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POROUS SETS AND ADDITIVITY OF LEBESGUE MEASURE

The aim of this paper is to compare some set-theoretic cardinal characteristics of the ideal of σ-porous sets with those similar cardinals associated with the notions of measure and category which have been extensively studied by A. W. Miller, J. Ihoda, S. Shelah, and others. We will prove that every set of reals of cardinality less than the additivity of the ideal of Lebesgue measure zero sets is σ-porous, the real line can be covered by a family of closed porous sets of cardinality of any cofinal family of the ideal of Lebesgue measure zero sets and it is consistent that the minimal cardinality of a set which is not σ-porous is greater than the minimal cardinality of an unbounded family of functions from ${}^{\omega}\omega$. In fact, we will prove all these for the ideal of σ-strongly symmetrically porous sets.

If A is a subset of the real line $\mathbf{R}$, $I = (a, b)$ is an open interval then we denote by $\lambda(A, I)$ the length of the largest open subinterval of I which does not intersect A and $\lambda^*(A, I)$ is the largest $\delta \geq 0$ such that $(a, a+\delta) \cup (b-\delta, b)$ is disjoint with A. The porosity and the symmetric porosity of A at $c \in \mathbf{R}$ is the number

$$p(A,c) = \limsup_{\varepsilon \to 0^+} \frac{\lambda(A, (c-\varepsilon, c+\varepsilon))}{\varepsilon} \quad \text{and}$$
$$s(A,c) = \limsup_{\varepsilon \to 0^+} \frac{\lambda^*(A, (c-\varepsilon, c+\varepsilon))}{\varepsilon},$$

respectively. We say that A is porous (resp. strongly porous, resp. strongly symmetrically porous) if $p(A, a) > 0$ (resp. $p(A, a) = 1$, resp. $s(A, a) = 1$) for every $a \in A$. We say that A is σ-porous (resp. σ-strongly porous, resp. σ-strongly symmetrically porous) if it is a countable union of porous (resp. strongly porous resp. strongly symmetrically porous) sets. See [11].

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Let us recall some notation and cardinal characteristics. By ω we denote the set of natural numbers, i.e. $\omega = \{0, 1, 2, \dots\}$. Each natural number is a von Neuman ordinal, i.e. $n = \{0, 1, \dots, n-1\}$, $0 = \emptyset$ and $n < m$ iff $n \in m$ (and $n \le m$ iff $n \subseteq m$). Let x, y be sets. Then ${}^x y$ is the set of all functions from x into y; $\mathcal{P}(x)$ is the power set of x, i.e. $\mathcal{P}(x)$ is the set of all subsets of x; $[x]^{<\omega}$ is the set of all finite subsets of x; ${}^{<\omega}x = \bigcup_{n\in\omega} {}^n x$; $|x|$ denotes the cardinality of x and observe that if $s \in {}^n x$ then $|s| = n$. But if I is an interval of reals then $|I|$ denotes the length of I.

$\mathbf{p}$ denotes the minimal cardinality of a family $F \subseteq \mathcal{P}(\omega)$ such that $\bigcap F_0$ is infinite for every finite $F_0 \subseteq F$ and for every infinite $x \subseteq \omega$ there is $y \in F$ such that $x - y$ is infinite (see [2]). Since uniform ultrafilter is a such family, $\mathbf{p} \le 2^\omega$.

Let $(P, \le)$ be a partially ordered set. A set $P_0 \subseteq P$ is said to be cofinal in P if for every $p \in P$ there is $q \in P_0$ such that $p \le q$; P_0 is unbounded if no element of P dominates all elements of P_0.

$$\begin{aligned}
\mathbf{b}(P, \le) &= \min\{|P_0|;\ P_0 \text{ is an unbounded subset of } P\},\\
\mathbf{d}(P, \le) &= \min\{|P_0|;\ P_0 \text{ is a cofinal subset of } P\}.
\end{aligned}$$

et $\mathcal{F} \subseteq {}^\omega([\omega]^{<\omega})$, i.e. $\mathcal{F}$ is a family of functions from ω into the family of finite subsets of ω. For $f, g \in \mathcal{F}$ we put $f \le g$ iff g eventually dominates f, i.e. $\forall^\infty n\ f(n) \subseteq g(n)$. We use symbols $\forall^\infty n$, $\exists^\infty n$ as abbreviations for $(\exists m)(\forall n > m)$, $(\forall m)(\exists n > m)$. We will simply write $\mathbf{b}(\mathcal{F})$, $\mathbf{d}(\mathcal{F})$ instead of $\mathbf{b}(\mathcal{F}, \le)$, $\mathbf{d}(\mathcal{F}, \le)$. Particulary we denote $\mathbf{b} = \mathbf{b}({}^\omega\omega)$ and $\mathbf{d} = \mathbf{d}({}^\omega\omega)$. This is well defined because $\omega \subseteq [\omega]^{<\omega}$.

