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ADDITIVITY OF POROUS SETS

Let $\mathbf{P}$ denote the ideal of σ-porous sets and let $\mathbf{K}$ be the ideal of first category sets on the real line $\mathbf{R}$. The aim of this note is to prove this theorem:

Theorem. *Let $\mathcal{I}$ be arbitrary ideal on $\mathbf{R}$ such that $\mathbf{P} \subseteq \mathcal{I} \subseteq \mathbf{K}$. Then $\mathrm{add}(\mathcal{I}) \leq \mathbf{b}$ and $\mathbf{d} \leq \mathrm{cof}(\mathcal{I})$.*

Let us recall some definitions (see e.g. [2] and [5]): A set $A \subseteq \mathbf{R}$ is porous if for every $b \in A$,

$$p(A, b) = \limsup_{\varepsilon \to 0^+} \frac{\lambda(A, (b-\varepsilon, b+\varepsilon))}{\varepsilon} > 0.$$

Here $\lambda(A, I)$ denotes the maximal length of a subinterval of the interval I which is disjoint with A. A set A is σ-porous if it is a countable union of porous sets.

$$\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{I}_0| : \mathcal{I}_0 \subseteq \mathcal{I} \ \&\ \bigcup \mathcal{I}_0 \notin \mathcal{I}\},$$
$$\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{I}_0| : \mathcal{I}_0 \subseteq \mathcal{I} \ \&\ (\forall A \in \mathcal{I})(\exists B \in \mathcal{I}_0)\, A \subseteq B\}.$$

For $f, g \in {}^\omega\omega$, $f \leq^* g$ iff $(\exists n)(\forall m > n)\, f(n) \leq g(n)$.

$$\mathbf{b} = \min\{|F| : F \subseteq {}^\omega\omega \ \&\ (\forall f \in {}^\omega\omega)(\exists g \in F)\, g \not\leq^* f\},$$

$$\mathbf{d} = \min\{|F| : F \subseteq {}^\omega\omega \ \&\ (\forall f \in {}^\omega\omega)(\exists g \in F)\, f \leq^* g\}.$$

See also [1].

If m, n are integers then $\langle m, n) = \{k \in \omega : m \leq k < n\}$. For $s \in {}^{<\omega}2$, $[s] = \{x \in {}^\omega 2 : s \subseteq x\}$ is a basic clopen set of the Cantor space ${}^\omega 2$. Let φ be the mapping from ${}^\omega 2$ onto the interval $\langle 0, 1\rangle$ defined by $\varphi(x) = \sum_{n \in \omega} x(n) 2^{-n-1}$. For $s \in {}^n 2$, $I_s = \varphi([s])$ is a closed subinterval of $\langle 0, 1\rangle$ of length 2^{-n}.

The proof of the Theorem is a slight modification of A. W. Miller's proof of $\mathrm{add}(\mathbf{K}) \leq \mathbf{b}$ (see [3]).

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Let us fix a sequence of integers $k(n)$, $n \in \omega$ such that

$$k(n+1) - k(n) > n, \quad \text{for all } n \in \omega.$$

For $f \in {}^{\omega}\omega$ let us denote

$$\alpha(f) = \{x \in {}^{\omega}2 : (\forall m) x{\restriction}\langle k(f(m)), k(f(m)+1)) \equiv 1\}. \tag{1}$$

Lemma 1. *If $f \in {}^{\omega}\omega$ is increasing then the set $A = \varphi(\alpha(f))$ is porous.*

Proof. Let $b \in A$ be arbitrary, $b = \varphi(x)$ for some $x \in \alpha(f)$. We will show that $p(A, b) = 1$.

Let $m \in \omega$ be arbitrary. Put $\varepsilon = 2^m |I_{x\restriction k(f(m)+1)}| < 2^{-m}$. For $s \in {}^{k(f(m))}2$ let s^* denote the sequence $s^\frown 1^\frown \ldots {}^\frown 1$ of length $k(f(m)+1)$. It is obvious that $(x{\restriction}k(f(m)))^* = x{\restriction}k(f(m)+1)$ and $A \cap I_s \subseteq I_{s^*}$. Since $|I_s| \geq 2^{f(m)}|I_{s^*}| \geq 2^m |I_{s^*}|$ and the distance between any two closest distinct intervals of the form I_{s^*} is $|I_s| - |I_{s^*}| > \varepsilon$ we have $A \cap (b - \varepsilon, b + \varepsilon) \subseteq I_{x\restriction k(f(m)+1)}$. Therefore $\lambda(A, (b - \varepsilon, b + \varepsilon)) \geq (1 - 2^{-m})\varepsilon$ and so $p(A, b) = 1$. □

It is easy to see that the set A is closed and even strongly symmetrically porous, see [5].

