STRONGLY DOMINATING SETS OF REALS

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Abstract. We analyze the structure of strongly dominating sets of reals introduced in [4]. We prove that for every $\kappa < \mathfrak{b}$ a κ-Suslin set $A \subseteq {}^\omega\omega$ is strongly dominating if and only if A has a Laver perfect subset. We also investigate the structure of the class l of Baire sets for the Laver category base and compare the σ-ideal of sets which are not strongly dominating with the Laver ideal l^0.

1. Introduction

Strongly dominating sets were introduced in the paper [4] where the Borel determinacy was used to show that for a Borel set $A \subseteq {}^\omega\omega$ the following conditions are equivalent: (1) A is strongly dominating; (2) A has a Laver perfect subset; and (3) player I has a winning strategy in the domination game for the set A. In the present paper we prove, by elementary means, that conditions (1)–(3) are equivalent for any analytic and more generally for any κ-Suslin set A with $\kappa < \mathfrak{b}$. A motivation for this research is the paper [6] in which it is proved that an analytic set $A \subseteq {}^\omega\omega$ is unbounded if and only if it has a superperfect (i.e., Miller perfect) subset. Let us mention also that the notion of a strongly dominating set is similar to but strictly stronger than the notion of a dominating set. In [12] it is proved that every analytic dominating set in ${}^\omega\omega$ contains a uniform superperfect set and that the concept of uniform superperfect set does not suffice to characterize dominating analytic sets in general. The question of the existence of perfect dominating sets in analytic dominating sets was solved in [2]. The paper is organized as follows:

In Section 2 we introduce the notion of a strongly dominating set. For practical reasons we define strongly dominating sets by using an equivalent expression from [4]. We discuss the original definition at the end of Section 3. We show that the system $\mathcal{D}$ of sets which are not strongly dominating is a σ-ideal orthogonal to the σ-ideals of meager sets and of measure zero sets. In Section 3 we examine some combinatorial properties of strongly dominating sets.

In Section 4 we obtain the equivalence of conditions (1) and (2) for κ-Suslin sets (Theorem 4.4) and in Section 5 we show also their equivalence to condition (3). We prove that every analytic domination game is determined.

In Section 6 we consider the Laver σ-ideal l^0 and the σ-algebra l. Assuming $\mathfrak{b} = \mathfrak{c}$, these systems have some regularity properties because the Laver perfect sets form a category base. We show that $\mathcal{D}$ is a proper subset of l^0 and we consider

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several subsystems of l whose intersection with l^0 coincide with $\mathcal{D}$. We show that, at least consistently, almost all of them are pairwise distinct.

In Section 7 we present properties of a natural partition of the Baire space into Laver perfect sets.

In Section 8 we consider several partial quasi-orderings of good sequences of Laver perfect sets. We show that the distributivity invariants for two of them coincide and provide a lower estimation for $\mathrm{add}(l^0)$.

We use the standard notation. By symbols $\mathrm{add}(\mathcal{I})$, $\mathrm{cov}(\mathcal{I})$, $\mathrm{non}(\mathcal{I})$, and $\mathrm{cof}(\mathcal{I})$ we denote respectively the additivity, the covering, the uniformity, and the cofinality of an ideal $\mathcal{I}$; $\mathfrak{b}$ is the least cardinality of an unbounded family and $\mathfrak{d}$ is the least cardinality of a dominating family in ${}^{\omega}\omega$ with respect to eventual dominance; and $\mathfrak{t}$ is the least cardinality of a tower in $[\omega]^{\omega}$ (see e.g., [1, 3]).

2. Strongly dominating sets and the ideal $\mathcal{D}$

A tree $q \subseteq {}^{<\omega}\omega$ is said to be a Laver tree, if there is $s \in q$ (a stem of q) such that for every $t \in q$ either $t \subseteq s$ or $s \subseteq t$ and for every $t \in q$ with $s \subseteq t$ the set $\mathrm{br}_q(t) = \{n \in \omega : t^\frown\langle n\rangle \in q\}$ is infinite. The set of branches

$$[q] = \{x \in {}^{\omega}\omega : (\forall k \in \omega)\ x{\restriction}k \in q\}$$

of a Laver tree q with stem s is called a Laver perfect set with stem s. For a tree $q \subseteq {}^{<\omega}\omega$ and $t \in q$ let $(q)_t$ be the subtree of q defined by

$$(q)_t = \{s \in q : s \subseteq t \text{ or } t \subseteq s\}.$$

Let us recall that a set $A \subseteq {}^{\omega}\omega$ is dominating, if for every $y \in {}^{\omega}\omega$ there is $x \in A$ such that $y(k) \le x(k)$ for all but finitely many $k \in \omega$. Let us consider the following dominating properties of a set $A \subseteq {}^{\omega}\omega$.

$$D(A) \leftrightarrow (\forall f : {}^{<\omega}\omega \to \omega)(\exists x \in A)(\forall^{\infty} k \in \omega)\ x(k) \ge f(x{\restriction}k),$$

$$D_s(A) \leftrightarrow (\forall f : {}^{<\omega}\omega \to \omega)(\exists x \in A \cap [s])(\forall k \ge |s|)\ x(k) \ge f(x{\restriction}k).$$

We say that a set $A \subseteq {}^{\omega}\omega$ is strongly dominating (see [4, Lemma 2.3 (4)]), if $D(A)$ holds. We discuss the original Definition 2.1 of [4] at the end of the next section.

Let us define

$$\mathcal{D} = \{A \subseteq {}^{\omega}\omega : A \text{ is not strongly dominating}\}$$

and for $f : {}^{<\omega}\omega \to \omega$ let

$$B(f) = \{x \in {}^{\omega}\omega : (\exists^{\infty} k \in \omega)\ x(k) < f(x{\restriction}k)\}.$$

Remark 2.1. Assume that $A, B \subseteq {}^{\omega}\omega$.

(1) If $D(B)$ holds for some $B \subseteq A$, then $D(A)$ holds.
(2) If $D(A)$ holds, then $D(A \setminus B(f))$ holds for every $f : {}^{<\omega}\omega \to \omega$.
(3) If $D(A)$ holds, then A is a dominating family.

Proof. (3) For $g \in {}^{\omega}\omega$ define $f(s) = g(|s|)$ for $s \in {}^{\omega}\omega$. There is $x \in A$ such that for all but finitely many $k \in \omega$, $x(k) \ge f(x{\restriction}k) = g(k)$. □

Example 2.2. The following dominating sets are not strongly dominating:

$$A_0 = \{x \in {}^{\omega}\omega : (\forall k \in \omega)\ x(2k+1) = x(2k)\},$$

$$A_1 = \{x \in {}^{\omega}\omega : (\exists^{\infty} k \in \omega)\ x(2k+1) \le x(2k)\}.$$

A_0 is a closed subset and A_1 is a G_δ dense subset of ${}^\omega\omega$. Actually, $A_0 \subseteq A_1$ and, if $f(s) = s(k) + 1$ for $s \in {}^{k+1}\omega$ and $k \in \omega$, then for every $x \in A_1$, $(\exists^\infty k \in \omega)$ $x(2k+1) \le x(2k) < f(x{\restriction}(2k+1))$.

Lemma 2.3 ([13])**.** *$\mathcal{D}$ is a σ-ideal on ${}^\omega\omega$ with a base consisting of G_δ sets.*

Proof. For each $f : {}^{<\omega}\omega \to \omega$ the set $B(f)$ is a G_δ set in $\mathcal{D}$ and, by definition, for each $A \in \mathcal{D}$ there is f such that $A \subseteq B(f)$.

If $f_n : {}^{<\omega}\omega \to \omega$ for $n \in \omega$ and $f : {}^{<\omega}\omega \to \omega$ is defined by $f(s) = \max\{f_n(s) : n \le |s|\}$, then $\bigcup_{n\in\omega} B(f_n) \subseteq B(f)$. □

Lemma 2.4. $\mathrm{add}(\mathcal{D}) = \mathrm{cov}(\mathcal{D}) = \mathfrak{b}$ *and* $\mathrm{cof}(\mathcal{D}) = \mathrm{non}(\mathcal{D}) = \mathfrak{d}$*.*

Proof. $\mathrm{cov}(\mathcal{D}) \ge \mathrm{add}(\mathcal{D}) \ge \mathfrak{b}$ and $\mathrm{non}(\mathcal{D}) \le \mathrm{cof}(\mathcal{D}) \le \mathfrak{d}$ because

$$f \le^* f' \to B(f) \subseteq B(f') \quad \text{for } f, f' : {}^{<\omega}\omega \to \omega.$$

Conversely, $\mathrm{cov}(\mathcal{D}) \le \mathfrak{b}$ and $\mathrm{non}(\mathcal{D}) \ge \mathfrak{d}$ because there is $\varphi : {}^\omega\omega \to \mathcal{D}$ such that

$$f' \notin \varphi(f) \to f \le^* f' \quad \text{for } f, f' \in {}^\omega\omega.$$

For $f \in {}^\omega\omega$ let $\varphi(f) = B(g)$ where $g(s) = f(|s|)$ for $s \in {}^{<\omega}\omega$. If $f' \notin \varphi(f) = B(g)$, then $(\forall^\infty k \in \omega)$ $f'(k) \ge g(f'{\restriction}k) = f(k)$. □

Denote by $\mathcal{M}$ the ideal of meager subsets of ${}^\omega\omega$ and by $\mathcal{N}_\mu$ the ideal of sets of measure 0 for a (σ-additive) Borel measure μ on ${}^\omega\omega$. The ideals $\mathcal{M}$ and $\mathcal{N}_\mu$ are known to be orthogonal for all "reasonable" μ, i.e., there are sets $M \in \mathcal{M}$ and $N \in \mathcal{N}_\mu$ such that $M \cup N = {}^\omega\omega$.

Lemma 2.5. *The ideal $\mathcal{D}$ is orthogonal to the ideals $\mathcal{M}$ and $\mathcal{N}_\mu$ for every finite Borel measure μ.*

Proof. $\mathcal{D}$ and $\mathcal{M}$ are orthogonal because $B(f) \in \mathcal{D}$ and $B(f)$ is a G_δ dense subset of ${}^\omega\omega$ whenever $f : {}^{<\omega}\omega \to \omega$ and $B(f) \ne \emptyset$. Assume that μ is a Borel measure and $\mu({}^\omega\omega) = 1$. Denote $A_{k,m} = \{x \in {}^\omega\omega : x(k) < m\}$ and find $g \in {}^\omega\omega$ such that $\mu(A_{k,g(k)}) > 1 - 2^{-k}$ for all $k \in \omega$. Then for $f(s) = g(|s|)$ we have $\mu(B(f)) = 1$ because

$$B(f) = \bigcap_{n\in\omega} \bigcup_{k\ge n} A_{k,g(k)}.$$

Therefore $\mathcal{D}$ and $\mathcal{N}_\mu$ are orthogonal. □

3. Combinatorial properties of strongly dominating sets

For a set $A \subseteq {}^\omega\omega$ and $s \in {}^{<\omega}\omega$ we define

$$\Phi(A) = \{x \in {}^\omega\omega : (\forall^\infty k \in \omega)\ D_{x\restriction k}(A)\},$$

$$\Phi_s(A) = \{x \in [s] : (\forall k \ge |s|)\ D_{x\restriction k}(A)\},$$

$$p_s(A) = \{x{\restriction}k : x \in \Phi_s(A) \text{ and } k \in \omega\}.$$

The set $\Phi_s(A)$ is closed and, if $\Phi_s(A) \ne \emptyset$, then $p_s(A) \subseteq {}^{<\omega}\omega$ is a tree and $\Phi_s(A) = [p_s(A)]$.