Let $\mathcal{I} \subseteq \mathcal{P}(\mathbf{R})$ be an ideal. Let us define:

$$\begin{aligned}
\operatorname{add}(\mathcal{I}) &= \min\{|\mathcal{I}_0|;\ \mathcal{I}_0 \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{I}_0 \notin \mathcal{I}\} = \mathbf{b}(\mathcal{I}, \subseteq),\\
\operatorname{cov}(\mathcal{I}) &= \min\{|\mathcal{I}_0|;\ \mathcal{I}_0 \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{I}_0 = \mathbf{R}\},\\
\operatorname{non}(\mathcal{I}) &= \min\{|A|;\ A \subseteq \mathbf{R} \text{ and } A \notin \mathcal{I}\},\\
\operatorname{cof}(\mathcal{I}) &= \min\{|\mathcal{I}_0|;\ \mathcal{I}_0 \subseteq \mathcal{I} \text{ and } \forall A \in \mathcal{I}\ \exists B \in \mathcal{I}_0\ A \subseteq B\} = \mathbf{d}(\mathcal{I}, \subseteq).
\end{aligned}$$

In the following $\mathbf{L}, \mathbf{K}, \mathbf{P}$ and $\mathbf{S}$ denote the ideal of Lebesgue measure zero sets, the ideal of sets of first category, the ideal of σ-strongly porous sets, and the ideal of σ-strongly symmetrically porous sets respectively. Some facts about the ideals $\mathbf{L}$ and $\mathbf{K}$ are summarized (see [3]) in the following diagram:

$$\begin{tikzcd}
& \operatorname{cov}\mathbf{L} \arrow[r] & \operatorname{non}\mathbf{K} \arrow[r] & \operatorname{cof}\mathbf{K} \arrow[r] & \operatorname{cof}\mathbf{L} \arrow[r] & 2^\omega \\
& & \mathbf{b} \arrow[r] \arrow[u] & \mathbf{d} \arrow[u] & & \\
\omega_1 \arrow[r] & \operatorname{add}\mathbf{L} \arrow[r] \arrow[uu] & \operatorname{add}\mathbf{K} \arrow[r] \arrow[u] & \operatorname{cov}\mathbf{K} \arrow[r] \arrow[u] & \operatorname{non}\mathbf{L} \arrow[uu] &
\end{tikzcd}$$

The cardinals increase (not necessarily strictly) from south-west to north-east. The diagram is called Cichoń's diagram in Europe and Kunen-Miller chart in the rest of the world.

Every porous set has measure zero and is nowhere dense. Both the ideals $\mathbf{P}$ and $\mathbf{S}$ are included in the ideal $\mathbf{P}^+$ of σ-porous sets. F. Galvin and A. W. Miller [5] noticed that $\mathbf{p}$ is the minimal cardinality of a subset of the real line which is not a γ-set. Let us recall the definition of γ-set. A family $\mathcal{A}$ is an ω-cover of a set A if for every finite $A_0 \subseteq A$ there is $V \in \mathcal{A}$ such that $A_0 \subseteq V$. A set A is a γ-set if for every open ω-cover $\mathcal{A}$ of A there is a sequence $\langle V_n;\, n \in \omega\rangle \in {}^{\omega}\mathcal{A}$ such that $X \subseteq \bigcup_{n\in\omega} \bigcap_{m>n} V_m$. I. Recław [10] proved that if $A \subseteq \mathbf{R}$ is a γ-set then A is σ-porous. Recław's proof can be slightly improved to show that every γ-set on $\mathbf{R}$ can be covered by countably many closed strongly symmetrically porous sets. The natural question is how large can $\mathrm{non}(\mathbf{P})$ and $\mathrm{non}(\mathbf{S})$ be. Of course $\mathbf{p} \le \mathrm{non}(\mathbf{S}) \le \mathrm{non}(\mathbf{P}) \le \min\{\mathrm{non}(\mathbf{L}), \mathrm{non}(\mathbf{K})\}$. It is well known that $\mathbf{p} \le \mathbf{b}$ (see e.g. [2]) and even $\mathbf{p} \le \mathrm{add}(\mathbf{K})$ since $\mathbf{p} \le \mathrm{cov}(\mathbf{K})$ (see e.g. [4]) and $\mathrm{add}(\mathbf{K}) = \min\{\mathrm{cov}(\mathbf{K}), \mathbf{b}\}$ (see [9] or [3]). The inequality $\mathbf{b} \le \mathrm{non}(\mathbf{P})$ is not provable in ZFC since in the generic extension by adding ω_2 Laver reals (see [6]), $\mathbf{b} = \omega_2$ and $\mathrm{non}(\mathbf{L}) = \omega_1$. Neither the inequality $\mathrm{non}(\mathbf{S}) \le \mathbf{b}$ is provable in ZFC (Theorem 7). On the other hand, $\mathrm{add}(\mathbf{L}) \le \mathbf{b}$ holds (see [8]) and $\mathrm{add}(\mathbf{L})$ is another lower estimate of $\mathrm{non}(\mathbf{S})$ (Theorem 3).

The inequality $\mathbf{p} < \mathrm{non}(\mathbf{S})$ is possible because the inequality $\mathbf{p} < \mathrm{add}(\mathbf{L})$ is consistent. But we cannot decide whether any of the cases $\mathrm{non}(\mathbf{S}) < \mathrm{non}(\mathbf{P})$ and $\mathbf{p} \le \mathrm{non}(\mathbf{P}) < \min\{\mathrm{non}(\mathbf{K}), \mathrm{non}(\mathbf{L})\}$ is possible and what is true for $\mathrm{add}(\mathbf{K})$ and $\mathbf{d}$. Can the additivity of these ideals be greater then ω_1? L. Zajíček [12] proved that $\mathbf{P} \ne \mathbf{P}^+$ and D. Preiss asks (oral comunication) whether the ideals $\mathbf{P}$ and $\mathbf{P}^+$ can be distinguished by some cardinal characteristic. The question is still open.

We will need some combinatorial characterizations of $\mathrm{add}(\mathbf{L})$ and $\mathrm{cof}(\mathbf{L})$. Let $g \in {}^{\omega}\omega$ be arbitrary. Let us denote

$$\begin{aligned}
\mathcal{L}_g &= \{h;\, h : \omega \to [\omega]^{<\omega} \text{ and } \forall n\ |h(n)| \le g(n)\},\\
\mathcal{F}_g &= \{f \in {}^{\omega}\omega;\, \lim_{n\to\infty} f(n)/g(n) = 0\},\\
\mathcal{S}_g &= \{h;\, h : \omega \to [\omega]^{<\omega} \text{ and } \lim_{n\to\infty} |h(n)|/g(n) = 0\}.
\end{aligned}$$

The next theorem is well known; its first part is completely proved in [1] and the second part can be proved by using the same ideas. In fact, both can be done uniformly at the same time. For our purpose the modification of this theorem formulated in Theorem 2 will be more useful.

