and $o_t : S_t^{[0,1]}[0, 1] \rightarrow^{[0,1]}(S_t\omega)$ are witnesses that the ideals J and I_t for $t \in T$ satisfy (M'') or (H'') and define $o : S^{[0,1]}[0, 1] \rightarrow^{[0,1]}(S\omega)$ by

$$o(f)(x)(t, s) = \max\{o_J(\bar{f})(x)(t), o_t(\hat{f}^t)(x)(s)\}$$

for $f \rightarrow_{I^*} 0$, $x \in [0, 1]$, $t \in T$ and $s \in S_t$. If o_J and o_t are all measurability preserving, then also o is such.

Let $f \rightarrow_{I^*} 0$ and $A \subseteq [0, 1]$ be such that $o(f)[A]$ is bounded in $(S\omega, \leq)$ by a function $\nu : S \rightarrow \omega$. Then $o_J(\bar{f})[A]$ is bounded in $(T\omega, \leq)$ by the function

$\bar{\nu} : T \rightarrow \omega$ defined by $\bar{\nu}(t) = \nu(t, 0)$ and $o_t(\hat{f}^t)[A]$ is bounded in $(S_t\omega, \leq)$ by the function $\hat{\nu}_t : S_t \rightarrow \omega$ defined by $\hat{\nu}_t(s) = \nu(t, s)$. It follows that $\bar{f} \rightrightarrows_{J^*} 0$ on A and $\hat{f}^t \rightrightarrows_{I_t^*} 0$ on A for all $t \in S$. Let $M = \{t \in T : \sup_{x \in A} \bar{f}_t(x) \geq 1\}$. Then $M \in J$, $M = \{t \in T : \sup_{x \in A} \bar{f}_t(x) > 0\}$, and $(\hat{f}^t)_s(x) = f_{t,s}(x)$ for all $t \in T \setminus M$, $s \in S_t$ and $x \in [0, 1]$. Since $\hat{f}^t \rightrightarrows_{I_t^*} 0$ on A for all $t \in T \setminus M$, by Lemma 3.14 (3) it follows that $f \rightrightarrows_{I^*} 0$ on A . \square

Lemma 3.16. *Let J be an ideal on T , let I_t be ideals on S_t for $t \in T$, where all sets T and S_t are infinite countable, let $S = \varinjlim_{t \in T} S_t$, and let $I = \varinjlim_{t \in T} I_t$. If $S \notin I$, then for the ideal I all conclusions of Lemma 3.15 hold.*

Proof. Applying direct sums $J \oplus \text{Fin}$, restrictions $J \upharpoonright E$, and isomorphisms for the ideals $J = I_t$, without loss of generality we can assume that all ideals I_t are ideals on S . Then by Remark 3.1, $\varinjlim_{t \in T} I_t \leq_{\text{RB}} \sum_{t \in T} I_t$, and by Lemma 3.9, the lemma is a consequence of Lemma 3.15. \square

For an ideal I on ω and a family of functions $G \subseteq {}^\omega\omega$ let $G^{\leftarrow}(I) = \{E \subseteq \omega : (\forall g \in G) g(E) \in I\}$. Then $G^{\leftarrow}(I)$ is an ideal if and only if $\omega \notin G^{\leftarrow}(I)$ if and only if $(\exists g \in G) \text{rng}(g) \notin I$. If $G^{\leftarrow}(I)$ is an ideal, then $G^{\leftarrow}(I) = G_0^{\leftarrow}(I)$ where $G_0 = \{g \in G : \text{rng}(g) \notin I\}$. If $G = \{g\}$, we write $g^{\leftarrow}(I)$ instead of $G^{\leftarrow}(I)$. Hence $G^{\leftarrow}(I) = \bigcap_{g \in G} g^{\leftarrow}(I)$.