Lemma 3.1. *If $A \subseteq {}^\omega\omega$, then for every set $B \in \mathcal{D}$ and for every $s \in {}^{<\omega}\omega$, $D_s(A) \leftrightarrow D_s(A \setminus B)$.*

Proof. Let $f_0 : {}^{<\omega}\omega \to \omega$ be such that $B \subseteq B(f_0)$. Assume that $D_s(A)$ holds and $f : {}^{<\omega}\omega \to \omega$. Find $x \in A \cap [s]$ such that $(\forall k \ge |s|)$ $x(k) \ge \max\{f(x{\restriction}k), f_0(x{\restriction}k)\}$. Then $x \notin B(f_0)$ and hence $D_s(A \setminus B)$ holds. □

Lemma 3.2. *Assume that* $A \subseteq {}^{\omega}\omega$.

(1) $A \setminus \Phi(A) \in \mathcal{D}$.
(2) $D(A) \leftrightarrow (\exists s \in {}^{<\omega}\omega)\ D_s(A)$.
(3) $D_s(A) \leftrightarrow (\exists^\infty n \in \omega)\ D_{s^\frown\langle n\rangle}(A)$.

Proof. Let $E_A = \{s \in {}^{<\omega}\omega : D_s(A)\}$ and $E_{A,s} = \{n \in \omega : s^\frown\langle n\rangle \in E_A\}$ for $s \in {}^{<\omega}\omega$. For every $s \in {}^{<\omega}\omega \setminus E_A$ choose $f_s : {}^{<\omega}\omega \to \omega$ such that $(\forall x \in A \cap [s])$ $(\exists k \geq |s|)\ x(k) < f_s(x{\restriction}k)$. Define

$$g_A(s) = \max(\{f_t(s) : t \notin E_A \text{ and } t \subseteq s\} \cup \{0\}),$$
$$h_A(s) = \begin{cases} 0, & \text{if } E_{A,s} \text{ is infinite or } E_{A,s} = \emptyset, \\ (\max E_{A,s}) + 1, & \text{if } E_{A,s} \neq \emptyset \text{ is finite} \end{cases}$$

for $s \in {}^{<\omega}\omega$.

(1) We prove that $A \setminus \Phi(A) \subseteq B(g_A)$. For $x \in A \setminus \Phi(A)$ we can derive the following conditions:

$$(\exists^\infty k \in \omega)\ x{\restriction}k \notin E_A,$$
$$(\exists^\infty k \in \omega)(\forall y \in A \cap [x{\restriction}k])(\exists l \geq k)\ y(l) < f_{x\restriction k}(y{\restriction}l) \leq g_A(y{\restriction}l),$$
$$(\exists^\infty l \in \omega)\ x(l) < g_A(x{\restriction}l).$$

(2) The implication $(\exists s \in {}^{<\omega}\omega)\ D_s(A) \to D(A)$ is obvious. If $E_A = \emptyset$, then for every $x \in A$, $n \in \omega$, and $s = x{\restriction}n$ there is $k \geq n$ such that $x(k) < f_s(x{\restriction}k) \leq g_A(x{\restriction}k)$. Therefore $A \subseteq B(g_A)$ and $D(A)$ does not hold.

(3) Assume that $E_{A,s}$ is infinite. For arbitrary $f : {}^{<\omega}\omega \to \omega$ there are $n \in E_{A,s}$ and $x \in A \cap [s^\frown\langle n\rangle]$ such that $n \geq f(s)$ and $(\forall k \geq |s| + 1)\ x(k) \geq f(x{\restriction}k)$. Clearly $x(k) \geq f(x{\restriction}k)$ also for $k = |s|$ and hence $D_s(A)$ holds.

Assume that $E_{A,s}$ is finite. Let $f(s) = \max\{g_A(s), h_A(s)\}$ and we show that $(\forall x \in A \cap [s])(\exists k \geq |s|)\ x(k) < f(x{\restriction}k)$. Therefore $D_s(A)$ does not hold. Let $x \in A \cap [s]$ and $n = x(|s|)$. If $n \in E_{A,s}$, then for $k = |s|$, $x(k) < h_A(x{\restriction}k) \leq f(x{\restriction}k)$. If $n \notin E_{A,s}$, then by the choice of $f_{s^\frown\langle n\rangle}$ there is $k \geq |s| + 1$ such that $x(k) < f_{s^\frown\langle n\rangle}(x{\restriction}k) \leq g_A(x{\restriction}k) \leq f(x{\restriction}k)$. □

Lemma 3.3. $\Phi(A \cap \Phi(A)) = \Phi(A)$ *and* $\Phi_s(A \cap \Phi(A)) = \Phi_s(A)$ *for all* $s \in {}^{<\omega}\omega$ *and* $A \subseteq {}^{\omega}\omega$.

Proof. The equalities follow by definitions of $\Phi(A)$ and $\Phi_s(A)$, by Lemma 3.1 and by Lemma 3.2 (1). □

Lemma 3.4.

(1) $D_s(A) \leftrightarrow \Phi_s(A) \neq \emptyset \leftrightarrow p_s(A)$ *is a Laver tree with stem* s.
(2) $D_s(A) \leftrightarrow D_s(A \cap \Phi_s(A))$.
(3) *If* $s \subseteq t$ *and* $D_{t\restriction k}(A)$ *holds for all* $|s| \leq k < |t|$, *then* $\Phi_s(A) \cap [t] = \Phi_t(A)$.
(4) *If* $A \subseteq {}^{\omega}\omega$ *is closed, then* $\Phi_s(A) \subseteq A$ *for all* $s \in {}^{<\omega}\omega$.
(5) $\Phi_s(A \cap \Phi_s(A)) = \Phi_s(A) = \Phi_s(\Phi_s(A))$.

Proof. (1) is a consequence of Lemma 3.2 (3).

(2) Assume that $D_s(A)$ holds and $D_s(A \cap \Phi_s(A))$ does not hold. Denote

$$U = \{t \supseteq s : A \cap [t] \neq \emptyset \text{ and } \neg D_t(A) \text{ and } D_u(A) \text{ holds for } s \subseteq u \subsetneq t\}.$$

Then $\{[t] : t \in U\}$ is a disjoint system of basic clopen sets. Choose $g, f_t : {}^{<\omega}\omega \to \omega$ for $t \in U$ such that

$$(\forall x \in A \cap [t])(\exists k \geq |t|)\ x(k) < f_t(x{\restriction}k),$$
$$(\forall x \in A \cap \Phi_s(A))(\exists k \geq |s|)\ x(k) < g(x{\restriction}k).$$

Define $f(u) = \min\{f_t(u) : [u] \cap [t] \neq \emptyset\}$ (where we let $\min \emptyset = 0$ if necessary). Since $D_s(A)$ holds, there is $x \in A \cap [s]$ such that

$$(\forall k \geq |s|)\ x(k) \geq \max\{f(x{\restriction}k), g(x{\restriction}k)\}.$$

If $x \notin \Phi_s(A)$, then $x \in [t]$ for some $t \in U$ and $(\forall k \geq |t|)\ x(k) \geq f(x{\restriction}k) = f_t(x{\restriction}k)$; this disagree with $x(k) < f_t(x{\restriction}k)$ for some $k \geq |t|$. If $x \in \Phi_s(A)$, then $x(k) < g(x{\restriction}k)$ for some $k \geq |s|$ which contradicts the choice of x.

(3) The inclusion $\Phi_s(A) \cap [t] \subseteq \Phi_t(A)$ is obvious. If $x \in \Phi_t(A)$, then $x \in [t]$ and $D_{x\restriction k}(A)$ holds for all $k \geq |t|$ and it holds for $|s| \leq k < |t|$ by the assumption. Therefore $x \in \Phi_s(A) \cap [t]$.

(4) If A is closed, then $\Phi_s(A) \subseteq \mathrm{cl}(A) = A$ because for every $t \in {}^{<\omega}\omega$ such that $[t] \cap \Phi_s(A) \neq \emptyset$ we have $[t] \cap A \neq \emptyset$.

(5) The inclusion $\Phi_s(A \cap \Phi_s(A)) \subseteq \Phi_s(\Phi_s(A))$ is obvious and $\Phi_s(\Phi_s(A)) \subseteq \Phi_s(A)$ because $\Phi_s(A)$ is closed. We prove $\Phi_s(A) \subseteq \Phi_s(A \cap \Phi_s(A))$. Let $x \in \Phi_s(A)$. Then for every $k \geq |s|$, $D_{x\restriction k}(A)$ holds, and then by (2) and (3), $D_{x\restriction k}(A \cap \Phi_{x\restriction k}(A))$ holds and $\Phi_{x\restriction k}(A) = \Phi_s(A) \cap [x{\restriction}k]$ because $s \subseteq x{\restriction}k$ and $D_{x\restriction l}(A)$ holds for all $|s| \leq l < k$. Therefore $D_{x\restriction k}(A \cap \Phi_s(A))$ holds and hence $x \in \Phi_s(A \cap \Phi_s(A))$. □

Remark 3.5. For $A \subseteq {}^{\omega}\omega$ denote

$$F(A) = {}^{\omega}\omega \setminus \bigcup\{[s] : s \in {}^{<\omega}\omega \text{ and } A \cap [s] \in \mathcal{D}\}.$$

If $F(A) \neq \emptyset$, then $F(A)$ is a perfect set, $\Phi(A)$ is an F_σ dense subset of $F(A)$, and $\mathrm{cl}(\Phi(A)) = \mathrm{cl}(A \cap \Phi(A)) = F(A)$. By Lemma 2.3 and Lemma 3.2 we have

$$\begin{aligned} D(A \cap [s]) &\leftrightarrow D(A \cap F(A) \cap [s]) \leftrightarrow D(A \cap \Phi(A) \cap [s]) \\ &\leftrightarrow F(A) \cap [s] \neq \emptyset \leftrightarrow \Phi(A) \cap [s] \neq \emptyset. \end{aligned}$$

Example 3.6. There is an open set $A \subseteq {}^{\omega}\omega$ and a Laver tree q with stem s such that $[q] \subseteq \Phi_s(A)$ and $[q] \cap A = \emptyset$. For example, if $A = {}^{\omega}\omega \setminus [q]$, where q is the Laver tree of sequences of even numbers, then $\Phi_\emptyset(A) = {}^{\omega}\omega$.

For the sake of completeness let us mention that in [4] the following equivalent expression for a strongly dominating set A (here denoted by $D'(A)$) was primarily considered:

$$D'(A) \leftrightarrow (\forall f \in {}^{\omega}\omega)(\exists x \in A)(\forall^\infty k \in \omega)\ x(k+1) \geq f(x(k)).$$

For $s \in {}^{<\omega}\omega$ let us define also

$$D'_s(A) \leftrightarrow (\forall f \in {}^{\omega}\omega)(\exists x \in A \cap [s])(\forall k \geq |s| \dot{-} 1)\ x(k+1) \geq f(x(k))$$

where $n \dot{-} 1 = n - 1$ if $n > 0$ and otherwise $n \dot{-} 1 = 0$.