Theorem 1. *Let $g \in {}^\omega\omega$ be monotone unbounded. Then*

$$\mathrm{add}(\mathbf{L}) = \min\{|\mathcal{F}|;\ \mathcal{F} \subseteq {}^\omega\omega \textit{ and } \forall x \in \mathcal{L}_g\ \exists y \in \mathcal{F}\ \exists^\infty n\ y(n) \notin x(n)\} \quad \textit{and}$$
$$\mathrm{cof}(\mathbf{L}) = \min\{|\mathcal{L}|;\ \mathcal{L} \subseteq \mathcal{L}_g \textit{ and } \forall y \in {}^\omega\omega\ \exists x \in \mathcal{L}\ \forall^\infty n\ y(n) \in x(n)\}.$$

A function $g \in {}^\omega\omega$ is said to be finite-to-one if the inverse image of any finite set is finite or, equivalently, if there is a permutation π of ω such that $g(\pi(.))$ is monotone unbounded.

Theorem 2. *Let $g \in {}^\omega\omega$ be finite-to-one. Then* $\mathrm{add}(\mathbf{L}) = \mathbf{b}(\mathcal{S}_g)$ *and* $\mathrm{cof}(\mathbf{L}) = \mathbf{d}(\mathcal{S}_g)$.

Notice that it is enough to prove this theorem in the case g is monotone unbounded. In the proof we will need the next lemma.

Lemma 2.1. *Let $g \in {}^\omega\omega$ be monotone unbounded. Then $\mathbf{b} = \mathbf{b}(\mathcal{F}_g)$ and $\mathbf{d} = \mathbf{d}(\mathcal{F}_g)$.*

Proof. Let $\mathcal{M}$ be the family of all unbounded functions from ω into ω. $\mathcal{M}$ is cofinal in ${}^\omega\omega$. We will define a family $\mathcal{F} \subseteq \mathcal{F}_g$ cofinal in $\mathcal{F}_g$ and mappings $\alpha : \mathcal{F} \to \mathcal{M}$ and $\beta : \mathcal{M} \to \mathcal{F}$ such that $\mathcal{F} = \beta(\mathcal{M})$ and

(a) if $f \in \mathcal{F}$ then $\beta(\alpha(f))(n) > f(n)$ for all but finitely many $n \in \omega$,

(b) if $h \in \mathcal{M}$ then $\alpha(\beta(h))(n) > h(n)$ for all n.

Let us define

$$\alpha(f)(n) = \min\{k;\ \forall m \geq k\ f(m) < g(m)/(n+1)\},$$
$$\beta(h)(n) = \min\{k;\ (m+1)k \geq g(n)\}, \quad \text{if } \max_{i \leq m} h(i) < n \leq \max_{i \leq m+1} h(i) \text{ and}$$
$$\beta(h)(n) = 0, \quad \text{otherwise.}$$

We will verify conditions (a), (b).

(a) Let $f \in \mathcal{F}$ and let $n \in \omega$. Then either $\beta(\alpha(f))(n) = 0$ and then $n \leq \alpha(f)(0)$ or there is $m \in \omega$ such that $\max_{i \leq m} \alpha(f)(i) < n \leq \max_{i \leq m+1} \alpha(f)(i)$. In the latter case by the defintion of β, $\beta(\alpha(f))(n) \geq g(n)/(m+1)$ and by the definition of α, $g(n)/(m+1) > f(n)$ since $\alpha(f)(m) \leq n$.

(b) Denote $m = \max_{i \leq n+1} h(i)$. Then $\beta(h)(m) \geq g(m)/(n+1)$. The definition of α implies $\alpha(\beta(h))(n) > m \geq h(n)$.

The mappings α, β are monotone (with respect to eventual dominance) and if $f \in \mathcal{F}$ and $h \in \mathcal{M}$ then using (a) and (b) we have:

$$\alpha(f) \leq h \text{ implies } f \leq \beta(h), \quad \text{and}$$
$$\beta(h) \leq f \text{ implies } h \leq \alpha(f).$$

To finish the proof it is sufficient to observe that the following lemma holds: $\square$

Lemma 2.2 ([3]). *Let* $(P, \leq_P)$, $(Q, \leq_Q)$ *be partially ordered sets and let* $\alpha : P \to Q$ *and* $\beta : Q \to P$ *be mappings such that* $\alpha(p) \leq_Q q$ *implies* $p \leq_P \beta(q)$ *for all* $p \in P$, $q \in Q$. *Then* $\mathbf{b}(P, \leq_P) \leq \mathbf{b}(Q, \leq_Q)$ *and* $\mathbf{d}(Q, \leq_Q) \leq \mathbf{d}(P, \leq_P)$.

Proof of Theorem 2. According to Theorem 1, since every element of ${}^\omega\omega$ we can identify with an element of $\mathcal{S}_g$, we have $\mathbf{b}(\mathcal{S}_g) \leq \mathrm{add}(\mathbf{L})$ and $\mathrm{cof}(\mathbf{L}) \leq \mathbf{d}(\mathcal{S}_g)$.