Lemma 3.17. *Let I be an ideal on ω and $G \subseteq {}^\omega\omega$ be a countable family of functions such that $\text{rng}(g) \notin I$ for all $g \in G$. The ideal $G^{\leftarrow}(I)$ has any of the properties (M') , (H') , (M'') , (H'') , whenever I has the same property.*

Proof. Since $G^{\leftarrow}(I) = \bigcap_{g \in G} g^{\leftarrow}(I)$, by Lemma 3.11 it is enough to prove the lemma in the case $|G| = 1$. So let $g \in {}^\omega\omega$ be such that $\text{rng}(g) \notin I$ and let $J = g^{\leftarrow}(I) = \{E \subseteq \omega : g(E) \in I\}$. For $f : \omega \rightarrow [0, 1]$ define $\bar{f} : \omega \rightarrow [0, 1]$ by $\bar{f}_k(x) = \sup\{f_m(x) : m \in g^{-1}(\{k\})\}$, if $k \in \text{rng}(g)$, and \bar{f}_k is the constant real function with value 0, if $k \notin \text{rng}(g)$.

If $f \rightarrow_J 0$, then $\bar{f} \rightarrow_I 0$ because for $x \in [0, 1]$ and $\varepsilon > 0$, $\{k \in \omega : \bar{f}_k(x) > \varepsilon\} = \{k \in \omega : (\exists m \in g^{-1}(\{k\})) f_m(x) > \varepsilon\} = g(\{m \in \omega : f_m(x) > \varepsilon\}) \in I$.

If $f \rightarrow_{J^*} 0$, then $\bar{f} \rightarrow_{I^*} 0$. To see this, for every $x \in [0, 1]$ choose $M_x \in J$ such that $\{m \in \omega : f_m(x) > \varepsilon\} \setminus M_x \in \text{Fin}$ for all $\varepsilon > 0$. Then $g(M_x) \in I$ and $\{k \in \omega : \bar{f}_k(x) > \varepsilon\} \setminus g(M_x) \subseteq \{k \in \omega : (\exists m \in g^{-1}(\{k\}) \setminus M_x) f_m(x) > \varepsilon\} = g(\{m \in \omega : f_m(x) > \varepsilon\} \setminus M_x) \in \text{Fin}$.

Assume that $o_I : \omega^{([0, 1][0, 1])} \rightarrow [0, 1]^{(\omega\omega)}$ witnesses any of the properties for I mentioned in the lemma. Define $o : \omega^{([0, 1][0, 1])} \rightarrow [0, 1]^{(\omega\omega)}$ by $o(f) = o_I(\bar{f})$. Clearly, if o_I is measurability preserving, then also o is measurability preserving.

Let $f \rightarrow_J 0$ and $A \subseteq [0, 1]$ be such that $o(f)[A] = o_I(\bar{f})[A]$ is bounded in ${}^\omega\omega$. By (H') for I (or by (M') in the measurability case), then $\bar{f} \rightrightarrows_I 0$ on A and hence for every $\varepsilon > 0$, $\{m \in \omega : \sup_{x \in A} f_m(x) > \varepsilon\} \subseteq g^{-1}(\{k \in \omega : \sup_{x \in A} \bar{f}_k(x) > \varepsilon\}) \in J$. Therefore $f \rightrightarrows_J 0$ on A and (H') (or (M')) holds for J .

Let $f \rightarrow_{J^*} 0$ and $A \subseteq [0, 1]$ be such that $o(f)[A] = o_I(\bar{f})[A]$ is bounded in ${}^\omega\omega$. By (H'') for I (or by (M'')), then $\bar{f} \rightrightarrows_{I^*} 0$ on A and hence there is $M \in I$ such that $\{k \in \omega : \sup_{x \in A} \bar{f}_k(x) > \varepsilon\} \setminus M \in \text{Fin}$ for all $\varepsilon > 0$. Then $g^{-1}(M) \in J$ and $\{m \in \omega : \sup_{x \in A} f_m(x) > \varepsilon\} \setminus g^{-1}(M) = g^{-1}(\{k \in \omega : \sup_{x \in A} \bar{f}_k(x) > \varepsilon\} \setminus M) \in \text{Fin}$. Therefore $f \rightrightarrows_{J^*} 0$ on A and (H'') (or (M'')) holds for J . \square

Lemma 3.9 concerns to preservation of properties of ideals \leq_{RK} -downward and \leq_{RB} -downward. In the upward direction we have the following lemma:

Lemma 3.18. *Let $I \leq_{\text{RK}} J$ be ideals on ω . There exists an ideal J_0 on ω such $I \leq_{\text{RK}} J_0 \subseteq J$ and J_0 has any of the properties (M') , (H') , (M'') , (H'') , whenever I has the same property.*