Lemma 3.7. *Assume that* $A \subseteq {}^{\omega}\omega$.

(1) $D'(A) \leftrightarrow D(A)$.
(2) $D'_s(A) \leftrightarrow D_s(A)$ *for all* $s \in {}^{<\omega}\omega \setminus \{\emptyset\}$.
(3) $D'(A) \leftrightarrow (\exists s \in {}^{<\omega}\omega)\ D'_s(A)$.

Proof. (1) Assume that $D(A)$ holds and let $f : \omega \to \omega$ be given. Define $F(\emptyset) = 0$ and $F(t) = f(t(k))$ for $t \in {}^{k+1}\omega$ and $k \in \omega$. Find $x \in A$ such that $(\forall^\infty k \in \omega)$ $x(k) \geq F(x\restriction k)$. Then $(\forall^\infty k \in \omega)$ $x(k+1) \geq F(x\restriction(k+1)) = f(x(k))$ and hence $D'(A)$ holds.

Assume that $D'(A)$ holds and let $F : {}^{<\omega}\omega \to \omega$ be given. Define $f(n) = \max\{F(t) : t \in {}^{\leq n}(n+1)\} + 2n + 1$ and find $x \in A$ such that $(\forall^\infty k \in \omega)$ $x(k+1) \geq f(x(k))$. Then for all but finitely many $k \in \omega$, $x(k+1) > x(k) \geq k+1$, then for all but finitely many $k \in \omega$, $x\restriction(k+1) \in {}^{\leq n}(n+1)$ for $n = x(k)$, and hence $x(k+1) \geq f(x(k)) \geq F(x\restriction(k+1))$. Therefore $D(A)$ holds.

(2) Assume that $D_s(A)$ holds and let $f : \omega \to \omega$ be given. Define $F(\emptyset) = 0$ and $F(t) = f(t(k))$ for $t \in {}^{k+1}\omega$ and $k \in \omega$. Find $x \in A \cap [s]$ such that $(\forall k \geq |s|)$ $x(k) \geq F(x\restriction k)$. Then $(\forall k \geq |s| \dot{-} 1)$ $x(k+1) \geq F(x\restriction(k+1)) = f(x(k))$ and hence $D'_s(A)$ holds.

Assume that $D'_s(A)$ holds for an $s \neq \emptyset$ and let $F : {}^{<\omega}\omega \to \omega$ be given. We can assume that $u(i) < F(u) \leq F(v)$ whenever $u \subseteq v$ and $i \in \mathrm{dom}(u)$. Find $m \in \omega$ such that $s \in {}^{<m}m$ and define $f(n) = \max\{F(t) : t \in {}^{n+m}((n+1)\cup m)\}$. Find $x \in A \cap [s]$ such that $(\forall k \geq |s| \dot{-} 1)$ $x(k+1) \geq f(x(k))$. Then for all $k \geq |s|-1$, $x(k+1) > x(k)$ and $k+1 \leq (x(k)+1)+(m-1)$, hence $x\restriction(k+1) \in {}^{\leq x(k)+m}((x(k)+1)\cup m)$. Therefore $(\forall k \geq |s|-1)$ $x(k+1) \geq f(x(k)) \geq F(x\restriction(k+1))$ and hence $D_s(A)$ holds.

(3) The implication $(\exists s \in {}^{<\omega}\omega)$ $D'_s(A) \to D'(A)$ is obvious. Assume that $D'(A)$ holds. Then $D(A)$ holds by (1), by Lemma 3.2, $D_s(A)$ holds for some $s \neq \emptyset$, and by (2), $D'_s(A)$ holds for the same s. □

For $f \in {}^\omega\omega$ let

$$B'(f) = \{x \in {}^\omega\omega : (\exists^\infty k \in \omega)\ x(k+1) < f(x(k))\}.$$

By Lemma 3.7 (1), the family $\{B'(f) : f \in {}^\omega\omega\}$ is a base of the ideal $\mathcal{D}$. Therefore for every $g : {}^{<\omega}\omega \to \omega$ there is $f : \omega \to \omega$ such that $B(g) \subseteq B'(f)$ and, vice versa, for every f there is g such that $B'(f) \subseteq B(g)$.

4. Analytic strongly dominating sets

A subset of a Polish space X is κ-Suslin, if it is a projection of a closed subset of $X \times {}^\omega\kappa$, and equivalently, if it is a continuous image of ${}^\omega\kappa$, see [10]. To simplify the notation we say that a set is $<\lambda$-Suslin, if it is κ-Suslin for some $\kappa < \lambda$.

Let $\mathbb{L}(s)$ denote the family of all Laver trees $q \subseteq {}^{<\omega}\omega$ with stem s and let $\mathbb{L}$ denote the family of all Laver trees.

For a family $\mathcal{A} \subseteq \mathcal{P}({}^\omega\omega)$ by induction on $\alpha < \omega_1$ we define

$$S_{\mathcal{A},0} = \{s \in {}^{<\omega}\omega : (\exists B \in \mathcal{A})\ D_s(B)\},$$
$$S_{\mathcal{A},\alpha} = \{s \in {}^{<\omega}\omega : (\exists^\infty k \in \omega)\ s^\frown\langle k\rangle \in \textstyle\bigcup_{\beta<\alpha} S_{\mathcal{A},\beta}\} \quad \text{for } \alpha > 0,$$
$$S_{\mathcal{A}} = \textstyle\bigcup_{\alpha<\omega_1} S_{\mathcal{A},\alpha},$$
$$\rho_{\mathcal{A}}(s) = \min\{\alpha \leq \omega_1 : s \in S_{\mathcal{A},\alpha} \text{ or } \alpha = \omega_1\} \quad \text{for } s \in {}^{<\omega}\omega.$$

Lemma 4.1. *If $\mathcal{A} \subseteq \mathcal{P}({}^\omega\omega)$ and $|\mathcal{A}| < \mathfrak{b}$, then $D_s(\bigcup\mathcal{A})$ holds if and only if $\rho_{\mathcal{A}}(s) < \omega_1$.*

Proof. Denote $A = \bigcup\mathcal{A}$. Assume $\rho_{\mathcal{A}}(s) < \omega_1$. If $\rho_{\mathcal{A}}(s) = 0$, then $D_s(A)$ follows by some $D_s(B)$ for a $B \in \mathcal{A}$; if $\rho_{\mathcal{A}}(s) > 0$, then $\rho_{\mathcal{A}}(s^\frown\langle k\rangle) < \rho_{\mathcal{A}}(s)$ for infinitely many $k \in \omega$ and, by induction hypothesis, $(\exists^\infty k \in \omega)$ $D_{s^\frown\langle k\rangle}(A)$ holds. Then $D_s(A)$ holds by Lemma 3.2.

To prove the converse implication by a contradiction, let us assume that $D_s(A)$ holds and $\rho_{\mathcal{A}}(s) = \omega_1$. By Lemma 3.2 and Lemma 2.4, $A \setminus \bigcup_{B\in\mathcal{A}} \Phi(B) \subseteq \bigcup_{B\in\mathcal{A}}(B \setminus \Phi(B)) \in \mathcal{D}$, and therefore, by lemma 3.1, $D_s(A \cap \bigcup_{B\in\mathcal{A}} \Phi(B))$ holds. Define $f : {}^{<\omega}\omega \to \omega$ by

$$f(t) = \begin{cases} \min\{m \in \omega : (\forall k \geq m)\ \rho_{\mathcal{A}}(t^\frown\langle k\rangle) = \omega_1\}, & \text{if } \rho_{\mathcal{A}}(t) = \omega_1, \\ 0, & \text{otherwise.} \end{cases}$$

There is $x \in A \cap \bigcup_{B\in\mathcal{A}} \Phi(B) \cap [s]$ such that $f(x{\restriction}k) \leq x(k)$ for all $k \geq |s|$. Since $\rho_{\mathcal{A}}(s) = \omega_1$ and $x \in [s]$, by definition of f it follows that $\rho_{\mathcal{A}}(x{\restriction}k) = \omega_1$ for all $k \geq |s|$. On the other hand $x \in \Phi(B)$ for some $B \in \mathcal{A}$ and hence $\rho_{\mathcal{A}}(x{\restriction}k) = 0$ for almost all $k \in \omega$. □

The lemma allows inductive constructions over rank $\rho_{\mathcal{A}}$. As an application we give the proof of the following theorem. Denote

$$\mathbb{M}_0 = \{A \subseteq {}^\omega\omega : (\forall s \in {}^{<\omega}\omega)(D_s(A) \to (\exists q \in \mathbb{L}(s))\ [q] \subseteq A)\}.$$

Theorem 4.2. *If $\mathcal{A} \subseteq \mathbb{M}_0$ has cardinality $< \mathfrak{b}$, then $\bigcup \mathcal{A} \in \mathbb{M}_0$.*

Proof. Let $\mathcal{A} \subseteq \mathbb{M}_0$ have cardinality $< \mathfrak{b}$ and let $A = \bigcup \mathcal{A}$. By Lemma 4.1 it is enough to prove by induction on $\alpha < \omega_1$ that if $\rho_{\mathcal{A}}(s) = \alpha$, then there is a Laver tree q with stem s such that $[q] \subseteq A$. If $\rho_{\mathcal{A}}(s) = 0$, then $D_s(B)$ holds for some $B \in \mathcal{A}$ and then the existence of a such q easily follows. Assume $0 < \rho_{\mathcal{A}}(s) < \omega_1$. There exists an infinite set $a \subseteq \omega$ such that $\rho_{\mathcal{A}}(s^\frown\langle k\rangle) < \rho_{\mathcal{A}}(s)$ for all $k \in a$. Applying the induction hypothesis we find Laver trees q_k with stem $s^\frown\langle k\rangle$ for $k \in a$ such that $[q_k] \subseteq A$. Then $q = \bigcup_{k\in a} q_k$ is a Laver tree with stem s and $[q] \subseteq A$. □

Let $\mathcal{A} \subseteq \mathcal{P}({}^\omega\omega)$. A tree $q \subseteq {}^{<\omega}\omega$ is called an $\mathcal{A}$-tree with stem s, if

(1) q is well-founded, i.e., q has no infinite branch,
(2) for every $t \in q$ either $t \subseteq s$ or $s \subseteq t$,
(3) for every $t \in q$ with $s \subseteq t$ either $\mathrm{br}_q(t) \in [\omega]^\omega$ or $\mathrm{br}_q(t) = \emptyset$, and
(4) for every $t \in q$ with $\mathrm{br}_q(t) = \emptyset$ there is $B \in \mathcal{A}$ such that $D_t(B)$ holds.