Lemma 2. *Let us denote $\varphi^{-1}(\mathcal{I}) = \{A \subseteq {}^{\omega}2 : \varphi(A) \in \mathcal{I}\}$. Then* $\mathrm{add}(\mathcal{I}) = \mathrm{add}(\varphi^{-1}(\mathcal{I}))$ *and* $\mathrm{cof}(\mathcal{I}) = \mathrm{cof}(\varphi^{-1}(\mathcal{I}))$.

Proof. The lemma is a simple consequence of these two implications (see e.g [4, Lemma 2.2]):

$$\varphi(A) \subseteq B \text{ implies } A \subseteq \varphi^{-1}(B), \text{ and}$$
$$\varphi^{-1}(B) \subseteq A \text{ implies } B \subseteq \varphi(A)$$

for $A \in \varphi^{-1}(\mathcal{I})$ and $B \in \mathcal{I}$. □

According to the previous two lemmas, for proving the Theorem it is enough to prove the following:

Claim. *Let $\mathcal{H} = \{\alpha(f) : f \in {}^{\omega}\omega$ is increasing$\}$ and let $\mathcal{J}$ be an arbitrary ideal on ${}^{\omega}2$ such that $\mathcal{H} \subseteq \mathcal{J} \subseteq \mathbf{K}({}^{\omega}2)$. Then* $\mathrm{add}(\mathcal{J}) \leq \mathbf{b}$ *and* $\mathbf{d} \leq \mathrm{cof}(\mathcal{J})$.

Proof. We will find two mappings

$$\alpha : {}^{\omega}\omega \to \mathcal{H} \quad \text{and} \quad \beta : \mathbf{K}({}^{\omega}2) \to {}^{\omega}\omega$$

such that

$$\alpha(f) \subseteq A \text{ implies } f \leq^* \beta(A) \text{ for every } A \in \mathbf{K}(2),\ f \in {}^{\omega}\omega. \tag{2}$$

This will conclude the proof because if $\mathcal{F} \subseteq {}^\omega\omega$ is any family such that $\alpha(f) \subseteq A$ for every $f \in \mathcal{F}$ then the family $\mathcal{F}$ is dominated by $\beta(A)$. Therefore $\mathrm{add}(\mathcal{J}) \leq \mathbf{b}$. The proof of $\mathbf{d} \leq \mathrm{cof}(\mathcal{J})$ is similar. Let us note that anologous details are omitted in [2] and [3].

The mapping α is already defined by (1). Let us define β. Let $A \subseteq {}^\omega 2$ be an arbitrary meager set. There is a sequence $C_0 \subseteq C_1 \subseteq C_2 \dots$ of closed nowhere dense subsets of ${}^\omega 2$ such that $A \subseteq \bigcup_{n\in\omega} C_n$. By induction define an increasing function $g \in {}^\omega\omega$ such that

$$(\forall s \in {}^{k(g(m)+1)}2)(\exists t \in {}^{k(g(m+1))}2)\, s \subseteq t \ \&\ [t] \cap C_m = \emptyset \tag{3}$$

for every m. Put $\beta(A)(n) = g(2n)$.

We shall verify (2). Let $f \in {}^\omega\omega$ be arbitrary and let $f \not\leq^* \beta(A)$. We can assume that f is increasing. Therefore

$$(\forall k)(\exists n > k)\, f(n) > \beta(A)(n) = g(2n).$$

Let us denote $A_m = \langle g(m), g(m+1)) \cap \mathrm{rng}(f)$. We claim that for all $k \in \omega$ there exists $m > k$ such that $A_m = \emptyset$. To see this choose $n > k+2$ such that $g(2n) < f(n)$. Then at most n sets among A_i, $i = 0, 1, \dots, 2n-1$ are nonempty. Hence there is an $m > n-2$ such that $A_m = \emptyset$.

Using this fact together with property (3) we can inductively define an $x \in {}^\omega 2$ and an increasing sequence n_i, $i \in \omega$ such that:

$$(\forall i)\, [x \restriction k(g(n_i+1))] \cap C_{n_i} = \emptyset, \text{ and} \tag{4}$$

$$(\forall m)\, x \restriction \langle k(f(m)), k(f(m)+1)) \equiv 1. \tag{5}$$

The condition (5) ensures that $x \in \alpha(f)$ and since the sequence C_n, $n \in \omega$ is increasing, (4) ensures that $x \notin A$. Therefore $\alpha(f) \not\subseteq A$ and (2) holds true.

This finishes the proof of the Claim and of the Theorem. □

On the one hand the Theorem gives some restrictions on the values of the cardinals $\mathrm{add}(\mathbf{P})$ and $\mathrm{cof}(\mathbf{P})$. On the other hand we still cannot decide whether inequalities $\mathrm{add}(\mathbf{P}) > \omega_1$ and $\mathrm{cof}(\mathbf{P}) < 2^\omega$ are possible.

An example of a σ-ideal which fulfils assuptions of the Claim is the σ-ideal generated by closed Lebesgue measure zero sets.

References

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