Proof. Let $g : \omega \rightarrow \omega$ be such that $I = \{E \subseteq \omega : g^{-1}(E) \in J\}$ and let $J_0 = g^{\leftarrow}(I) = \{E \subseteq \omega : g(E) \in I\}$. Then $I = \{E \subseteq \omega : g^{-1}(E) \in J_0\}$ and $I \leq_{\text{RK}} J_0 \subseteq J$. The remaining part of the lemma follows by Lemma 3.17. \square

4. NEGATIVE RESULTS ON (M) AND (H)

An ideal I on ω is said to be thick, if there is a measurable function $g : [0, 1] \rightarrow I$ such that $\mu(\{x \in [0, 1] : g(x) \subseteq E\}) = 0$ for all $E \in I$. In this definition the interval $[0, 1]$ can be replaced by ${}^\omega 2$ since there exists a measure preserving Borel isomorphism between $[0, 1]$ and ${}^\omega 2$. Recall that functions $g : [0, 1] \rightarrow \mathcal{P}(\omega)$ and $h : [0, 1] \rightarrow {}^\omega \mathcal{P}(\omega)$ are measurable, if the sets $\{x \in [0, 1] : m \in g(x)\}$ and $\{x \in [0, 1] : m \in h_n(x)\}$ are measurable for all $m, n \in \omega$.

Lemma 4.1. *If I is a thick ideal on ω , then neither the classical Egoroff's theorem nor the generalized Egoroff's theorem between \rightarrow_I and \rightrightarrows_I and between \rightarrow_{I^*} and \rightrightarrows_{I^*} holds.*

Proof. Let $g : [0, 1] \rightarrow I$ be measurable such that $\mu(\{x \in [0, 1] : g(x) \subseteq E\}) = 0$ for all $E \in I$. Define a sequence of measurable functions $f : \omega \rightarrow [0, 1]$ by

$$f_m(x) = \begin{cases} 1, & \text{if } m \in g(x), \\ 0, & \text{otherwise.} \end{cases} \quad (*)$$

Then $f \rightarrow_I 0$ and $f \rightarrow_{I^*} 0$ on $[0, 1]$ because $\{m \in \omega : f_m(x) \geq \varepsilon\} = g(x)$ for all $\varepsilon > 0$. If $A \subseteq [0, 1]$ is arbitrary such that $f \rightrightarrows_I 0$ on A , then $\mu(A) = 0$ because the set $E = \{m \in \omega : \sup_{z \in A} f_m(z) \geq 1\}$ is in I and $g(x) \subseteq E$ for every $x \in A$. The same argument works for \rightrightarrows_{I^*} because it implies \rightrightarrows_I . \square

Due to Lemma 4.1 the following is a consequence of Theorem 1.3:

Corollary 4.2. *Let I be a thick ideal on ω .*

- (1) (M) holds neither between \rightarrow_I and \rightrightarrows_I , nor between \rightarrow_{I^*} and \rightrightarrows_{I^*} .
- (2) If $\text{non}(\mathcal{N}) < \mathfrak{b}$, then (H) holds neither between \rightarrow_I and \rightrightarrows_I , nor between \rightarrow_{I^*} and \rightrightarrows_{I^*} . \square

Lemma 4.3. *Let I, J , and I_n for $n \in \omega$ be ideals on ω and let $B \in \mathcal{P}(\omega) \setminus I$.*

- (1) *If $I \restriction B$ is thick, then also I is thick.*
- (2) *If $J \leq_{\text{RK}} I$ and J is thick, then also I is thick.*
- (3) *If $J \leq_{\text{RK}} I \restriction B$ and J is thick, then also I is thick.*
- (4) *If J is thick, then also $\sum_{n \in \omega}^J I_n, J \times I, I \times J$ are thick.*

Proof. (1) is easy; (3) is a consequence of (1) and (2); and by Remark 3.1, (4) is a consequence of (2). We prove (2).