For every $\mathcal{A}$-tree q with stem s there is a rank function $\rho : \{t \in q : s \subseteq t\} \to \omega_1$ inductively defined by

$$\rho(t) = \sup(\{0\} \cup \{\rho(t^\frown\langle n\rangle) + 1 : n \in \mathrm{br}_q(t)\}).$$

Then $\rho_{\mathcal{A}}(s) < \omega_1$ because $\rho_{\mathcal{A}}(s) \leq \rho(s) < \omega_1$ ($\rho_{\mathcal{A}}$ is defined before Lemma 4.1). Conversely, if $\rho_{\mathcal{A}}(s) < \omega_1$, then

$$q = \{t \in {}^{<\omega}\omega : t \subseteq s \text{ or } [s \subseteq t \text{ and } \forall k\ |s| \leq k < |t| \to \rho_{\mathcal{A}}(t{\restriction}(k+1)) < \rho_{\mathcal{A}}(t{\restriction}k)]\}$$

is an $\mathcal{A}$-tree. Therefore we can express Lemma 4.1 in the following form:

Lemma 4.3. *If $\mathcal{A} \subseteq \mathcal{P}({}^\omega\omega)$ and $|\mathcal{A}| < \mathfrak{b}$, then $D_s(\bigcup \mathcal{A})$ holds if and only if there is an $\mathcal{A}$-tree with stem s.* □

Theorem 4.4. *If A is a $<\mathfrak{b}$-Suslin set, then $A \in \mathbb{M}_0$.*

Proof. Assume that $A \subseteq {}^\omega\omega$ is a strongly dominating κ-Suslin set with $\kappa < \mathfrak{b}$. Let $f : {}^\omega\kappa \to A$ be a continuous surjection and let $s \in {}^{<\omega}\omega$ be such that $D_s(A)$ holds. We find a Laver tree q with stem s such that $[q] \subseteq A$.

Denote $A_\sigma = f([\sigma])$ for $\sigma \in {}^{<\omega}\kappa$; hence $A_\emptyset = A$. By induction on $n \in \omega$ define well-founded trees $q_n \subseteq {}^{<\omega}\omega$ with the set of maximal nodes $M_n = \{t \in q_n : (\forall k \in \omega)\ t^\frown\langle k\rangle \notin q_n\}$ and $\sigma(t) \in {}^{<\omega}\kappa$ for $t \in M_n$ so that the following conditions hold:

(1) $q_0 = \{t \in {}^{<\omega}\omega : t \subseteq s\}$, $M_0 = \{s\}$, and $\sigma(s) = \emptyset$.
(2) $q_{n-1} \subseteq q_n$ and $q_n = \bigcup\{(q_n)_t : t \in M_{n-1}\}$.
(3) $(\forall u \in M_n)\ D_u\big(A_{\sigma(u)}\big)$ holds.
(4) $(\forall t \in M_{n-1})\ (q_n)_t$ is an $\{A_{\sigma(u)} : u \in M_n \cap (q_n)_t\}$-tree with stem t.
(5) If $t \in M_{n-1}$, $u \in M_n$, and $t \subseteq u$, then $t \neq u$ and $\sigma(t) \subsetneq \sigma(u)$.

Let $n > 0$ and assume that q_{n-1}, M_{n-1}, and $\sigma(t)$ for $t \in M_{n-1}$ have been defined. For each $t \in M_{n-1}$ denote

$$S_t = \{\sigma \in {}^{<\omega}\kappa : \sigma(t) \subseteq \sigma \text{ and } (\exists i \in \omega)\ A_\sigma \subseteq [t^\frown\langle i\rangle]\}.$$

Then $A_{\sigma(t)} \cap [t] = \bigcup\{A_\sigma : \sigma \in S_t\}$ by continuity of f. Since $|S_t| < \mathfrak{b}$, by Lemma 4.3 there is an $\{A_\sigma : \sigma \in S_t\}$-tree p_t with stem t. Set $q_n = \bigcup\{p_t : t \in M_{n-1}\}$ and for each $u \in M_n \cap p_t$ with $t \in M_{n-1}$ choose $\sigma(u) \in S_t$ so that $D_u\big(A_{\sigma(u)}\big)$ holds. Conditions (2)–(4) follow by definitions of q_n and $\sigma(u)$ because $(q_n)_t = p_t$. For $t \in M_{n-1}$, $D_t\big(A_{\sigma(t)}\big)$ holds by induction hypothesis but $D_t(A_\sigma)$ does not hold for $\sigma \in S_t$ because $A_\sigma \subseteq [t^\frown\langle i\rangle]$ for some $i \in \omega$. Therefore $\sigma(t) \neq \sigma(u)$ for $u \in M_n \cap p_t$ and hence condition (5) is fulfilled.

Set $q = \bigcup_{n\in\omega} q_n$. By (2) and (5), q is a tree with no finite branches and, by (1) and (4), q is a Laver tree with stem s. Let $x \in [q]$. By (4) and (5) there are $t_n \in M_n$ such that $t_{n-1} \subsetneq t_n \subseteq x$ for all $n > 0$, and then, $\sigma(t_{n-1}) \subsetneq \sigma(t_n)$ for all $n > 0$. Denote $y = \bigcup_{n\in\omega} \sigma(t_n)$. Then $y \in {}^{\omega}\kappa$. For each $n > 0$, $\sigma(t_n) \in S_{t_{n-1}}$ and, for some $i \in \omega$, $f([\sigma(t_n)]) = A_{\sigma(t_n)} \subseteq [t_{n-1}{}^\frown\langle i\rangle] \subseteq [t_{n-1}]$. It follows that $f(y) = x$ and hence $x \in A$. Therefore $[q] \subseteq A$. $\square$

Recall that a tree $q \subseteq {}^{<\omega}\omega$ is a Hechler tree if $(\forall s \in q)(\forall^\infty n \in \omega)\ s^\frown\langle n\rangle \in q$. It is easy to see that for every set $A \in \mathcal{D}$ there exists a Hechler tree q with $[q] \cap A = \emptyset$. Therefore we obtain (it was the referee who drew our attention to this result):

Corollary 4.5 (A. W. Miller [8, Theorem 3]). *For any $<\mathfrak{b}$-Suslin set $A \subseteq {}^{\omega}\omega$ either there exists a Hechler tree q with $[q] \cap A = \emptyset$ or there exists a Laver tree q with $[q] \subseteq A$.* $\square$

5. Domination game

Let us recall the domination game $G(A)$ introduced in [4] (and denoted $D(A)$ there) as an instance of a two-person game $G_X(B)$ with perfect information where $X = \omega$ and $A \subseteq {}^{\omega}\omega$ is a projection of the set $B \subseteq {}^{\omega}\omega$. The initial move of player I in $G(A)$ is a finite sequence $s \in {}^{<\omega}\omega$ and subsequent kth move of player II is $x_{2k} \in \omega$ and $(k+1)$th move of player I is $x_{2k+1} \in \omega$ for $k \in \omega$. Player I wins if $(\forall k \in \omega)$ $x_{2k+1} \geq x_{2k}$ and $s^\frown\langle x_{2k+1} : k \in \omega\rangle \in A$.

Lemma 5.1 ([4, Lemma 2.3]). *Let $A \subseteq {}^{\omega}\omega$.*

(1) *Player I has a winning strategy in the game $G(A)$ if and only if there exists $p \in \mathbb{L}$ such that $[p] \subseteq A$.*
(2) *Player II has no winning strategy in the game $G(A)$ if and only if A is strongly dominating.*

Proof. (1) If p is a Laver tree with stem s such that $[p] \subseteq A$, then player I wins by opening the game with stem s of p and then following this strategy: To play $x_{2k+1} \geq x_{2k}$ so that $s^\frown\langle x_{2i+1} : i \leq k\rangle \in p$.

Let τ be a winning strategy of player I in the game $G(A)$ and let $p_\tau = \{\tau^*(u) : u \in {}^{<\omega}\omega\}$ where $\tau^*(u) = \tau(\emptyset)^\frown\langle\tau(u{\restriction}k) : 1 \leq k \leq |u|\rangle$. Then p_τ is a Laver tree with stem $s = \tau(\emptyset)$ and $[p_\tau] \subseteq A$.

(2) By a strategy of player II in the game $G(A)$ we understand a function $\sigma : {}^{<\omega}\omega \times {}^{<\omega}\omega \to \omega$ having this meaning: If $s \in {}^{<\omega}\omega$ is the initial move of player I and $t \in {}^{<\omega}\omega$ is an extension of s by a sequence of subsequent moves of player I, then $\sigma(s,t)$ is the response of player II by σ. The strategy σ is a winning strategy of player II in the game $G(A)$ if and only if for every $s \in {}^{<\omega}\omega$,

$$(\forall x \in A \cap [s])(\exists k \geq |s|)\; x(k) < \sigma(s, x{\restriction}k).$$

It is evident that player II has a winning strategy if and only if $D_s(A)$ does not hold for any $s \in {}^{<\omega}\omega$, i.e., if and only if A is not strongly dominating. □

Now we have the following consequence of Theorem 4.4:

Corollary 5.2. *Let $A \subseteq {}^{\omega}\omega$ be a $<\mathfrak{b}$-Suslin set. Then:*

(1) *The domination game $G(A)$ is determined.*
(2) *Player I has a winning strategy in the game $G(A)$ if and only if A is strongly dominating.* □

6. Strongly dominating sets and Laver category base

Let us recall that a pair $(X, \mathcal{P})$ where X is a set and $\mathcal{P} \subseteq \mathcal{P}(X)$ is a family of regions of X is called a category base (see [9]), if the following conditions are satisfied:

(1) $X = \bigcup \mathcal{P}$.
(2) Let $A \in \mathcal{P}$ and $\mathcal{Q} \subseteq \mathcal{P}$ be a disjoint family with $|\mathcal{Q}| < |\mathcal{P}|$.
  (a) If $A \cap \bigcup \mathcal{Q}$ contains a region, then there is a region $B \in \mathcal{Q}$ such that $A \cap B$ contains a region.
  (b) If $A \cap \bigcup \mathcal{Q}$ contains no region, then there is a region $B \subseteq A$ such that $B \cap \bigcup \mathcal{Q} = \emptyset$.

Let $\mathbb{L}$ denote the family of Laver perfect subtrees of ${}^{<\omega}\omega$ ordered by $p \leq q$ if $p \subseteq q$ and let $\mathcal{P}_{\mathbb{L}}$ be the family of all Laver perfect sets on ${}^{\omega}\omega$. Denote (see [4])

$$l = \{X \subseteq {}^{\omega}\omega : (\forall A \in \mathcal{P}_{\mathbb{L}})(\exists B \in \mathcal{P}_{\mathbb{L}})\; B \subseteq A \text{ and } (B \subseteq X \text{ or } B \cap X = \emptyset)\},$$
$$l^0 = \{X \subseteq {}^{\omega}\omega : (\forall A \in \mathcal{P}_{\mathbb{L}})(\exists B \in \mathcal{P}_{\mathbb{L}})\; B \subseteq A \text{ and } B \cap X = \emptyset\}.$$

It is easy to see that $\mathcal{D} \subseteq l^0$ and by Theorem 4.4, $\mathcal{D} \cap \mathbf{\Sigma}^1_1 = l^0 \cap \mathbf{\Sigma}^1_1$.

Theorem 6.1 ([4, Theorem 1.1 (1)])**.** $\mathfrak{t} \leq \mathrm{add}(l^0) \leq \mathrm{cov}(l^0) \leq \mathfrak{b}$ *and* $\mathrm{non}(l^0) = \mathfrak{c}$.

