

**A PROOF OF THE INDEPENDENCE
OF THE AXIOM OF CHOICE
FROM THE BOOLEAN PRIME IDEAL THEOREM**

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ABSTRACT. We present a proof of the Boolean Prime Ideal Theorem in a transitive model of ZF in which the Axiom of Choice does not hold. We omit the argument based on the full Halpern-Läuchli partition theorem and instead we reduce the proof to its elementary case.

Theorem 1 (Halpern and Lévy [2]). *There is a transitive model of ZF in which the Boolean Prime Ideal Theorem holds and the Axiom of Choice fails.*

We assume $V \models \text{ZFC}$ and we consider the following transitive model M (see [3, pages 184–187] or [4, pages 221–223]). Let P be the set of finite functions p such that $\text{dom}(p) \subseteq \omega \times \omega$ and $\text{rng}(p) \subseteq \{0, 1\}$. Let $G \subseteq P$ be a generic set of conditions. For $i \in \omega$ let

$$a_i(n) = \begin{cases} 1, & \text{if } (\exists p \in G) p(i, n) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$A = \{a_i : i \in \omega\},$$

$$M = \text{HOD}^{V[G]}(A).$$

M is a transitive model of ZF and $A \in M$. The Axiom of Choice does not hold in M because the set A is infinite and has no countable subset in M (see [3]).

We prove the Boolean Prime Ideal Theorem in $M = \text{HOD}^{V[G]}(A)$. The present proof uses the same ideas as the proof in [2] but its exposition relies on [3]. We also omit the argument from [2] based on the full Halpern-Läuchli partition theorem [1] and instead we reduce the proof to its elementary case substantiated in [2].

Recall that $[u] = \{x \in {}^\omega 2 : u \subseteq x\}$ for any finite function u such that $\text{dom}(u) \subseteq \omega$ and $\text{rng}(u) \in \{0, 1\}$. For $t \in {}^m(\omega 2)$ and $k \in \omega$, $[t \upharpoonright k] = \prod_{i < m} [t(i) \upharpoonright k]$ denotes a basic clopen set in ${}^m(\omega 2)$.

Lemma 2 (Schema of continuity). *Let $\varphi(x_1, \dots, x_n, s, A)$ be a formula of ZF with no free variables other than x_1, \dots, x_n, s, A . If $x_1, \dots, x_n \in V$, $m \in \omega$, $s \in {}^m A$ is a sequence of distinct members of A , and $\varphi(x_1, \dots, x_n, s, A)$ holds in $V[G]$, then there is a basic clopen set $U \subseteq {}^m(\omega 2)$ with pairwise disjoint projections in ${}^\omega 2$ such that $s \in U$ and $\varphi(x_1, \dots, x_n, t, A)$ holds in $V[G]$ for every $t \in U \cap {}^m A$.*

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Proof. Let W be the set of all one-to-one functions in ${}^m\omega$. For $h \in W$ let $h^* \in {}^mA$ be defined by $h^*(i) = a_{h(i)}$. For $h \in W$ let

$$\begin{aligned} b(h) &= \|\varphi(x_1, \dots, x_n, \dot{h}^*, \dot{A})\|, \\ c(h) &= \bigvee_{k \in \omega} \bigwedge_{z \in W} \neg \|\dot{z}^* \in [\dot{h}^* \upharpoonright k]\| \vee \|\varphi(x_1, \dots, x_n, \dot{z}^*, \dot{A})\| \\ &= \|(\exists k \in \omega)(\forall z \in W) \dot{z}^* \in [\dot{h}^* \upharpoonright k] \rightarrow \varphi(x_1, \dots, x_n, \dot{z}^*, \dot{A})\| \end{aligned}$$

where \dot{h}^* , \dot{z}^* and \dot{A} denote the canonical names for h^* , z^* , and A constructed by means of the canonical names \dot{a}_i for $i \in \omega$. The inequality $b(h) \leq c(h)$ means that if $\varphi(x_1, \dots, x_n, s, A)$ holds in $V[G]$ for $s = h^*$, then there is $k \in \omega$ such that the conclusion of the lemma holds for the clopen set $U = [s \upharpoonright k]$. Then, since s is one-to-one, the projections of U are pairwise disjoint if k is sufficiently large. We prove $b(h) \leq c(h)$ for all $h \in W$.

Let $p' \in P$ be arbitrary such that $p' \leq b(h)$ and we find $p \leq p'$ such that $p \leq c(h)$. Extend p' to a condition $p \supseteq p'$ so that $\text{dom}(p) = k \times k$ for some $k \in \omega$, $\text{rng}(h) \subseteq k$, and for all $i < j < k$ there is $l < k$ such that $p(i, l) \neq p(j, l)$. For every $q \in P$ let q_i be defined by $q_i(j) = q(i, j)$. Then $p_i \in {}^k 2$ for $i < k$ are pairwise incompatible and $p \upharpoonright [\dot{h}^* \upharpoonright k] = \prod_{i < m} [p_{h(i)}]$. We prove that $p \leq c(h)$.