Let $g : \omega \rightarrow \omega$ be such that $J = \{E \subseteq \omega : g^{-1}(E) \in I\}$. Assume that $h : [0, 1] \rightarrow J$ is measurable and $\mu(\{x \in [0, 1] : h(x) \subseteq E\}) = 0$ for all $E \in J$. The function $g^{-1} : \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$ is continuous and therefore the function $\bar{h} : [0, 1] \rightarrow I$ defined by $\bar{h}(x) = g^{-1}(h(x))$ is measurable. Given set $F \in I$ let $E = \{n \in \omega :$

$g^{-1}(\{n\}) \in F\}$, i.e., $E \in J$ and E is the largest subset of ω such that $g^{-1}(E) \subseteq F$. Then $\{x \in [0, 1] : \bar{h}(x) \subseteq F\} = \{x \in [0, 1] : g^{-1}(h(x)) \subseteq F\} = \{x \in [0, 1] : g^{-1}(h(x)) \subseteq g^{-1}(E)\} = \{x \in [0, 1] : h(x) \subseteq E\}$ because $\omega \setminus \text{rng}(g) \subseteq E$. Therefore $\mu(\{x \in [0, 1] : \bar{h}(x) \subseteq F\}) = 0$. \square

Now we introduce several ideals and we prove that they are thick.

Example 4.4. (a) Let $\varphi : {}^{<\omega}2 \rightarrow \omega$ be a bijection such that $\varphi(s) < \varphi(t)$ whenever $|s| < |t|$. Then $2^{|s|} \leq \varphi(s) + 1 < 2^{|s|+1}$ for $s \in {}^{<\omega}2$, i.e., φ maps n2 onto the interval $K_n = [2^n - 1, 2^{n+1} - 1)$ and $2^{-(n+1)} < 1/(k+1) \leq 2^{-n}$ for $k \in K_n$. For $x \in {}^\omega 2$ let $E_x = \{\varphi(x \upharpoonright n) : n \in \omega\}$. The mapping $x \mapsto E_x$ is a homeomorphism of ${}^\omega 2$ onto the family $C = \{E_x : x \in {}^\omega 2\} \subseteq [\omega]^\omega$, $\omega = \bigcup C$, and C is a perfect compact almost disjoint family on ω (shortly, a. d. family, i.e., $|E_x \cap E_y| < \omega$ for $x \neq y$). For $A \subseteq {}^\omega 2$ denote $E(A) = \bigcup_{x \in A} E_x$. Consider the ideals

$$\begin{aligned} I_0 &= \{E \subseteq \omega : (\exists A \in [{}^\omega 2]^{<\omega}) E \subseteq E(A)\}, \\ I_1 &= \{E \subseteq \omega : (\exists A \in [{}^\omega 2]^\omega) \{E_x : x \in {}^\omega 2 \setminus A\} \cup \{E\} \text{ is an a. d. family}\}, \\ I_2 &= \{E \subseteq \omega : (\exists A \in \mathcal{N}) \{E_x : x \in {}^\omega 2 \setminus A\} \cup \{E\} \text{ is an a. d. family}\}, \\ I_3 &= \{E \subseteq \omega : \sum_{k \in E} 1/(k+1) < \infty\}, \\ I_4 &= \{E \subseteq \omega : \lim_{n \rightarrow \infty} |E \cap n|/n = 0\} \\ &= \{E \subseteq \omega : \lim_{n \rightarrow \infty} |E \cap [2^n, 2^{n+1})|/2^n = 0\}. \end{aligned}$$

(b) A family $D \subseteq [\omega]^\omega$ is an independent family, if $|\bigcap D_0 \cap \bigcap_{F \in D_1} (\omega \setminus F)| = \omega$ for any two disjoint finite subfamilies $D_0, D_1 \subseteq D$. Let $D = \{F_x : x \in {}^\omega 2\}$ be a perfect compact independent family on ω such that $\omega = \bigcup D$ and the mapping $x \mapsto F_x$ is a homeomorphism. (For example, on the countable set $S = \{(n, u) : n \in \omega \text{ and } u \subseteq {}^n 2\}$ the family of sets $F'_x = \{(n, u) \in S : n \in \omega \text{ and } x \upharpoonright n \in u\}$ for $x \in {}^\omega 2$ is a perfect compact independent family. This is a modification of the Hausdorff's construction, see [4].) For $A, B_1, \dots, B_k \subseteq {}^\omega 2$ denote $F(A) = \bigcup_{x \in A} F_x$ and $F(A; B_1, \dots, B_k) = F(A) \cup \bigcup_{i=1}^k (\omega \setminus F(B_i))$. Let \mathcal{N}^+ denote the family of sets of positive measure in ${}^\omega 2$. Consider the ideals