Let π be a fixed one-to-one mapping from $[\omega]^{<\omega}$ into ω. Let us define several mappings:

$$\begin{aligned} &\beta : \mathcal{F}_g \to {}^\omega\omega \quad \text{by } \beta(f)(n) = \max\{m;\, \forall k > n\ f(k)m^2 < g(k)\}, \\ &\delta : \mathcal{S}_g \to {}^\omega\omega \quad \text{by } \delta(h)(n) = \pi(h(n)), \end{aligned}$$

and for every $f \in \mathcal{F}_g$ let $\varepsilon_f : \mathcal{L}_{\beta(f)} \to \mathcal{S}_g$ be defined by

$$\varepsilon_f(x)(n) = \bigcup\{\pi^{-1}(k);\, k \in x(n) \text{ and } |\pi^{-1}(k)| < f(n)\}.$$

If $f \in \mathcal{F}_g$ then $\beta(f)$ is monotone unbounded and $f \cdot \beta(f) \in \mathcal{F}_g$. Thus the mappings ε_f are well defined. Moreover, if $f \in \mathcal{F}_g$ and $x \in \mathcal{L}_{\beta(f)}$ then

$$\text{if } h \in \mathcal{L}_f \text{ and } \forall^\infty n\ \delta(h)(n) \in x(n) \text{ then } h \leq \varepsilon_f(x). \tag{$*$}$$

We will show that $\mathrm{add}(\mathbf{L}) \leq \mathbf{b}(\mathcal{S}_g)$. Let $\mathcal{S} \subseteq \mathcal{S}_g$ and $|\mathcal{S}| < \mathrm{add}(\mathbf{L})$. Since $\mathrm{add}(\mathbf{L}) \leq \mathbf{b}$, by Lemma 2.1, there is $f \in \mathcal{F}_g$ such that $\mathcal{S} \subseteq \mathcal{L}_f$ and, by Theorem 1, there is $x \in \mathcal{L}_{\beta(f)}$ such that for every $h \in \mathcal{S}$, $\forall^\infty n\ \delta(h)(n) \in x(n)$. Then, by $(*)$, $h \leq \varepsilon_f(x)$ holds for every $h \in \mathcal{S}$. Therefore $\mathrm{add}(\mathbf{L}) = \mathbf{b}(\mathcal{S}_g)$.

Let $\mathcal{F} \subseteq \mathcal{F}_g$ be a family of cardinality $\mathbf{d}$ such that every element of $\mathcal{F}_g$ is dominated by an element of $\mathcal{F}$. By Lemma 2.1 such family exists. Using Theorem 1, assign to each $f \in \mathcal{F}$ a $\mathcal{K}_f \subseteq \mathcal{L}_{\beta(f)}$ of cardinality $\mathrm{cof}(\mathbf{L})$ such that $\forall y \in {}^\omega\omega\ \exists x \in \mathcal{K}_f\ \forall^\infty n\ y(n) \in x(n)$. Then the family $\mathcal{S} = \{\varepsilon_f(x);\, f \in \mathcal{F} \text{ and } x \in \mathcal{K}_f\} \subseteq \mathcal{S}_g$ has cardinality $\mathrm{cof}(\mathbf{L})$ since $\mathbf{d} \leq \mathrm{cof}(\mathbf{L})$ (see [8] or [3]), and by $(*)$, every element of $\mathcal{S}_g$ is dominated by a member of $\mathcal{S}$. Therefore $\mathbf{d}(\mathcal{F}_g) \leq \mathrm{cof}(\mathbf{L})$ and so $\mathbf{d}(\mathcal{F}_g) = \mathrm{cof}(\mathbf{L})$. □

Theorem 3. $\mathrm{add}(\mathbf{L}) \leq \mathrm{non}(\mathbf{S})$ *and* $\mathrm{cov}(\mathbf{S}) \leq \mathrm{cof}(\mathbf{L})$.

Proof. We will show that the ideal $\mathbf{S}$ contains some ideal $\mathbf{Sm}_\rho$ such that $\mathrm{add}(\mathbf{L}) \leq \mathrm{add}(\mathbf{Sm}_\rho)$ and $\mathrm{cof}(\mathbf{Sm}_\rho) \leq \mathrm{cof}(\mathbf{L})$ and $\mathbf{Sm}_\rho$ contains all singletons. Instead of $\mathbf{R}$ we can and we will confine ourselves to the closed interval $\langle 0, 1\rangle 1$.

Let us denote:

$$W = \{\rho \in {}^\omega\omega;\, \rho \text{ is finite-to-one and } \forall n\ \rho(n) > 1\}.$$

If $\rho \in W$ then $T_\rho = \{s \in {}^{<\omega}\omega;\ \forall n \in \mathrm{dom}(s)\ s(n) \in \rho(n)\}$ and $X_\rho = \{x \in {}^\omega\omega;\ \forall n\ x(n) \in \rho(n)\}$. Let φ_ρ be a mapping from X_ρ onto $\langle 0,1\rangle$ defined by

$$\varphi_\rho(x) = \sum_{n\in\omega} \frac{x(n)}{\rho(0)\rho(1)\ldots\rho(n)}$$

i.e. the sequence x is the Cantor expansion of the real $\varphi_\rho(x)$. If $s \in T_\rho \cap {}^n\omega$ then $I^\rho_s = \{\varphi_\rho(x);\ x \in X_\rho \text{ and } s \subseteq x\}$ is a closed subinterval of $\langle 0,1\rangle$ of length $1/(\rho(0)\rho(1)\ldots\rho(n-1))$. (Let us recall that $\emptyset$ is the empty sequence and if s is a sequence of length n with values $s(0), s(1), \ldots, s(n-1)$ then $s^\frown k$ denotes the sequence of length $n+1$ which extends s and $s(n) = k$). Thus $I^\rho_\emptyset = \langle 0,1\rangle$ and if $|s| = n$ then I^ρ_s is divided into $\rho(n)$ intervals $I^\rho_{s^\frown k}$, $k \in \rho(n)$, with disjoint interiors and of the same length, i.e. $I^\rho_s = \bigcup\{I^\rho_{s^\frown k};\ k \in \rho(n)\}$ and $\varphi_\rho(x)$ is the unique element of $\bigcap\{I^\rho_{x\restriction n};\ n \in \omega\}$. For $h : T_\rho \to [\omega]^{<\omega}$ let us denote $h^*(n) = \sup\{|h(s)|;\ s \in T_\rho \cap {}^n\omega\}$. Notice that $h^*(n) < \infty$. Let $H_\rho = \{h : T_\rho \to [\omega]^{<\omega};\ \lim_{n\to\infty} h^*(n)/\log(\rho(n)) = 0\}$. For $\rho \in W$, $h \in H_\rho$ and $m \in \omega$ put