To get a contradiction assume that for some $z \in W$ there is $r \leq p$ such that $r \upharpoonright \dot{z}^* \in [\dot{h}^* \upharpoonright k]$ and $r \upharpoonright \neg \varphi(x_1, \dots, x_n, \dot{z}^*, \dot{A})$; the former assumption is equivalent to saying that $r_{z(i)} \upharpoonright k = p_{h(i)}$ for all $i < m$. If $z(i) \neq h(i)$, then $z(i) > h(i)$ because p_j for $j < k$ are pairwise incompatible. Let π be the permutation of ω that interchanges $h(i)$ and $z(i)$ for all $i < m$ and $\pi(j) = j$ otherwise. The permutation π induces an automorphism of P and an automorphism of V^P : For $p, q \in P$, $q = \pi(p)$ if $q(\pi(i), j) = p(i, j)$. By the symmetry lemma $\pi(r) \upharpoonright \neg \varphi(x_1, \dots, x_n, \pi(\dot{z}^*), \pi(\dot{A}))$ which is impossible because $\pi(r)$ and p are compatible, $\pi(\dot{z}^*) = \dot{h}^*$, $\pi(\dot{A}) = \dot{A}$, and $p \upharpoonright \varphi(x_1, \dots, x_n, \dot{h}^*, \dot{A})$. This contradiction proves that there is no such r and hence $p \leq c(h)$. \square

Let $F \in [A]^m$. We say that a sequence $\langle U_i : i < m \rangle$ of pairwise disjoint basic open sets in ${}^\omega 2$ distinguishes F , if $|F \cap U_i| = 1$ for all $i < m$.

Corollary 3. *Let $\varphi(x_1, \dots, x_n, F)$ be a formula of ZF with no free variables other than x_1, \dots, x_n, F . If $s \in {}^{<\omega}A$, $x_1, \dots, x_n \in \text{OD}^{V[G]}[A, s]$, $F' \subseteq A \setminus \text{rng}(s)$ is a finite set, $m = |F'|$, and $\varphi(x_1, \dots, x_n, F')$ holds in $V[G]$, then there is a sequence of basic open sets $\langle U_i : i < m \rangle$ in ${}^\omega 2$ disjoint from $\text{rng}(s)$ and distinguishing members of F' such that $\varphi(x_1, \dots, x_n, F)$ holds in $V[G]$ for every $F \in [A]^m$ such that $|F \cap U_i| = 1$ for all $i < m$.*

Proof. Assume $|s| = k$ and let $t' : m \rightarrow F'$ be any one-to-one enumeration. There is a formula ψ such that for some ordinals $\alpha_1, \dots, \alpha_r$,

$$\begin{aligned} V[G] \models (\forall t) \psi(\alpha_1, \dots, \alpha_r, s \hat{\ } t, A) \rightarrow \varphi(x_1, \dots, x_n, \text{rng}(t)), \text{ and} \\ V[G] \models \psi(\alpha_1, \dots, \alpha_r, s \hat{\ } t', A). \end{aligned}$$

By Lemma 2 there is a disjoint sequence of basic open sets $\langle V_i : i < k + m \rangle$ in ${}^\omega 2$ such that $s \frown t' \in \prod_{i < k+m} V_i$ and $\psi(\alpha_1, \dots, \alpha_r, t, A)$ holds in $V[G]$ for every $t \in \prod_{i < k+m} V_i$. Take $U_i = V_{k+i}$ for $i < m$. \square

Now we prove the Boolean Prime Ideal Theorem in $M = \text{HOD}^{V[G]}(A)$.

Let $(B, \vee, \wedge, -, 0, 1)$ be a Boolean algebra in M . Then there is $f \in {}^{<\omega} A$ such that $B \in \text{OD}^{V[G]}[A, f]$. The class $\text{OD}^{V[G]}[A, f]$ has a well-ordering ordinal-definable from A and f . Using this well-ordering by transfinite recursion we can define a proper ideal $I \subseteq B$ maximal ordinal-definable from A and f . Hence, for every $x \in B$ which is ordinal-definable from A and f , either $x \in I$ or $-x \in I$. Clearly $I \in M$ because $I \subseteq B \subseteq M$. We prove that I is a prime ideal of B in M .