$$\begin{aligned} I'_0 &= \{E \subseteq \omega : (\exists A \in [{}^\omega 2]^{<\omega}) E \subseteq F(A)\}, \\ I'_1 &= \{E \subseteq \omega : (\exists A \in [{}^\omega 2]^{<\omega}) (\exists B_1, \dots, B_k \in \mathcal{N}^+) E \subseteq F(A; B_1, \dots, B_k)\}, \\ I'_2 &= \{E \subseteq \omega : (\exists A \in [{}^\omega 2]^{<\omega}) (\exists B_1, \dots, B_k \in [{}^\omega 2]^\omega) E \subseteq F(A; B_1, \dots, B_k)\}. \end{aligned}$$

In fact, I_0 and I'_i are Σ_1^1 , I_1 and I_2 are Σ_2^1 , I_3 is Σ_2^0 , and I_4 is Π_3^0 . It follows that $I_0, I_3, I_4, I'_0, I'_1, I'_2$ are meager and have measure zero.

For $E \subseteq \omega$ let $X_E = \{x \in {}^\omega 2 : E_x \subseteq E\}$ and $Z_E = \{x \in {}^\omega 2 : F_x \subseteq E\}$. Then X_E is closed, $X_\emptyset = \emptyset$, $X_\omega = {}^\omega 2$, and $X_{\bigcap_i E_i} = \bigcap_i X_{E_i}$. The same holds for Z_E .

Lemma 4.5. (1) $I_0, I_1, I_2, I_3, I_4, I'_0, I'_1, I'_2$ are ideals and I_2, I_3, I_4 are P -ideals.
(2) $I_0 \subseteq I_1 \subseteq I_2, I_0 \subseteq I_3 \subseteq I_4$, and $I'_0 \subseteq I'_1 \subseteq I'_2$.

Proof. (1) We show that I'_1 and I'_2 are ideals and that I_2 is a P -ideal. The other facts are obvious or known.

By definition I'_1 and I'_2 are closed under finite unions and subsets. If $A \in [{}^\omega 2]^{<\omega}$ and B_1, \dots, B_k are infinite, then there are distinct reals $x_1 \in B_1 \setminus A, \dots, x_k \in B_k \setminus A$. Since D is independent, the set $\bigcap_{i=1}^k F_{x_i} \cap \bigcap_{x \in A} (\omega \setminus F_x)$ is infinite and disjoint from $F(A; B_1, \dots, B_k)$. Therefore $\omega \notin I'_1$ and $\omega \notin I'_2$.

The sets X_E are closed and therefore one can easily verify that

$$I_2 = \{E \subseteq \omega : \lim_{n \in \omega} \mu(X_{(\omega \setminus (E \setminus n))}) = 1\}.$$

Assume that $E_n \subseteq E_{n+1} \in I_2$ for all $n \in \omega$. Find an increasing sequence $\langle k_n \rangle_{n \in \omega}$ in ω such that $\mu(X_{(\omega \setminus (E_n \setminus k_n))}) \geq 1 - 2^{-(n+1)}$ and let $E = \bigcup_{n \in \omega} (E_n \setminus k_n)$. Then $\mu(X_{(\omega \setminus (E \setminus k_n))}) \geq \mu(X_{(\omega \setminus \bigcup_{i \geq n} (E_i \setminus k_i))}) = \mu(\bigcap_{i \geq n} X_{(\omega \setminus (E_i \setminus k_i))}) \geq 1 - \sum_{i \geq n} 2^{-(i+1)} = 1 - 2^{-n}$. This proves that I_2 is a P -ideal.