$$X^m_{\rho,h} = \{x \in X_\rho;\ \forall n > m\ x(n) \in h(x\restriction n)\},$$

$$A^m_{\rho,h} = \varphi_\rho(X^m_{\rho,h}) \quad \text{and} \quad A_{\rho,h} = \bigcup_{m\in\omega} A^m_{\rho,h}.$$

A set $A \subseteq \langle 0,1\rangle$ is said to be ρ-small if $A \subseteq A_{\rho,h}$ for some $h \in H_\rho$. Let $\mathbf{Sm}_\rho$ denote the family of all ρ-small sets. $\mathbf{Sm}_\rho$ is an ideal since the set H_ρ is upward directed (in the ordering $f \le h$ iff $f(s) \subseteq h(s)$ for all but finitely many $s \in T_\rho$) and $f \le h$ implies $A_{\rho,f} \subseteq A_{\rho,h}$ for $f, h \in H_\rho$. Since for every $x \in X_\rho$ there is an $h \in H_\rho$ with $x(n) \in h(x\restriction n)$ for all n, $\mathbf{Sm}_\rho$ contains all singletons. □

Lemma 3.1. *For each $\rho \in W$,* $\mathrm{add}(\mathbf{L}) \le \mathrm{add}(\mathbf{Sm}_\rho)$ *and* $\mathrm{cof}(\mathbf{Sm}_\rho) \le \mathrm{cof}(\mathbf{L})$.

Proof. For every $A \in \mathbf{Sm}_\rho$ fix some $\alpha(A) \in H_\rho$ such that $A \subseteq A_{\rho,\alpha(A)}$. Then $\alpha(A) \le h$ implies $A \subseteq A_{\rho,h}$. By Lemma 2.2, $\mathbf{b}(H_\rho) \le \mathrm{add}(\mathbf{Sm}_\rho)$ and $\mathrm{cof}(\mathbf{Sm}_\rho) \le \mathbf{d}(H_\rho)$. Since the function

$$g_1(s) = \text{“the greatest integer lower than } \log(\rho(|s|))\text{”}, \quad \text{for } s \in T_\rho$$

is finite-to-one, $H_\rho = \{h : T_\rho \to [\omega]^{<\omega};\ \lim_{|s|\to\infty} h(s)/g_1(s) = 0\}$ and $|T_\rho| = \omega$, H_ρ corresponds to some $\mathcal{S}_g$ with a g finite-to-one and $\mathbf{b}(H_\rho) = \mathbf{b}(\mathcal{S}_g)$ and $\mathbf{d}(H_\rho) = \mathbf{d}(\mathcal{S}_g)$ and so Theorem 2 concludes the proof. □

Lemma 3.2. *For every $\rho \in W$,* $\mathbf{Sm}_\rho \subseteq \mathbf{S}$.

Proof. Let $h \in H_\rho$ be arbitrary. We will prove that $A_{\rho,h}$ is σ-strongly symmetrically porous. Since $A^m_{\rho,h} = A^0_{\rho,g}$ for some $g \in H_\rho$, it is enough to show that $A = A^0_{\rho,h}$ is strongly symmetrically porous. Let $a = \varphi_\rho(x)$ be an arbitrary element of A. We will show that $s(A,a) = 1$.

Let $m \in \omega$ be arbitrary. Let us denote $k(i) = 4h^*(i)+1$. There is $n \in \omega$ such that $|I^\rho_{x\restriction n}| < 1/m$ and $(2m+1)^{k(n)} \leq \rho(n)$ since $\lim_{n\to\infty} h^*(n)/\log(\rho(n)) = 0$. Denote $k = k(n)$. Let $v = \{-m, -m+1, \dots, m-1, m\}$. Then $|v| = 2m+1$. Let $\{J_t;\ t \in \bigcup_{i\leq k} {}^i v\}$ be a family of closed intervals such that if $r = 0^\frown 0^\frown \dots {}^\frown 0$ (k times) then $J_r = I^\rho_{x\restriction(n+1)}$; $J_t = \bigcup\{J_{t^\frown j};\ j \in v\}$ when $t \in {}^i v$ for $i < k$ and the intervals $J_{t^\frown j}$, $j \in v$ have disjoint interiors, the same length and $J_{t^\frown j}$ is on the left of $J_{t^\frown j+1}$. Thus $|J_\emptyset| = |J_r|(2m+1)^k \leq |J_r|\rho(n) = |I^\rho_{x\restriction n}|$. Therefore, there is $s \in T_\rho$, $|s| = n$ such that I^ρ_s, $I^\rho_{x\restriction n}$ are neighboring intervals and they cover $J_\emptyset \cap \langle 0,1\rangle$ jointly. Each interval $J_t \subseteq \langle 0,1\rangle$ with $|t| = k$ is an interval I^ρ_s for some s, $|s| = n+1$. If $s \in T_\rho$, $|s| = n$ then A intersects at most $2h^*(n)$ intervals among $I^\rho_{s^\frown i}$, $i \in \rho(n)$ (neighboring intervals have non-empty intersection). Therefore, A intersects at most $4h^*(n) < k$ intervals among J_t, $|t| = k$. Let us denote $J_i = J_{r\restriction i}$ for $i \leq k$. Then $J_{i+1} \subseteq J_i$ have the same centre, $|J_i| = (2m+1)|J_{i+1}|$ and for some $i < k$, $A \cap (J_i - J_{i+1}) = \emptyset$. Fix such an i and put $\varepsilon = m|J_{i+1}|$. Then $\varepsilon < |J_\emptyset| < 1/m$ and $A \cap (a-\varepsilon, a+\varepsilon) \subseteq A \cap J_i \subseteq J_{i+1}$. That is why $\lambda^*(A,(a-\varepsilon), a+\varepsilon)) \geq \varepsilon - |J_{i+1}| = (1-1/m)\varepsilon$. Since m was arbitrary, $s(A,a) = 1$. This concludes the proofs of the lemma and Theorem 3. □