Proof. Since $\mathcal{D} \subseteq l^0$, by Lemma 2.4, $\mathrm{add}(l^0) \leq \mathrm{cov}(l^0) \leq \mathrm{cov}(\mathcal{D}) = \mathfrak{b}$.

Each Laver perfect set is a union of $\mathfrak{c}$ many disjoint Laver perfect sets and some of them must avoid a set of cardinality $< \mathfrak{c}$. Therefore $\mathrm{non}(l^0) = \mathfrak{c}$.

The inequality $\mathfrak{t} \leq \mathrm{add}(l^0)$ is proved in [4]. It follows also by Lemma 8.4 below. □

Part of the following theorem has its origin in [4, Lemma 2.5]; see also [7, 11].

Theorem 6.2. *Assume $\mathfrak{b} = \mathfrak{c}$. Then $({}^\omega\omega, \mathcal{P}_{\mathbb{L}})$ is a category base and l is closed under the κ-Suslin operation for all $\kappa < \mathrm{add}(l^0)$.*

Proof. We verify condition (2) of the definition of category base. Let $\mathcal{Q} \subseteq \mathcal{P}_{\mathbb{L}}$ have cardinality $< \mathfrak{c}$ and let $A \in \mathcal{P}_{\mathbb{L}}$.

If $A \cap \bigcup \mathcal{Q}$ contains a Laver perfect set, then $A \cap \bigcup \mathcal{Q} \notin \mathcal{D}$, then there is $B \in \mathcal{Q}$ such that $A \cap B \notin \mathcal{D}$ because $|\mathcal{Q}| < \mathfrak{b} = \mathrm{add}(\mathcal{D})$, and since $A \cap B$ is closed, it contains a Laver perfect subset. If $A \cap \bigcup \mathcal{Q}$ contains no Laver perfect set, then $A \cap C \in \mathcal{D}$ for all $C \in \mathcal{Q}$, then by additivity of $\mathcal{D}$ there is $f : {}^{<\omega}\omega \to \omega$ such that $A \cap \bigcup \mathcal{Q} \subseteq B(f)$, and then there is a Laver perfect set $B \subseteq A \setminus B(f)$ because $A \setminus B(f)$ is an F_σ set and $A \setminus B(f) \notin \mathcal{D}$.

Since l^0 is a σ-ideal, l is the family of Baire sets in the category base $\mathcal{P}_{\mathbb{L}}$ and therefore l is $< \mathrm{add}(l^0)$-complete algebra and is closed under the κ-Suslin operation for all $\kappa < \mathrm{add}(l^0)$ (see [9]; in fact Morgan's exposition concerns the countable case only, but if we replace in this exposition meager sets as countable unions of singular sets by κ-meager sets as unions of κ many singular sets, the definition of Baire sets will be converted into the definition of κ-Baire sets which are closed under κ-Suslin operation by the same arguments). □

Let us note that the conclusion of Theorem 4.4 is not a consequence of Theorem 6.2. This is due to the fact $\mathcal{D} \subsetneq l^0$. To prove that $\mathcal{D} \neq l^0$ we need some notation. Let G be the set of functions $g : {}^{<\omega}\omega \to \{0,1\}$. For a function $g \in G$ let $p(g)$ be the Laver tree with stem $\emptyset$ recursively defined by letting $\emptyset \in p(g)$ and, if $t \in p(g)$ and $m \in \omega$, then $t^\frown\langle m\rangle \in p(g)$ if and only if $m \equiv g(t) \mod 2$.

Lemma 6.3. *Let $G_0 \subseteq G$ be a set of functions of cardinality less than $\mathfrak{c}$ and let $A = {}^\omega\omega \setminus \bigcup_{g \in G_0}[p(g)]$. Then $D_s(A)$ holds for all $s \in {}^{<\omega}\omega$. If $\mathfrak{c} = \omega_1$, then, moreover, there is $\widetilde{g} \in G$ such that $[p(\widetilde{g})] \subseteq A$.*

Proof. Let $f : {}^{<\omega}\omega \to \omega$ and $s \in {}^{<\omega}\omega$ be given. The set

$$C_{s,f} = \{x \in [s] : (\forall k \geq |s|)\ x(k) = f(x{\restriction}k) \text{ or } x(k) = f(x{\restriction}k) + 1\}$$

is a Cantor perfect set such that $|C_{s,f} \cap p(g)| \leq 1$ for every $g \in G$, and for every $x \in C_{s,f}$, $(\forall k \geq |s|)\ x(k) \geq f(x{\restriction}k)$. Since $|G_0| < \mathfrak{c} = |C_{s,f}|$ it follows that $C_{s,f} \cap A \neq \emptyset$.

If G_0 is countable and $G_0 = \{g_n : n \in \omega\}$, then let $\widetilde{g}(s) = 1 - g_n(s)$ for $s \in {}^n\omega$. □

Theorem 6.4. *$\mathcal{D} \neq l^0$.*

Proof. Let $\{f_\alpha : \alpha < \mathfrak{c}\}$ and $\{g_\alpha : \alpha < \mathfrak{c}\}$ be enumerations of the set of functions $f : {}^{<\omega}\omega \to \omega$ and the set of functions $g : {}^{<\omega}\omega \to \{0,1\}$, respectively. By Lemma 6.3 for each $\alpha < \mathfrak{c}$ pick $x_\alpha \in {}^\omega\omega \setminus \bigcup_{\beta<\alpha}[p(g_\beta)]$ such that $(\forall k \in \omega)\ x_\alpha(k) \geq f_\alpha(x_\alpha{\restriction}k)$. Clearly the set $X = \{x_\alpha : \alpha < \mathfrak{c}\}$ is strongly dominating; actually $D_\emptyset(X)$ holds. We prove that $X \in l^0$. Let $p \in \mathbb{L}$ be arbitrary. Take $\alpha < \mathfrak{c}$ such that $p' = p \cap p(g_\alpha)$ is a Laver tree with the same stem as p has. Since $[p'] \cap X \subseteq [p(g_\alpha)] \cap X \subseteq \{x_\beta : \beta \leq \alpha\}$ and $\mathrm{non}(l^0) = \mathfrak{c}$ there is $q \in \mathbb{L}$ below p' such that $[q] \cap X = \emptyset$. □

Corollary 6.5.

(1) *There is $A \in l^0 \setminus \mathcal{D}$ such that $D_s(A)$ holds for every $s \in {}^{<\omega}\omega$.*
(2) *If $\mathfrak{d} = \mathfrak{c}$, then there is $A \in l^0 \setminus \mathcal{D}$ such that $D_s(A)$ holds for every $s \in {}^{<\omega}\omega$ and $A \cap [p(g)] \in \mathcal{D}$ for every $g : {}^{<\omega}\omega \to \{0,1\}$.*

Proof. Let $X \in l^0 \setminus \mathcal{D}$ be the set constructed in the proof of Theorem 6.4 and let $A = \{s^\frown x : s \in {}^{<\omega}\omega \text{ and } x \in X\}$. Then $A \in l^0$ and $D_s(A)$ holds for every s. Moreover, for every $g : {}^{<\omega}\omega \to \{0,1\}$, $|A \cap [p(g)]| < \mathfrak{c}$; hence if $\mathrm{non}(\mathcal{D}) = \mathfrak{c}$ (see Lemma 2.4), then $A \cap [p(g)] \in \mathcal{D}$. □

Let us recall from the previous section that

$$\mathbb{M}_0 = \{A \subseteq {}^\omega\omega : (\forall s \in {}^{<\omega}\omega)\ D_s(A) \to (\exists q \in \mathbb{L}(s))\ [q] \subseteq A\}.$$

The class $\mathbb{M}_0$ may be not closed under intersections and complements. Therefore we introduce the following derived classes:

$$\mathbb{M}_1 = \{A \subseteq {}^\omega\omega : (\forall F \subseteq {}^\omega\omega \text{ closed})\ F \cap A \in \mathbb{M}_0\},$$
$$\mathbb{M}_2 = \{A \subseteq {}^\omega\omega : (\forall F \subseteq {}^\omega\omega <\!\mathfrak{b}\text{-Suslin})\ F \cap A \in \mathbb{M}_0\},$$
$$\mathbb{M}_{1i} = \{A \subseteq {}^\omega\omega : (\forall B \in \mathbb{M}_i)\ B \cap A \in \mathbb{M}_i\}, \quad \text{for } i = 0,\ 1,\ 2,$$
$$\mathbb{M}_i^* = \{A \in \mathbb{M}_i : {}^\omega\omega \setminus A \in \mathbb{M}_i\}, \quad \text{for } i = 0,\ 1,\ 2,\ 10,\ 11,\ 12.$$

We have the following inclusions:

Lemma 6.6.
(1) $\mathbb{M}_{10} \subseteq \mathbb{M}_{11} \subseteq \mathbb{M}_{12} \subseteq \mathbb{M}_2 \subseteq \mathbb{M}_1 \subseteq \mathbb{M}_0$.
(2) $\mathbb{M}_{10}^* \subseteq \mathbb{M}_{11}^* \subseteq \mathbb{M}_{12}^* \subseteq \mathbb{M}_2^* \subseteq \mathbb{M}_1^* \subseteq \mathbb{M}_0^*$.
(3) $\mathcal{D} \subseteq \mathbb{M}_i^* \subseteq \mathbb{M}_i$ *for* $i = 0,\ 1,\ 2,\ 10,\ 11,\ 12$.
(4) $l^0 \setminus \mathcal{D} = l^0 \setminus \mathbb{M}_i$ *for* $i = 0,\ 1,\ 2,\ 10,\ 11,\ 12$.
(5) $\mathbb{M}_1 \subsetneq l$, $l \nsubseteq \mathbb{M}_0$, $\mathbb{M}_0^* \nsubseteq l$, $\mathbb{M}_{11}^* \nsubseteq \mathbb{M}_{10}$.
(6) $\mathbb{M}_{10} \subsetneq \mathbb{M}_{11}$, $\mathbb{M}_{10}^* \subsetneq \mathbb{M}_{11}^*$, $\mathbb{M}_1 \subsetneq \mathbb{M}_0$. $\mathbb{M}_1^* \subsetneq \mathbb{M}_0^*$.

Proof. (1) We show $\mathbb{M}_{11} \subseteq \mathbb{M}_{12}$ (the proof of $\mathbb{M}_{10} \subseteq \mathbb{M}_{11}$ is similar). Assume that $A \subseteq {}^\omega\omega$ and $A \notin \mathbb{M}_{12}$. There is $B \in \mathbb{M}_2$ such that $B \cap A \notin \mathbb{M}_2$ and then $F \cap (B \cap A) \notin \mathbb{M}_0$ for some $<\!\mathfrak{b}$-Suslin set $F \subseteq {}^\omega\omega$. Then $(F \cap B) \cap A \notin \mathbb{M}_1$ and $F \cap B \in \mathbb{M}_1$ because $B \in \mathbb{M}_2$ and the intersection of F with any closed set is $<\!\mathfrak{b}$-Suslin. Therefore $A \notin \mathbb{M}_{11}$. The other inclusions are obvious.

(2) follows by (1); (3) is easy.