(2) For every $x \in {}^\omega 2$ we have $1 = \sum_{n \in \omega} 2^{-(n+1)} < \sum_{n \in \omega} 1/(\varphi(x \upharpoonright n) + 1) = \sum_{k \in E_x} 1/(k+1) < \sum_{n \in \omega} 2^{-n} = 2$ and hence $E_x \in I_3$. Therefore $I_0 \subseteq I_3$. The inclusion $I_3 \subseteq I_4$ follows by the fact that for $E \subseteq \omega$ and $n \in \omega$, $\sum_{k \in E \cap K_n} 1/(k+1) \geq \sum_{k \in E \cap K_n} 2^{-n} = |E \cap K_n|/2^n$. The remaining inclusions are obvious. \square

An ideal I on ω is said to be simply summable in a set $B \subseteq \omega$, if there is a function $a : B \rightarrow [0, 1]$ such that $\sum_{k \in B} a(k) = \infty$, $\lim_{k \in B} a(k) = 0$, and $I \upharpoonright B = S(a)$, where

$$S(a) = \{E \subseteq \text{dom}(a) : \sum_{k \in E} a(k) < \infty\}.$$

If I is a summable ideal on ω (i.e., $I = S(a)$ for a divergent series $a : \omega \rightarrow [0, 1]$), then there is $B \subseteq \omega$ such that either $I = \langle B \rangle$ (this includes $I = \text{Fin}$), or I is simply summable in B . Really, if $I = S(a)$ for a divergent series $a : \omega \rightarrow (0, 1]$, then either there is $\delta > 0$ such that $\sum \{a(k) : a(k) < \delta\} < \infty$ and then $I = \langle B \rangle$ for $B = \{k \in \omega : a(k) < \delta\}$, or otherwise by induction find a decreasing sequence of positive reals $\delta_n < 2^{-n}$ and pairwise disjoint finite sets B_n such that $\sum_{k \in B_n} a_k \geq 1$ and $\delta_{n+1} \leq a_k < \delta_n$ for $k \in B_n$. Then I is simply summable in $B = \bigcup_{n \in \omega} B_n$.

By next theorem simply summable ideals are thick and therefore by Lemma 3.6, a summable ideal on ω is thick if and only if it is simply summable in a subset of ω .

Theorem 4.6. *Let I be an ideal on ω .*

- (1) *If $I_0 \subseteq I \subseteq I_2$ or $I_0 \subseteq I \subseteq I_4$, then I is thick.*
- (2) *If I is simply summable in a subset of ω , then I is thick.*
- (3) *If $I'_0 \subseteq I \subseteq I'_2$ or $I'_1 \subseteq I$, then I is thick. In particular, there are prime thick ideals on ω .*

Proof. (1) Since $I_0 \subseteq I$ we can define $g : {}^\omega 2 \rightarrow I$ by $g(x) = E_x$. The function g is continuous and $X_E = \{x \in {}^\omega 2 : g(x) \subseteq E\}$ for $E \subseteq \omega$.

If $E \in I_2$, then $X_E \in \mathcal{N}$, i.e., $\mu(X_E) = 0$, by definition of I_2 .

If $E \in I_4$, then $\mu(X_E) = 0$ because $X_E \subseteq \bigcup \{[s] : \varphi(s) \in E \cap K_n\}$ for all $n \in \omega$, $\mu([s]) = 1/2^n$ for $\varphi(s) \in K_n$, and $\lim_{n \in \omega} |E \cap K_n|/2^n = 0$.

(2) Assume that an ideal I on ω is simply summable in a set $B' \subseteq \omega$ by a series $a : B' \rightarrow [0, 1]$. We find an infinite set $B \subseteq B'$ and an ideal $J \leq_{\text{RB}} I \upharpoonright B$ on ω such that $I_0 \subseteq J \subseteq I_4$. Then I is thick by (1) and by Lemma 4.3.

Since a is converging to 0 and $\sum_{k \in B} a_k$ is divergent, by induction we can construct a system of pairwise disjoint nonempty finite sets $B_{n,s} \subseteq B'$ for $n \in \omega$ and $s \in {}^{n_2}$ such that $2^{-(n+1)} \leq \sum_{k \in B_{n,s}} a(k) < 2^{-n}$. Let $B = \bigcup_{n \in \omega} \bigcup_{s \in {}^{n_2}} B_{n,s}$ and let $\varphi : {}^{<\omega} 2 \rightarrow \omega$ be the bijection from Example 4.4. Define $g : B \rightarrow \omega$ by $g(k) = \varphi(s)$ for $k \in B_{|s|,s}$ and $s \in {}^{<\omega} 2$ and let $J = \{E \subseteq \omega : g^{-1}(E) \in I \upharpoonright B\}$ where $I \upharpoonright B = S(a \upharpoonright B)$. Then $J \leq_{\text{RB}} I \upharpoonright B$ because g is finite-to-one.