Corollary 4. *Every subset of the real line of cardinality less than* $\mathrm{add}(\mathbf{L})$ *can be covered by countably many closed strongly symmetrically porous sets and the real line can be covered by* $\mathrm{cof}(\mathbf{L})$ *closed strongly symmetrically porous sets.*

Proof. The sets $A^m_{\rho,h}$ are closed strongly symmetrically porous. □

Both $\mathbf{p}$ and $\mathrm{add}(\mathbf{L})$ are lower estimates of $\mathrm{non}(\mathbf{S})$, but neither of them is better than another since the next two consistencies hold:

$\mathrm{Con}(\mathrm{add}(\mathbf{L}) < \mathbf{p})$. Start with CH (in the ground model) and iterate forcing ω_2 times with σ-centered partial orders of size ω_1. In the generic extension, $\mathbf{p} = \omega_2$ and, because no random real is added, $\mathrm{cov}(\mathbf{L}) = \omega_1$.

$\mathrm{Con}(\mathbf{p} < \mathrm{add}(\mathbf{L})$. Start with $\omega_1 < 2^\omega < 2^{\omega_1}$ and 2^ω is a regular cardinal number. Let $\mathcal{K} = \{f \in {}^\omega\omega;\ \sum_{n\in\omega} 1/f(n) < 1\}$ be ordered by $f \leq g$ iff $\forall n\ f(n) \leq g(n)$ (f is stronger than g). $\mathcal{K}$ is a c.c.c. notion of forcing (see 33C of [4]). In the generic extension by $\mathcal{K}$, all the convergent series of the ground model are majorized by the generic series. Iteratively, with finite support, add 2^ω generic series. In this extension $\omega_1 < 2^\omega < 2^{\omega_1}$ remains true and so $\mathbf{p} = \omega_1$ (see [2]), and any family of less than 2^ω convergent series is majorized by a single convergent series and so $\mathrm{add}(\mathbf{L}) = 2^\omega$ (see [1]).

It is consistent that $\mathrm{cof}(\mathbf{L}) < 2^\omega$ (see [9]). Thus it is cosistent that $\mathrm{cov}(\mathbf{S}) < 2^\omega$.

$A \subseteq \langle 0,1\rangle$ is said to be small if it is ρ-small for some $\rho \in R$. We have already shown that every small set is σ-strongly symmetrically porous. The next theorem strengthens the Recław's result [10].

Theorem 5. *If $A \subseteq \langle 0,1\rangle$ is a γ-set then A is small.*

Proof. Let $A \subseteq \langle 0,1\rangle$ be a γ-set. We will find $\rho \in R$ and $h \in H_\rho$ such that $A \subseteq A_{\rho,h}$. Let $\sigma(n) = 2^{n^2}$. Let us fix a sequence $\langle y_n;\, n \in \omega\rangle$ of distinct members of A. For every $n \in \omega$ and $B \in [A]^n$ let

$$g(B) = \{s \in T_\sigma;\ |s| = n \text{ and } B \cap I^\sigma_s \neq \emptyset\}.$$

Of course $|g(B)| \leq 2n$ since each $x \in A$ can be in two neighboring I^σ_s. Let $V(B) = \mathrm{Int}\bigcup\{I^\sigma_s;\, s \in g(B)\}$; $B \subseteq V(B)$. Let $\mathcal{A}_n = \{V(B) - \{y_n\};\, B \in [A]^n\}$ and let $\mathcal{A} = \bigcup_{n\in\omega} \mathcal{A}_n$. $\mathcal{A}$ is an open ω-cover of A. Therefore there is a sequence $\langle V_k;\, k \in \omega\rangle \in {}^\omega\mathcal{A}$ such that $A \subseteq \bigcup_{m\in\omega}\bigcap_{k>m} V_k$. Since $y_n \in A$, only finitely many sets V_k may belong to $\mathcal{A}_n$. Thus we can assume that from every family $\mathcal{A}_n$ at most one element was chosen. Let n_k, $k \in \omega$ be an increasing sequence of natural numbers such that $n_0 = 0$ and $V_k \in \mathcal{A}_{n_k}$ and $V_k \subseteq V(B_k)$ for some B_k with $|B_k| = n_k$ for $k > 0$. Let Π_k be the family of all functions s with domain $\{i \in \omega;\, n_k \leq i < n_{k+1}\}$ such that $s(i) \in \sigma(i)$ for all i, i.e. $\Pi_k = \prod\{\sigma(i);\, n_k \leq i < n_{k+1}\}$. Put $\rho(k) = |\Pi_k|$. For $s \in \Pi_k$ let

$$\pi_k(s) = \sum\{s(n)\sigma(n+1)\sigma(n+2)\ldots\sigma(n_{k+1}-1);\ n_k \leq n < n_{k+1}\}.$$