(4) The inclusion $l^0 \setminus \mathcal{D} \subseteq l^0 \setminus \mathbb{M}_0$ follows by definitions. Hence for all i, $l^0 \setminus \mathcal{D} \subseteq l^0 \setminus \mathbb{M}_i$ by (1), and the inclusion $l^0 \setminus \mathcal{D} \supseteq l^0 \setminus \mathbb{M}_i$ holds because $\mathcal{D} \subseteq \mathbb{M}_i$ by (3).

(5) If $A \in \mathbb{M}_1$ and $q \in \mathbb{L}$, then $[q] \cap A \in \mathbb{M}_0$ and, either there is $s \in q$ such that $D_s([q] \cap A)$ holds, or $[q] \cap A \in \mathcal{D} \subseteq l^0$. Therefore, there is $p \in \mathbb{L}$ such that in the former case, $[p] \subseteq [q] \cap A$ and in the latter case, $[p] \subseteq [q] \setminus A$. Hence $\mathbb{M}_1 \subseteq l$. By Theorem 6.4 and (4), $\emptyset \neq l^0 \setminus \mathcal{D} \subseteq l \setminus \mathbb{M}_0 \subseteq l \setminus \mathbb{M}_1$ and hence $\mathbb{M}_1 \neq l$ and $l \nsubseteq \mathbb{M}_0$. To see that $\mathbb{M}_0^* \nsubseteq l$ and $\mathbb{M}_{11}^* \nsubseteq \mathbb{M}_{10}$ choose $q \in \mathbb{L}$ and $q_s^i \in \mathbb{L}(s)$ for $s \in {}^{<\omega}\omega$ and $i = 0,\ 1$ such that $[q_s^i] \cap [q] = \emptyset$ and $[q_s^0] \cap [q_s^1] = \emptyset$. Let $X \subseteq [q]$ be a Bernstein set in $[q]$ and let $A = X \cup \bigcup_{s \in {}^{<\omega}\omega} [q_s^0]$. Then $A \in \mathbb{M}_0^* \setminus l$ and $[q] \in \mathbb{M}_{11}^* \setminus \mathbb{M}_{10}$; $[q] \notin \mathbb{M}_{10}$ because $[q] \cap A = X$ and $X \notin \mathbb{M}_0$.

(6) follows by (5). □

The results of the previous section have the following consequences:

Corollary 6.7.
(1) $\mathbb{M}_i$ *is closed under unions of* $< \mathfrak{b}$ *sets for* $i = 0,\ 1,\ 2,\ 10,\ 11,\ 12$.
(2) $\mathbb{M}_{1i}$ *is closed under finite intersections and* $\mathbb{M}_{1i}^*$ *is an algebra of sets on* ${}^\omega\omega$ *for* $i = 0,\ 1,\ 2$.
(3) $\mathbb{M}_i$ *contains all* $<\!\mathfrak{b}$*-Suslin sets for* $i = 0,\ 1,\ 2$.

(4) $\mathbb{M}_i^*$ *contains all Borel sets and, if* $\mathfrak{b} > \omega_1$*, then* $\mathbb{M}_i^*$ *contains all analytic sets for* $i = 0, 1, 2$.

Proof. (1) follows by Theorem 4.2; (2) is an easy consequence of definitions; (3) follows by Theorem 4.4; (4) holds because the Borel sets are closed under complements and analytic and co-analytic sets (and $\mathbf{\Sigma}^1_2$ sets) are ω_1-Suslin. □

Modifying definitions of $\mathbb{M}_1$ and $\mathbb{M}_2$ let us define

$$\mathbb{M}_1' = \{A \subseteq {}^\omega\omega : (\forall F \subseteq {}^\omega\omega \text{ closed})\ D(F \cap A) \to (\exists q \in \mathbb{L})\ [q] \subseteq F \cap A\},$$
$$\mathbb{M}_2' = \{A \subseteq {}^\omega\omega : (\forall F < \mathfrak{b}\text{-Suslin})\ D(F \cap A) \to (\exists q \in \mathbb{L})\ [q] \subseteq F \cap A\}.$$

Lemma 6.8. $\mathbb{M}_1 = \mathbb{M}_1'$ *and* $\mathbb{M}_2 = \mathbb{M}_2'$.

Proof. Obviously, $\mathbb{M}_1 \subseteq \mathbb{M}_1'$ and $\mathbb{M}_2 \subseteq \mathbb{M}_2'$. We prove that $\mathbb{M}_1' \subseteq \mathbb{M}_1$ ($\mathbb{M}_2' \subseteq \mathbb{M}_2$). Let $A \subseteq {}^\omega\omega$ be such that $A \notin \mathbb{M}_1$ ($A \notin \mathbb{M}_2$). There are $s \in {}^{<\omega}\omega$ and a closed ($<\mathfrak{b}$-Suslin) set $F \subseteq [s]$ such that $D_s(F \cap A)$ holds and $(\forall q \in \mathbb{L}(s))\ [q] \nsubseteq F \cap A$. Denote

$$T = \{t \in {}^{<\omega}\omega : (\forall q \in \mathbb{L}(t))\ [q] \nsubseteq F \cap A\}.$$

Then $s \in T$ and for $t \in T$, $(\forall^\infty n \in \omega)\ t^\frown\langle n\rangle \in T$. Let $f : {}^{<\omega}\omega \to \omega$ be such that, if $t \supseteq s$ and $t \in T$, then $(\forall n \geq f(t))\ t^\frown\langle n\rangle \in T$. The set

$$F_0 = \{x \in F : (\forall k \geq |s|)\ x(k) \geq f(x{\restriction}k)\}$$

is closed ($<\mathfrak{b}$-Suslin) and $D_s(F_0 \cap A)$ holds, but there is no $q \in \mathbb{L}$ such that $[q] \subseteq F_0 \cap A$. Therefore $A \notin \mathbb{M}_1'$ ($A \notin \mathbb{M}_2'$). □

Lemma 6.9.

(1) $\mathbb{M}_1 = \{A \in l : (\forall r \in \mathbb{L})\ [r] \cap A \notin l^0 \setminus \mathcal{D}\}$.
(2) $\mathbb{M}_2 = \{A \in l : (\forall F < \mathfrak{b}\text{-}Suslin)\ F \cap A \notin l^0 \setminus \mathcal{D}\}$.
(3) $\mathbb{M}_{11} = \{A \in l : (\forall B \in \mathbb{M}_1)\ B \cap A \notin l^0 \setminus \mathcal{D}\}$.
(4) $\mathbb{M}_{12} = \{A \in l : (\forall B \in \mathbb{M}_2)\ B \cap A \notin l^0 \setminus \mathcal{D}\}$.

Proof. Apply characterizations in Lemma 6.8. For the equality (1) use the fact that the strongly dominating part $\Phi(F)$ of a closed set F is a countable union of Laver perfect sets and for (3) and (4) notice that $\mathbb{M}_1$ is closed under intersections by closed sets and $\mathbb{M}_2$ is closed under intersections by $<\mathfrak{b}$-Suslin sets. □

Question 6.10. Which classes in the chains of the inclusions $\mathbb{M}_{11} \subseteq \mathbb{M}_{12} \subseteq \mathbb{M}_2 \subseteq \mathbb{M}_1$ and $\mathbb{M}_{11}^* \subseteq \mathbb{M}_{12}^* \subseteq \mathbb{M}_2^* \subseteq \mathbb{M}_1^*$ are different?

At present we are not able to distinguish $\mathbb{M}_{11}$ from $\mathbb{M}_{12}$ and $\mathbb{M}_2$ from $\mathbb{M}_1$. Also we are not able to distinguish $\mathbb{M}_{11}^*$ from $\mathbb{M}_{12}^*$ and $\mathbb{M}_2^*$ from $\mathbb{M}_1^*$. We show that, at least consistently, $\mathbb{M}_{12} \subsetneq \mathbb{M}_2$ and $\mathbb{M}_{12}^* \subsetneq \mathbb{M}_2^*$.

Theorem 6.11. *Assume* $\mathfrak{c} = \omega_1$. *Let* $2 \leq \kappa \leq \omega_1$ *be a cardinal number. For every* $X \in l^0$ *there is a system of pairwise disjoint sets* $\{A_\xi : \xi < \kappa\} \subseteq \mathbb{M}_2 \setminus \mathcal{D}$ *such that the set* $X_0 = {}^\omega\omega \setminus \bigcup_{\xi<\kappa} A_\xi$ *is in* l^0, $X \subseteq X_0$ *and for every* $Y \subseteq X_0$ *and every* $\xi < \kappa$, $Y \cup A_\xi \in \mathbb{M}_2$.

Proof. We prove theorem for $\kappa = \omega_1$; the proof for $\kappa < \omega_1$ is similar. Let $X \in l^0$. Let $\{F_\alpha : \alpha < \omega_1\}$ be an enumeration of the family of analytic subsets of ${}^\omega\omega$ with ω_1 repetitions. By induction on $\alpha < \omega_1$ for all $\xi < \omega_1$ we define $\mathcal{A}_{\xi,\alpha} \subseteq \mathbb{L}$, $\mathcal{A}_{\xi,\alpha}^* = \bigcup_{\beta<\alpha} \mathcal{A}_{\xi,\beta}$, $A_{\xi,\alpha} = \bigcup\{[r] : r \in \mathcal{A}_{\xi,\alpha}\}$, and $A_{\xi,\alpha}^* = \bigcup_{\beta<\alpha} A_{\xi,\beta}$ such that for all $\alpha < \omega_1$ and $\xi < \omega_1$ the following conditions are satisfied:

(1) $\mathcal{A}^*_{\xi,\alpha} \subseteq \mathcal{A}_{\xi,\alpha}$ and $\{[r] : r \in \mathcal{A}_{\xi,\alpha}\}$ is a system of pairwise disjoint sets.
(2) $A^*_{\xi,\alpha} \subseteq A_{\xi,\alpha}$, $A_{\xi_1,\alpha} \cap A_{\xi_2,\alpha} = \emptyset$ for any $\xi_1 \neq \xi_2$.
(3) $\mathcal{A}^*_{\xi,\alpha} = \emptyset$ and $A^*_{\xi,\alpha} = \emptyset$ whenever $\xi \geq \alpha$, i.e., $\bigcup_{\xi<\omega_1} A^*_{\xi,\alpha} = \bigcup_{\xi<\alpha} A^*_{\xi,\alpha}$.
(4) If $F_\alpha \setminus \bigcup_{\xi<\omega_1} A^*_{\xi,\alpha} \notin \mathcal{D}$, then there are $r^\xi_\alpha \in \mathbb{L}$ for $\xi \leq \alpha$ such that $[r^\xi_\alpha] \subseteq F_\alpha \setminus (X \cup \bigcup_{\xi<\omega_1} A^*_{\xi,\alpha})$, $[r^{\xi_1}_\alpha] \cap [r^{\xi_2}_\alpha] = \emptyset$ for any $\xi_1 \neq \xi_2$, and $\mathcal{A}_{\xi,\alpha} = \mathcal{A}^*_{\xi,\alpha} \cup \{r^\xi_\alpha\}$ for $\xi \leq \alpha$ and $\mathcal{A}_{\xi,\alpha} = \mathcal{A}^*_{\xi,\alpha}$ for $\xi > \alpha$.
(5) If $F_\alpha \setminus \bigcup_{\xi<\omega_1} A^*_{\xi,\alpha} \in \mathcal{D}$, then $A_{\xi,\alpha} = A^*_{\xi,\alpha}$ for all $\xi < \omega_1$.
(6) $X \cap \bigcup_{\xi<\omega_1} A_{\xi,\alpha} = \emptyset$.
(7) For every $r \in \mathcal{A}^*_{\xi,\alpha}$ there is $g : {}^{<\omega}\omega \to \{0,1\}$ such that $r \subseteq p(g)$.