We show that $I_0 \subseteq J \subseteq I_4$. For every $x \in {}^\omega 2$, $E_x \in J$ because $g^{-1}(E_x) = \bigcup_{n \in \omega} B_{n,x \upharpoonright n}$ and $\sum_{n \in \omega} \sum_{k \in B_{n,x \upharpoonright n}} a_k < \sum_{n \in \omega} 2^{-n} < \infty$. It follows that $I_0 \subseteq J$.

Let $E \in J$ be arbitrary, i.e., $g^{-1}(E) = \bigcup_{\varphi(s) \in E} B_{|s|,s} = \bigcup_{n \in \omega} \bigcup_{\varphi(s) \in E \cap K_n} B_{n,s} \in I$. Then

$$\sum_{n \in \omega} |E \cap K_n|/2^{n+1} = \sum_{n \in \omega} \sum_{\varphi(s) \in E \cap K_n} 2^{-(n+1)} \leq \sum_{n \in \omega} \sum_{\varphi(s) \in E \cap K_n} \sum_{k \in B_{n,s}} a_k < \infty$$

and hence $\lim_{n \rightarrow \infty} |E \cap K_n|/2^n = 0$. Therefore $J \subseteq I_4$.

(3) The function $g : {}^\omega 2 \rightarrow I'_0$ defined by $g(x) = F_x$ is continuous. Then $Z_E = \{x \in {}^\omega 2 : g(x) \subseteq E\}$ for $E \subseteq \omega$.

Let $E \in I'_2$ be arbitrary, i.e., $E \subseteq F(A; B_1, \dots, B_k)$ for some $A \in [{}^\omega 2]^{<\omega}$ and $B_1, \dots, B_k \in [{}^\omega 2]^\omega$. We show that $Z_E \subseteq A$. Let $x \in {}^\omega 2 \setminus A$. Choose distinct $x_i \in B_i \setminus (A \cup \{x\})$ for $i = 1, \dots, k$ and denote $G = \bigcap_{y \in A} (\omega \setminus F_y) \cap \bigcap_{i=1}^k F_{x_i}$. Then $G \cap F_x \neq \emptyset$ because D is independent. Since $G \cap F(A; B_1, \dots, B_k) = \emptyset$ it follows that $F_x \not\subseteq F(A; B_1, \dots, B_k)$ and hence $x \in {}^\omega 2 \setminus Z_E$. Therefore Z_E is finite for every $E \in I'_2$. It follows that every ideal I with $I'_0 \subseteq I \subseteq I'_2$ is thick.

Let $I \supseteq I'_1$ be an arbitrary ideal. Then $\mu(Z_E) = 0$ for every $E \in I$ since otherwise for a closed set $B = Z_E$ of positive measure we have $F(B) \subseteq E \in I$ and $\omega \setminus F(B) = F(\emptyset; B) \in I'_1 \subseteq I$. Therefore I is thick. \square

Remark 4.7. The conclusion of Corollary 4.2 in case of the thick ideals $I = I_0$ and $I = I'_0$ does not require the assumption $\text{non}(\mathcal{N}) < \mathfrak{b}$ and does not need Theorem 1.2. This is due to the fact that for every function $\Phi : [0, 1] \rightarrow {}^\omega \omega$ there is an infinite set $A \subseteq [0, 1]$ such that $\Phi[A]$ is bounded (but the sequences of functions $(*)$ from the proof of Lemma 4.1 witnessing the thickness of I , I -uniformly converges on a set of the form $A = X_E$ and $A = Z_E$, respectively, only if A is finite): If $Z = \Phi([0, 1])$ is an uncountable subset of ${}^\omega \omega$, then Z has an accumulation point z_0 which together with a sequence converging to z_0 gives an infinite countable compact set $Z_0 \subseteq Z$. Then for the infinite set $A = \Phi^{-1}(Z_0)$, $\Phi[A]$ is bounded. If Z is countable, then for some $z_0 \in Z$, the set $A = \Phi^{-1}(\{z_0\})$ is infinite and $\Phi[A]$ is bounded.

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