It is not dificult to verify that π_k is a one-to-one function from Π_k onto $\rho(k)$. Moreover, if $x \in X_\sigma$ and $y \in X_\rho$ is defined by $y(k) = \pi_k(x\restriction\langle n_k, n_{k+1}))$ then $\varphi_\sigma(x) = \varphi_\rho(y)$. For $s \in T_\rho$, $|s| = k$ let us define

$$h(s) = \{n \in \rho(k);\ \pi_0^{-1}(s(0)) \cup \cdots \cup \pi_{k-1}^{-1}(s(k-1)) \cup \pi_k^{-1}(n) \in g(B_{k+1})\}.$$

Then $h^*(k) \leq 2n_{k+1}$ and

$$\lim_{k\to\infty} h^*(k)/\log(\rho(k)) \leq \lim_{k\to\infty} 2n_{k+1}/\log(\sigma(n_{k+1}-1)) = 0$$

and so $h \in H_\rho$. For every $B \in [A]^n$ and $n \in \omega$ if $\varphi_\sigma(x) \in V(B)$ then $x\restriction n \in g(B)$. Therefore we have: $\bigcap_{k\geq m} V_{k+1} \subseteq \bigcap_{k\geq m} V(B_{k+1}) \subseteq \varphi_\sigma(\{x \in X_\sigma;\ \forall k > m\ x\restriction n_{k+1} \in g(B_{k+1})\}) = \varphi_\rho(\{y \in X_\rho;\ \forall k > m\ y(k) \in h(y\restriction k)\}) = A^m_{\rho,h}$ and so $A \subseteq A_{\rho,h}$. $\square$

Theorem 6. *Let κ be a cardinal number such that $\kappa^\omega = \kappa$. Then there is a generic extension in which $2^\omega = \kappa$, $\mathbf{b} = \omega_1$ and* $\mathrm{non}(\mathbf{S}) = \mathrm{cov}(\mathbf{S}) = \mathrm{cf}(\kappa)$.

Moreover, in this generic extension every subset of the real line of cardinality less than $\mathrm{cf}(\kappa)$ *can be covered by countably many closed strongly symmetrically porous sets and the real line can be covered by* $\mathrm{cf}(\kappa)$ *closed strongly symmetrically porous sets.*

In the proof of this theorem we will use the notion of forcing introduced by J. I. Ihoda and S. Shelah [6]. It is called the meager forcing:

For $\tau \subseteq {}^{<\omega}2$ denote $n(\tau) = \sup\{|s|; s \in \tau\}$. Let T be the set of all $\tau \subseteq {}^{<\omega}2$ such that:

(i) $n(\tau) < \omega$,

(ii) if $s \in \tau$ and $k < |s|$ then $s \restriction k \in \tau$,

(iii) if $s \in \tau$ and $|s| < n(\tau)$ then $s^\frown 0 \in \tau$ or $s^\frown 1 \in \tau$.

For $\tau \in T$ let $[\tau] = \{x \in {}^{\omega}2;\ x\restriction n(\tau) \in \tau\}$ and $\overline{\tau} = \{s \in \tau;\ |s| = n(\tau)\}$. The meager forcing is the set $M = \{(\tau, H);\ \tau \in T$ and H is a finite subset of $[\tau]\}$ ordered by $(\tau, H) \le (\sigma, K)$ iff $\overline{\sigma} = \tau \cap {}^{n(\sigma)}2$ and $K \subseteq H$.

For each $n \in \omega$ let $D(n)$ be the set of all conditions $(\tau, H) \in M$ such that $|\overline{\tau}| < (n(\tau) - n)/n$.

Lemma 6.1. *For each* $n \in \omega$, $D(n)$ *is a dense subset of* M.

Proof. Let $(\sigma, K) \in M$ be arbitrary. There are integers k, m such that $n(\tau) < m < k$, all $x\restriction m$ for $x \in K$ are different, and $2^m < (k - n)/n$. Choose $\tau \in T$ such that $n(\tau) = k$, $\tau \cap {}^{n(\sigma)}2 = \overline{\sigma}$, and $|\overline{\tau}| = |{}^{m}2 \cap \tau|$ (i.e. every $s \in {}^{m}2 \cap \tau$ has exactly one extension in $\overline{\tau}$). Then $(\tau, K) \le (\sigma, K)$ and $(\tau, K) \in D(n)$ since $|\overline{\tau}| \le 2^m$. □

Lemma 6.2. *In the generic extension over the meager forcing, the set of reals of the ground model can be covered by countably many closed strongly symmetrically porous sets.*

Proof. To prove the lemma it is enough to show that the set of reals of the ground model is ρ-small for some ρ in the generic extension (see the proof of Lemma 3.2).

Let $G \subseteq M$ be a V-generic filter over M. We are working in $V[G]$. The generic tree $S = \bigcup\{\tau;\ (\tau, \emptyset) \in G\}$ has no endpoints and $(\tau, \emptyset) \in G$ iff $\tau = \sigma(n(\tau))$ where $\sigma(n) = \{t \in S;\ |t| \le n\}$ for $n \in \omega$. The set $C = \{x \in {}^{\omega}2;\ \forall n\ x\restriction n \in S\}$ of all branches of S is a closed nowhere dense subset of ${}^{\omega}2$ and for every $x \in {}^{\omega}2 \cap V$ there is $y \in C$ such that $x(n) = y(n)$ for all but finitely many $n \in \omega$.