At the induction step conditions (4) and (5) can be fulfilled because by (3), the set $F_\alpha \setminus \bigcup_{\xi<\omega_1} A^*_{\xi,\alpha}$ is analytic, and if it is strongly dominating, it has a Laver perfect subset avoiding the set $X \in l^0$. Let $A_\xi = \bigcup_{\alpha<\omega_1} A_{\xi,\alpha}$ for $\xi < \omega_1$ and let $X_0 = {}^\omega\omega \setminus \bigcup_{\xi<\omega_1} A_\xi$. Then $X \subseteq X_0$ by (6). Condition (7) by Lemma 6.3 ensures that condition (4) applies ω_1 times and hence each A_ξ has a Laver perfect subset and $A_\xi \notin \mathcal{D}$. By (1) and (2), the system $\mathcal{A} = \bigcup_{\xi<\omega_1} \bigcup_{\alpha<\omega_1} \mathcal{A}_{\xi,\alpha}$ is an antichain in $\mathbb{L}$. It is a maximal antichain because whenever $p \in \mathbb{L}$ and $[p] = F_\alpha$ for some α, then $[r^0_\alpha] \subseteq [p]$ and $r^0_\alpha \in \mathcal{A}$ whenever condition (4) takes place, and p is compatible with some $r \in \bigcup_{\xi<\alpha} \mathcal{A}^*_{\xi,\alpha}$ whenever condition (5) takes place. Therefore $X_0 \in l^0$. We prove that $Y \cup A_\eta \in \mathbb{M}'_2$ for all $\eta < \omega_1$ and $Y \subseteq X_0$. Let $F \subseteq {}^\omega\omega$ be an analytic set such that $F \cap (Y \cup A_\eta) \notin \mathcal{D}$ and let $F = F_\alpha$ for some $\alpha > \eta$. Then

$$F_\alpha \cap (Y \cup A_\eta) \subseteq F_\alpha \cap (X_0 \cup A_\eta) \subseteq (F_\alpha \cap A^*_{\eta,\alpha}) \cup (F_\alpha \setminus \bigcup_{\xi<\omega_1} A^*_{\xi,\alpha}).$$

Therefore, if $F_\alpha \setminus \bigcup_{\xi<\omega_1} A^*_{\xi,\alpha} \in \mathcal{D}$, then $F_\alpha \cap A^*_{\eta,\alpha} \notin \mathcal{D}$, and this set being analytic has a Laver perfect subset. If $F_\alpha \setminus \bigcup_{\xi<\omega_1} A^*_{\xi,\alpha} \notin \mathcal{D}$, then by (4), $[r^\eta_\alpha] \subseteq F_\alpha \cap A_{\eta,\alpha}$. Therefore $F \cap A_\eta$ has a Laver perfect subset. □

Corollary 6.12. *If* $\mathfrak{c} = \omega_1$, *then* $\mathbb{M}^*_2 \nsubseteq \mathbb{M}_{12}$. *Consequently* $\mathbb{M}_{12} \subsetneq \mathbb{M}_2$ *and* $\mathbb{M}^*_{12} \subsetneq \mathbb{M}^*_2$.

Proof. Let $X \in l^0 \setminus \mathcal{D}$ and let A_0 and A_1 be the sets from Theorem 6.11 found for the set X and for $\kappa = 2$. Then $X \cup A_i \in \mathbb{M}^*_2$ and $X \cup A_i \notin \mathbb{M}_{12}$ because $(X \cup A_0) \cap (X \cup A_1) = X \notin \mathbb{M}_0$. □

7. A partition of Laver perfect sets

Let us consider the following partition of ${}^\omega\omega$ into Laver perfect sets:

$$L_x = \{y \in {}^\omega\omega : (\forall n \in \omega)\ y(n) \equiv x(n) \mod 2\}, \quad x \in {}^\omega 2. \tag{$*$}$$

This partition is not a maximal antichain. To see this consider any homeomorphism ψ between $H = \{x \in {}^\omega 2 : (\exists^\infty n \in \omega)\ x(n) = 1\}$ and ${}^{({}^{<\omega}\omega)}\omega$. Let $f : H \to {}^\omega\omega$ be defined by

$$f(x) = y \leftrightarrow (\forall n \in \omega)\ y(n) = \min\{k \geq \psi(x)(y{\restriction}n) : k \equiv x(n) \mod 2\}.$$

The set $A = \mathrm{rng}(f)$ is a Borel set because it is a continuous one-to-one image of the Polish space H. The set A is a Borel strongly dominating selector of $\{L_x : x \in H\}$ and A is disjoint from L_x for $x \notin H$. Since A is analytic, by Theorem 4.4, A contains a Laver perfect set. Therefore the disjoint partition $(*)$ is not a maximal antichain.

Let G be the set of functions $g : {}^{<\omega}\omega \to \{0, 1\}$. Recall that for a function $g \in G$, $p(g)$ is the Laver tree with stem $\emptyset$ recursively defined by letting $\emptyset \in p(g)$ and, if $t \in p(g)$ and $m \in \omega$, then $t^\frown\langle m\rangle \in p(g)$ if and only if $m \equiv g(t) \mod 2$.

Lemma 7.1.

(1) *There is $g \in G$ such that the Laver perfect set $[p(g)]$ is a selector for the partition $(*)$.*

(2) *There is $g \in G$ such that $(\forall x \in {}^\omega 2)\ [p(g)] \cap L_x \in \mathcal{D}$ and $(\forall q \in \mathbb{L})(\forall x \in {}^\omega 2)$ if $q \subseteq p(g)$ and $[q] \cap L_x \neq \emptyset$, then $[q] \cap L_x$ is a perfect set.*

Proof. (1)–(2) For $a \in [\omega]^\omega$ consider a one-to-one function $\pi : a \times \omega \to a$ such that $n < \pi(n, m)$ for all $(n, m) \in a \times \omega$. For example, let $\pi(a(n), m) = a(2^n(2m+1))$ for $(n, m) \in \omega \times \omega$ where $a(n)$ denotes the nth member of the set a in the increasing enumeration. Let us define $g : {}^{<\omega}\omega \to \{0, 1\}$ by induction: Set $g(\emptyset) = 0$. Assume that $k > 0$ and $g(s)$ have been defined for $s \in {}^{<k}\omega$ and define $g(s) \in \{0, 1\}$ for $s \in {}^k\omega$ so that

(i) if $k \notin \mathrm{rng}(\pi)$, then $g(s) = 0$, and

(ii) if $\pi(n, m) = k$ for $(n, m) \in a \times \omega$, then $g(s) = 0$ if and only if $s(n) = 2m + g(s{\restriction}n)$.

If $y \in [p(g)] \cap L_x$, then

$$(\forall (n, m) \in a \times \omega)\ x(\pi(n, m)) = 0 \text{ if and only if } y(n) = 2m + g(y{\restriction}n) \qquad (**)$$

recursive formula $(**)$ uniquely defines $y{\restriction}a$. Therefore $|\{y{\restriction}a : y \in [p(g)] \cap L_x\}| \leq 1$, hence the closed set $[p(g)] \cap L_x$ contains no Laver perfect set, and so $[p(g)] \cap L_x \in \mathcal{D}$ for all $x \in {}^\omega 2$. In the case when $a = \omega$ the Laver perfect set $[p(g)]$ is a selector for the partition $(*)$ and we obtain assertion (1) of the lemma. We verify (the second part of) assertion (2) of the lemma in the case when the set $\omega \setminus a$ is infinite. Let $q \in \mathbb{L}$ be such that $q \subseteq p(g)$ and let $y \in [q] \cap L_x$ for some $x \in {}^\omega 2$. Let $q_0 \subseteq q$ be the tree such that $[q_0] = [q] \cap L_x$. If $s \in q_0$ is a splitting node of q and $|s| \notin a$, then by case (i) in the definition of π, $\{n \in \omega : s^\frown\langle n\rangle \in q_0\} = \{n \in \omega : s^\frown\langle n\rangle \in q\} \subseteq \{2n : n \in \omega\}$, and hence, s is a splitting node of q_0. Therefore q_0 is a perfect tree and $[q] \cap L_x$ is a perfect set. □

Assume that $q \in \mathbb{L}$ and $q \subseteq p(g)$ for some $g : {}^{<\omega}\omega \to \{0, 1\}$. The sets

$$E(q) = \{x \in {}^\omega 2 : [q] \cap L_x \notin \mathcal{D}\} \quad \text{and} \quad F(q) = \{x \in {}^\omega 2 : [q] \cap L_x \neq \emptyset\}$$

are analytic, $E(q) \subseteq F(q)$, $|E(q)| \leq \omega$ (because $[q] \cap L_x$ contains a relatively open subset of $[q]$ for $x \in E(q)$), and, if $E(q) = \emptyset$, then $|F(q)| = \mathfrak{c}$ (because, then $F(q)$ is an uncountable analytic set by σ-additivity of $\mathcal{D}$).

8. Sequences of Laver perfect sets

Let Q be the set of all functions $g : {}^{<\omega}\omega \to [\omega]^\omega$. For $g \in Q$ and $s \in {}^{<\omega}\omega$ let

$$L_s(g) = \{x \in [s] : (\forall k \geq |s|)\ x(k) \in g(x{\restriction}k)\},$$

$$L(g) = \{x \in {}^\omega\omega : (\forall^\infty k \in \omega)\ x(k) \in g(x{\restriction}k)\} = \textstyle\bigcup_{s \in {}^{<\omega}\omega} L_s(g).$$

and let $p_s(g)$ be the unique Laver perfect tree with stem s such that $L_s(g) = [p_s(g)]$.

For $f, g \in Q$ we define

$$f \leq g \leftrightarrow (\forall s \in {}^{<\omega}\omega)\ f(s) \subseteq g(s),$$

$$f \subseteq^* g \leftrightarrow (\forall s \in {}^{<\omega}\omega)\ f(s) \subseteq^* g(s),$$

$$f \subseteq_* g \leftrightarrow (\forall^\infty s \in {}^{<\omega}\omega)\ f(s) \subseteq g(s),$$
$$f \subseteq^{**} g \leftrightarrow (\forall^\infty s \in {}^{<\omega}\omega)\ f(s) \subseteq^* g(s),$$
$$f \leq^* g \leftrightarrow f \subseteq^* g \text{ and } f \subseteq_* g.$$

All these relations are quasi-orderings. The relation $\subseteq^*$ was considered in [5] and the relation $\leq^*$ was considered in [4]. We compare these quasi-orderings and refine some results from [4].