Suppose that I is not prime and let $k \in \omega$ be the least natural number such that for some $h' \in {}^{k+1} A$ there is an $x \in \text{OD}^{V[G]}[A, f \frown h']$ such that $x \in B \setminus I$ and $-x \in B \setminus I$. Let $a' = h'(k)$ and $h = h' \upharpoonright k$. Then $B \in \text{OD}^{V[G]}[A, f \frown h]$ and by minimality of k it is obvious that $a' \notin \text{rng}(f) \cup \text{rng}(h)$ and I is a maximal ideal of B in $\text{OD}^{V[G]}[A, f \frown h]$ because I is a prime ideal there. There is a formula φ such that

$$x = \{u \in V[G] : V[G] \models \varphi(u, \alpha_1, \dots, \alpha_n, f \frown h, a', A)\}$$

for some ordinals $\alpha_1, \dots, \alpha_n$. Since $f \frown h, \alpha_1, \dots, \alpha_n$ are fixed throughout the proof we shall denote

$$d(a) = \{u \in V[G] : V[G] \models \varphi(u, \alpha_1, \dots, \alpha_n, f \frown h, a, A)\}.$$

Hence

$$d(a') \in B \setminus I \quad \text{and} \quad -d(a') \in B \setminus I.$$

By Corollary 3 there is a basic open set $U \subseteq {}^\omega 2$ such that $a' \in U, U \cap \text{rng}(f \frown h) = \emptyset$, and

$$(1) \quad (\forall a \in U \cap A) \quad -d(a) \in B \setminus I \quad \text{and} \quad d(a) \in B \setminus I.$$

The ideal of B generated by $I \cup \{d(a) : a \in U \cap A\}$ is in $\text{OD}^{V[G]}[A, f \frown h]$ and it coincides with B by maximality of I . Therefore for some finite set $F'_1 \subseteq U \cap A$ we have $\bigwedge_{a \in F'_1} -d(a) \in I$. Similarly, if we consider the ideal generated by $I \cup \{-d(a) : a \in U \cap A\}$ we obtain a finite set $F'_2 \subseteq U \cap A$ such that $\bigwedge_{a \in F'_2} d(a) \in I$. Denote $F' = F'_1 \cup F'_2$ and $m = |F'|$. Then

$$\bigwedge_{a \in F'} -d(a) \in I \quad \text{and} \quad \bigwedge_{a \in F'} d(a) \in I.$$

By Corollary 3, there is a sequence of basic open sets $\langle U_i : i < m \rangle$ distinguishing F' , such that each set U_i is a subset of U (this is possible because $F' \subseteq U$), hence disjoint from $\text{rng}(f \frown h)$, and for every $F \in [A]^m$ such that $(\forall i < m) F \cap U_i \neq \emptyset$,

$$(2) \quad \bigwedge_{a \in F} -d(a) \in I \quad \text{and} \quad \bigwedge_{a \in F} d(a) \in I.$$

For every $i < m$, (1) holds with U replaced by U_i because $U_i \subseteq U$. Replacing U by U_i in the argument that leads to (2) we obtain a sequence of pairwise disjoint

basic open sets $\langle U_{i,j} : j < m_i \rangle$ which are subsets of U_i such that for every $i < m$, and for every $F \subseteq A \cap U$ such that $(\forall j < m_i) F \cap U_{i,j} \neq \emptyset$,

$$(3) \quad \bigwedge_{a \in F} \neg d(a) \in I \text{ and } \bigwedge_{a \in F} d(a) \in I.$$

The system $S = \{U_{i,j} : i < m \text{ and } j < m_i\}$ is a pairwise disjoint system of basic clopen sets in ${}^\omega 2$ and A is a dense subset of ${}^\omega 2$. Let $y \subseteq A \cap U$ be a finite set of the size $|S|$ such that $(\forall V \in S) |y \cap V| = 1$. Then for every $z \subseteq y$,

$$(4) \quad \bigwedge_{a \in z} d(a) \wedge \bigwedge_{a \in y \setminus z} \neg d(a) \in I.$$

To prove this let us consider these two possibilities:

(i) For every $i < m$, $z \cap U_i \neq \emptyset$. Then by (2), $\bigwedge_{a \in z} d(a) \in I$ and hence (4) holds.

(ii) There is $i < m$ such that $z \cap U_i = \emptyset$. Then $(\forall j < m_i) (y \setminus z) \cap U_{i,j} \neq \emptyset$, by (3), $\bigwedge_{a \in y \setminus z} \neg d(a) \in I$, and hence (4) holds.

Using (4) we obtain a contradiction as follows: $1 = \bigwedge_{a \in y} (d(a) \vee \neg d(a)) = \bigvee_{z \subseteq y} [\bigwedge_{a \in z} d(a) \wedge \bigwedge_{a \in y \setminus z} \neg d(a)] \in I$. This contradiction proves that I is prime in \bar{M} .

REFERENCES

- [1] J. D. Halpern and H. Läuchli, *A partition theorem*, Trans. Amer. Math. Soc. **124** (1966), 360–367.
- [2] J. D. Halpern and A. Lévy, *The Boolean Prime Ideal Theorem does not imply the Axiom of Choice*, In: *Axiomatic Set Theory, Proceedings of Symposia in Pure Mathematics*, vol. XIII, Part I, pp. 83–134, AMS, Providence, 1971.
- [3] T. Jech, *Set theory*, Academic Press, 1978.
- [4] T. Jech, *Set theory*, the third millennium edition, revised and expanded, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003

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