By Lemma 6.1, we can find in $V[G]$ an increasing sequence $n_k,\ k \in \omega$ of integers such that $n_0 = 0$ and $n_{k+1} = \min\{n > 2n_k;\ (\sigma(n), \emptyset) \in D(n_k)\}$. Let us define $\rho(k) = 2^{n_{k+1}-n_k}$. Of course $\rho \in W$. For every k let π_k be the one-to-one function from ${}^{\langle n_k, n_{k+1})}2$ onto $\rho(k)$ which is defined in the proof of Theorem 5 (for $\sigma \equiv 2$). Let $H^*(k) = \{n \in \rho(k);\ (\exists t \in {}^{n_k}2)\ t \cup \pi_k^{-1}(n) \in \overline{\sigma(n_{k+1})}\}$ and let $h(s) = h^*(|s|)$ for $s \in T_\rho$. Since $(\sigma(n_{k+1}), \emptyset) \in D(n_k)$ we have: $|h^*(k)| \le |\overline{\sigma(n_{k+1})}| < (n_{k+1} - n_k)/n_k$ and so $\lim_{k\to\infty} |h^*(k)|/\log(\rho(k)) = 0$. Therefore $h \in H_\rho$.

At last $\langle 0,1\rangle \cap V = \varphi_2({}^\omega 2 \cap V) \subseteq \varphi_2(\{x \in {}^\omega 2;\ \exists y \in C\ \forall^\infty n\ x(n) = y(n)\}) \subseteq \varphi_2(\{x \in {}^\omega 2;\ \forall^\infty k\ \exists t\ t \cup x{\restriction}\langle n_k, n_{k+1}) \in \sigma(n_{k+1})\}) = \varphi_\rho(\{y \in X_\rho;\ \forall^\infty k\ y(k) \in h(y{\restriction}k)\}) = A_{\rho,h} \in \mathbf{Sm}_\rho$. □

Lemma 6.3. *The predicate "y codes a closed strongly symmetrically porous set" is* Π^1_1.

Proof. Let $\{r_n;\ n \in \omega\}$ be some standard enumeration of the set of all rational numbers. Let $y \in {}^{\omega\times\omega}2$. Then y codes a closed strongly symmetrically porous set C, $C = \mathbf{R} - \bigcup\{(r_i, r_j);\ y(i,j) = 1\}$ iff

$$\begin{aligned}
&(\forall a \in \mathbf{R})[(\forall i,j)(y(i,j) = 1 \rightarrow a \notin (r_i, r_j)) \rightarrow \\
&(\forall n \in \omega)(\exists i_1, i_2, j_1, j_2, i, j \in \omega)(y(i_1, j_1) = y(i_2, j_2) = 1 \text{ and} \\
&0 < r_i < r_j < 1/n \text{ and } (r_j - r_i)/r_j > (n-1)/n \text{ and} \\
&(a - r_j, a - r_i) \subseteq (r_{i_1}, r_{j_1}) \text{ and } (a + r_i, a + r_j) \subseteq (r_{i_2}, r_{j_2}))].
\end{aligned}$$

□

Proof of Theorem 6. Let κ be an arbitrary cardinal number such that $\kappa^\omega = \kappa > \omega_1$. Let M_α be a finite support iteration of the meager forcing M of length α, $\alpha \le \kappa$. Let G be a V-generic filter over M_κ and let G_α be the restriction of G to M_α. On every step $\alpha + \omega$, a Cohen real c_α is added and no function from $V[G_{\omega_1}]$ dominates all functions of the family $\mathcal{F} = \{c_\alpha;\ \alpha \in \omega_1\}$. Since finite support iterations of M preserve unbounded families (see [6]), the family $\mathcal{F}$ remains unbounded and so $\mathbf{b} = \omega_1$ in $V[G]$. By c.c.c. of M, V and $V[G]$ have the same cardinals and cofinalities. One can easily verify (using c.c.c. and $|M| = 2^\omega$) that 2^ω is κ in $V[G]$. Let $X \in V$ be a cofinal subset of κ of cardinality $\mathrm{cf}(\kappa)$. The set $A = \{c_\alpha;\ \alpha \in X\}$ is not of first category and so $A \notin \mathbf{S}$ and $\mathrm{non}(\mathbf{S}) \le \mathrm{cf}(\kappa)$. Similarly, the set A cannot be covered by less than $\mathrm{cf}(\kappa)$ sets of first category. Therefore $\mathrm{cf}(\kappa) \le \mathrm{cov}(\mathbf{S})$. For $\alpha \in X$ let us denote $\mathbf{R}_\alpha = \mathbf{R} \cap V[G_\alpha]$. Then $\mathbf{R} = \bigcup\{\mathbf{R}_\alpha;\ \alpha \in X\}$ and $\mathbf{R}_\alpha \subseteq \mathbf{R}_\beta$ whenever $\alpha \le \beta$. By Lemma 6.2, for every $\alpha \in X$ there are countably many closed strongly symmetrically porous sets $B_{\alpha,n}$, $n \in \omega$ in $V[G_{\alpha+1}]$ which cover $\mathbf{R}_\alpha$. Let $B^*_{\alpha,n}$ be the closed set in $V[G]$ with the same code as $B_{\alpha,n}$ has. Then by Lemma 6.3 and by Shoenfield's absoluteness lemma (see [7]), $B^*_{\alpha,n}$ are closed strongly symmetrically porous in V[G] and

$B^*_{\alpha,n} \cap V[G_{\alpha+1}] = B_{\alpha,n}$. This proves $\mathrm{cf}(\kappa) \le \mathrm{non}(\mathbf{S})$ and $\mathrm{cov}(\mathbf{S}) \le \mathrm{cf}(\kappa)$. Therefore $\mathrm{non}(\mathbf{S}) = \mathrm{cov}(\mathbf{S}) = \mathrm{cf}(\kappa)$. $\square$

In the opinion of the referee and also of me it should be interesting to know if the last theorem is true when instead of meager reals we iterate eventually different reals (and Hechler reals respectively) (see [9]).

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