Lemma 8.1. *Let $f, g \in Q$.*

(1) $f \leq g \to f \leq^* g \to f \subseteq_* g \to f \subseteq^{**} g$.
(2) $f \subseteq_* g \to L(f) \subseteq L(g)$.
(3) $f \subseteq^{**} g \to L(f) \setminus L(g) \in l^0$.
(4) $L(f) \setminus L(g) \in l^0 \leftrightarrow (\forall f' \leq f)(\exists f'' \leq f')\ f'' \leq g$.

Proof. (2) Assume that $f \subseteq_* g$. There is $n \in \omega$ such that $f(s) \subseteq g(s)$ whenever $|s| \geq n$. If $|s| \geq n$, then $L_s(f) \subseteq L_s(g) \subseteq L(g)$. If $|s| < n$, then $L_s(f) = \bigcup\{L_t(f) : t \in p_s(f) \cap {}^n\omega\} \subseteq L(g)$. Therefore $L(f) \subseteq L(g)$.

(3) If $L(f) \setminus L(g) \notin l^0$, then there is $u \in {}^{<\omega}\omega$ such that $f(t) \subseteq^* g(t)$ for all $t \supseteq u$ and $L_u(f) \setminus L(g) \notin l^0$. Let $f' \leq f$ and $s \supseteq u$ be such that $L_s(f') \subseteq L_s(f) \setminus L(g)$. Since $f(t) \subseteq^* g(t)$ for all $t \supseteq s$ there is $f'' \leq f'$ such that $L_s(f'') \subseteq L_s(g)$. This is a contradiction because $L_s(f'') \subseteq L_s(f')$.

(4) Assume that $L(f) \setminus L(g) \in l^0$ and let $f' \leq f$. Without loss of generality we can assume $L(f') \subseteq L(g)$. Assume that there is $s \in {}^{<\omega}\omega$ such that $(\forall h \leq f')$ $L_s(h) \setminus L_s(g) \neq \emptyset$. Then $(\forall^\infty n \in f'(s))(\forall h \leq f')\ L_{s^\frown\langle n\rangle}(h) \setminus L_{s^\frown\langle n\rangle}(g) \neq \emptyset$ and by induction we can define $g' \leq f'$ such that for all $t \in p_s(g')$ above s, $(\forall h \leq f')$ $L_t(h) \setminus L_t(g) \neq \emptyset$. Since $L_s(g') \subseteq L_s(f') \subseteq L(g)$, by the Baire category theorem there is $t \in p_s(g')$ above s such that $L_t(g') \subseteq L_t(g)$. This is a contradiction. Therefore $(\forall s \in {}^{<\omega}\omega)(\exists h \leq f')\ L_s(h) \subseteq L_s(g)$ and by induction we can define $f'' \leq f'$ such that $f'' \leq g$.

To see the inverse implication notice that l is a σ-field (see a general argument in [7, Lemma 4.1]) and therefore $L(f) \in l$ for every $f \in Q$. Also, if $A \subseteq L_s(f)$ is a Laver perfect set with stem $s \in {}^{<\omega}\omega$, then there is $f' \leq f$ such that $L_s(f') = A$. □

A part of Theorem 1.1 of [4] says that $\mathfrak{t} \leq \text{add}(l^0)$ and the proof is based on the fact that the partially ordered set $(Q, \leq^*)$ is κ-closed for every $\kappa < \mathfrak{t}$. We can say a bit more. Let us recall that a partially ordered set $(P, \leq)$ is κ-distributive, if the intersection of κ many open dense subsets of P is dense. $(P, \leq)$ is nowhere κ-distributive, if there is a family of open dense sets of cardinality κ with the empty intersection.

Lemma 8.2. *Let κ be a cardinal number.*

(1) *$(Q, \leq^*)$ is κ-closed if and only if $(Q, \subseteq^*)$ is κ-closed if and only if $\kappa < \mathfrak{t}$.*
(2) *$(Q, \leq^*)$ is κ-distributive if and only if $(Q, \subseteq^*)$ is κ-distributive.*
(3) *$(Q, \leq^*)$ and $(Q, \subseteq^*)$ are nowhere $\mathfrak{h}$-distributive.*

Proof. (1) We present the proof for $(Q, \leq^*)$; the proof for $(Q, \subseteq^*)$ is the same.

Suppose that $\gamma \leq \kappa < \mathfrak{t}$ and $\langle f_\alpha : \alpha < \gamma\rangle$ is a decreasing sequence in $(Q, \leq^*)$. For each $s \in {}^{<\omega}\omega$ there is $f'(s) \in [\omega]^\omega$, such that $(\forall \alpha < \gamma)\ f'(s) \subseteq^* f_\alpha(s)$. Find $g_\alpha : {}^{<\omega}\omega \to \omega$ such that $f'(s) \setminus g_\alpha(s) \subseteq f_\alpha(s)$ for all s. Since $\mathfrak{t} \leq \mathfrak{b}$, there exists $g : {}^{<\omega}\omega \to \omega$ such that $g_\alpha \leq^* g$ for all $\alpha < \gamma$, Now let $f(s) = f'(s) \setminus g(s)$. It is easy to check that $f \in Q$ and $f \leq^* f_\alpha$ for all $\alpha < \gamma$.

Let $\langle a_\alpha : \alpha < \mathfrak{t}\rangle$ be a tower in $([\omega]^\omega, \subseteq^*)$, i.e., $a_\alpha \subseteq^* a_\beta$ for $\beta \le \alpha < \mathfrak{t}$ and there is no $a \in [\omega]^\omega$ such that $a \subseteq^* a_\alpha$ for all $\alpha < \mathfrak{t}$. Fix a one-to-one enumeration ${}^{<\omega}\omega = \{s_n : n \in \omega\}$ and define $f_\alpha(s_n) = a_\alpha \setminus n$ for $s_n \in {}^{<\omega}\omega$ and $\alpha < \mathfrak{t}$. Then $\langle f_\alpha : \alpha < \mathfrak{t}\rangle$ is a tower in $(Q, \le^*)$.

(2) Every open dense subset of $(Q, \subseteq^*)$ is an open dense subset of $(Q, \le^*)$ and the relation $\le^*$ is included in $\subseteq^*$. Therefore the κ-distributivity of $(Q, \le^*)$ implies the κ-distributivity of $(Q, \subseteq^*)$.

Assume that $(Q, \subseteq^*)$ is κ-distributive and let $\langle D_\alpha : \alpha < \kappa\rangle$ be a sequence of open dense subsets of $(Q, \le^*)$. Every open dense subset of $(Q, \le^*)$ is a dense subset of $(Q, \subseteq^*)$. Therefore $D'_\alpha = \{f \in Q : (\exists g \in D_\alpha)\ f \subseteq^* g\}$ are open dense subsets of $(Q, \subseteq^*)$ for all $\alpha < \kappa$ and hence $D' = \bigcap_{\alpha<\kappa} D'_\alpha$ is an open dense subset of $(Q, \subseteq^*)$. Let $f \in Q$ be arbitrary. There is $f' \in D'$ such that $f' \le f$. For each $\alpha < \kappa$ choose $f_\alpha \in D_\alpha$ and $h_\alpha : {}^{<\omega}\omega \to \omega$ such that $(\forall s \in {}^{<\omega}\omega)\ f'(s) \setminus h_\alpha(s) \subseteq f_\alpha(s)$. By (3), $\kappa < \mathfrak{h} \le \mathfrak{b}$ and therefore there exists $h : {}^{<\omega}\omega \to \omega$ such that $h_\alpha \le^* h$ for all $\alpha < \kappa$. Define $g \in Q$ by $g(s) = f'(s) \setminus h(s)$ for $s \in {}^{<\omega}\omega$. Then $g \le f$ and for all $\alpha < \kappa$, $g \le^* f_\alpha$, hence $g \in \bigcap_{\alpha<\kappa} D_\alpha$. Therefore $\bigcap_{\alpha<\kappa} D_\alpha$ is a dense subset of $(Q, \le^*)$ and $(Q, \le^*)$ is κ-distributive.

(3) Let $\{D_\alpha : \alpha < \mathfrak{h}\}$ be a family of open dense subsets of $([\omega]^\omega, \subseteq^*)$ such that $\bigcap_{\alpha<\mathfrak{h}} D_\alpha = \emptyset$. For every $\alpha < \mathfrak{h}$ the set $D'_\alpha = \{f \in Q : (\forall s \in {}^{<\omega}\omega)\ f(s) \in D_\alpha\}$ is an open dense subset of Q and $\bigcap_{\alpha<\mathfrak{h}} D'_\alpha = \emptyset$. □

Denote

$$\mathfrak{h}_{\mathbb{L}} = \min\{\kappa : (Q, \le^*) \text{ is not } \kappa\text{-distributive}\}.$$

By Lemma 8.2, $\mathfrak{t} \le \mathfrak{h}_{\mathbb{L}} \le \mathfrak{h}$.

Lemma 8.3. *For every $X \in l^0$ and $g \in Q$ there exists $f \in Q$ such that $f \le g$ and $L(f) \cap X = \emptyset$.*

Proof. Assume that there is $s \in {}^{<\omega}\omega$ such that $(\forall f \le g)\ L_s(f) \cap X \ne \emptyset$. Then the set $\{n \in g(s) : (\exists f \le g)\ L_{s^\frown\langle n\rangle}(f) \cap X = \emptyset\}$ is finite and by induction we can define $g' \le g$ such that $(\forall t \in p_s(g'))(\forall f \le g)\ L_t(f) \cap X \ne \emptyset$. Since $X \in l^0$ there is a Laver tree $p \subseteq p_s(g')$ with stem $t \in p_s(g')$ such that $[p] \cap X = \emptyset$ and we can define $f \le g' \le g$ such that $L_t(f) = [p]$ contradicting $t \in p_s(g')$. Therefore for every s there is $h \le g$ such that $L_s(h) \cap X = \emptyset$ and by induction we can define $f \in Q$ such that $L_s(f) \cap X = \emptyset$ for all $s \in {}^{<\omega}\omega$. □

Lemma 8.4. $\mathrm{add}(l^0) \ge \mathfrak{h}_{\mathbb{L}}$.

Proof. Let $\kappa < \mathfrak{h}_{\mathbb{L}}$ and let $X = \bigcup_{\alpha<\kappa} X_\alpha$ with all $X_\alpha \in l^0$. By Lemma 8.3 for every $\alpha < \kappa$ the set $D_\alpha = \{f \in Q : L(f) \cap X_\alpha = \emptyset\}$ is an open dense subset of $(Q, \le^*)$. By κ-distributivity of $(Q, \le^*)$, the set $D = \bigcap_{\alpha<\kappa} D_\alpha$ is an open dense subset of $(Q, \le^*)$ and $L(f) \cap \bigcup_{\alpha<\kappa} X_\alpha = \emptyset$ for every $f \in D$.

We show that $X \in l^0$. For $A \in \mathcal{P}_{\mathbb{L}}$ find $g \in Q$ and $s \in {}^{<\omega}\omega$ such that $L_s(g) = A$ and find $f \in D$ such that $f \le^* g$. There is $r \in p_s(f)$ above s such that for all $t \in p_r(f)$ above r we have $f(t) \subseteq g(t)$. Set $B = L_r(f)$. Then $B = L_r(f) \subseteq L_r(g) \subseteq L_s(g) = A$ and $B \cap X = \emptyset$ because $f \in D$. □

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