On Dobrakov net submeasures

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Abstract. I. Dobrakov introduced in [2] a notion of submeasure defined on a ring of sets. This submeasure type is now known as the Dobrakov submeasure. In this paper we will develop some limit techniques to create new Dobrakov submeasures from the old ones in the case when elements of the ring $\mathcal{R}$ are subsets of the real line.

1 Examples of Dobrakov net submeasures

In [2] I. Dobrakov has initiated a theory of monotone set functions intended to be "a non-additive generalization of the theory of finite non-negative countably additive measures". Thus, he has introduced the following notion of a submeasure:

Definition 1.1 (Dobrakov, [2]) Let $\mathcal{R}$ be a ring of subsets of a set $\mathcal{T} \neq \emptyset$. A set function $\mu : \mathcal{R} \to [0, \infty)$ is said to be a *submeasure*, if it is

(1) *monotone*: if $E, F \in \mathcal{R}$ such that $E \subset F$, then $\mu(E) \leq \mu(F)$;

(2) *subadditively continuous*: for every $F \in \mathcal{R}$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $E \in \mathcal{R}$ with $\mu(E) < \delta$ there holds

1. $\mu(E \cup F) \leq \mu(F) + \varepsilon$, and
2. $\mu(F) \leq \mu(F \setminus E) + \varepsilon$;

(3) *continuous at* $\emptyset$ (shortly *continuous*): if $\mu(E_n) \to 0$ for any sequence $E_n \in \mathcal{R}$, $n = 1, 2, \ldots$, such that $E_n \searrow \emptyset$ (i.e., $E_n \supset E_{n+1}$ and $\bigcap E_n = \emptyset$).

Such a set function μ is now known as the *Dobrakov submeasure* (D-submeasure, for short). If instead of (2) we have $\mu(E \cup F) \leq \mu(E) + \mu(F)$ for every $E, F \in \mathcal{R}$, or $\mu(E \cup F) = \mu(E) + \mu(F)$ for every $E, F \in \mathcal{R}$ with $E \cap F = \emptyset$, then we say that μ is a subadditive, or an additive D-submeasure, respectively. Therefore, the condition (2) is a useful generalization of the classical subadditivity.

Further, in paper [4], I. Dobrakov studied tools of enlargement of such D-submeasures to the σ-ring $\sigma(\mathcal{R})$ generated by $\mathcal{R}$. In paper [11] V. M. Klimkin and M. G. Svistula considered the Darboux property of non-additive set functions, in particular, the D-submeasure. In [12], we can find the D-submeasure in the context of fuzzy sets and systems. Note that there are two qualitative different types of continuity of μ in the definition. In literature, for miscellaneous reasons, some additional properties of continuity (or exhaustivity) are

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sometimes added to the property (1) in Definition 1.1 when defining the notion of a submeasure, cf. [6]. There are also many papers where authors consider various generalized settings (e.g. [3], [7], [8] and [13]).

In this paper we extend the notion of D-submeasure to nets and consider techniques based on limit methods to create new D-submeasures from the old ones parametrized with an l-group of real functions in the case when elements of the ring $\mathcal{R}$ are subsets of the real line. If functions in the limit are monotone and approximately continuous in a generalized sense, then we obtain a recursive process.

By *a net* (with values in a set S) we mean a function from Ω to S, where Ω is a directed partially ordered set. A net a_ω, $\omega \in \Omega$, is *eventually in* a set A if and only if there is an element $\omega_0 \in \Omega$ such that if $\omega \in \Omega$ and $\omega \geq \omega_0$, then $a_\omega \in A$. Also other terminology about nets (the notion of the subnet, etc.) is used in the standard sense, cf. [10].

Definition 1.2 We say that a set function $\mu : 2^{\mathbb{R}} \to [0, \infty)$ is a *Dobrakov net submeasure* (D-net-submeasure, for short), if it is

(1) *monotone*, i.e. if $E, F \in 2^{\mathbb{R}}$ such that $E \subset F$, then $\mu(E) \leq \mu(F)$;

(2) *subadditively continuous*, i.e., for every $F \in 2^{\mathbb{R}}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $E \in 2^{\mathbb{R}}$ with $\mu(E) < \delta$, then

 (a) $\mu(E \cup F) \leq \mu(F) + \varepsilon$, and
 (b) $\mu(F) \leq \mu(F \setminus E) + \varepsilon$;

(3) *continuous*, i.e., if $E_\omega \searrow \emptyset$ ($E_\omega \supset E_{\omega'}$, for $\omega \prec \omega'$, $\omega \in \Omega$, $\omega' \in \Omega$, and $\bigcap_{\omega \in \Omega} E_\omega = \emptyset$), then $\mu(E_\omega) \to 0$, where Ω is a directed set.

Note that if the δ in condition (2) is uniform with respect to $F \in 2^{\mathbb{R}}$, then we say that μ is a *uniform D-net-submeasure*.

The following few examples describe some simple tools how to create new D-net-submeasures from old ones.

Example 1.3 Let $(\mathbb{R}, \Sigma, \lambda)$ be the Lebesgue measure space. For every λ-integrable function f, the set function

$$\mu_f(E) = \inf_{A \in \Sigma, E \subset A} \int_A |f| \, d\lambda, \quad E \subset \mathbb{R},$$

is a D-net-submeasure.

Example 1.4 If f is a function, then a set function

$$\mu_f(E) = \sup_{t \in E} |f(t)|, \quad E \subset \mathbb{R}$$

is a D-net-submeasure.

Example 1.5 If $\lambda_1, \lambda_2, \ldots, \lambda_N$, are D-net-submeasures, then a set function

$$\mu(E) = \sqrt{\sum_{n=1}^{N} \lambda_n^2(E)}, \quad E \subset \mathbb{R}$$

is a D-net-submeasure.

Example 1.6 If $f : \mathbb{R} \to \mathbb{R}$ is a function, $\delta > 0$ is a positive real number and λ is a D-net-submeasure, then a set function

$$\mu_{\delta,f}(E) = \lambda(\{t \in E; |f(t)| \geq \delta\}), \quad E \subset \mathbb{R}$$

is a D-net-submeasure.

Example 1.7 Let λ be a D-net-submeasure. Let $\mathcal{F}$ be a set of all non-decreasing real functions f on $\mathbb{R}$, such that $f(0) = 0$ and $x \geq y \geq 0 \Rightarrow f(x) - f(y) \leq f(x - y)$ (e.g. $f(x) = \arctan x$). Then the set function

$$\mu_f(E) = f(\lambda(E)), \quad E \subset \mathbb{R}$$

is a D-net-submeasure.

Remark 1.8 The D-net-submeasure $\mu(\cdot) = \arctan(\lambda(\cdot))$ in Example 1.7 gives the same ring topology on $2^{\mathbb{R}}$, cf. [13], as the D-net-submeasure λ, because $\arctan(\cdot)$ is a continuous function. A linear combination of D-net-submeasures (if it is a D-net-submeasure) yields a new ring topology on $2^{\mathbb{R}}$ if the components in it are linearly independent. To obtain new ring topologies on $2^{\mathbb{R}}$, non-linear operations (cf. Examples 1.3, 1.4, 1.5, 1.6), a non-continuous function (in Example 1.7), or a limit process may be used when creating new D-net-submeasures.

The rest three examples show such monotone and subadditive set functions with 0 in $\emptyset$ which need not be continuous even in the case of sequences (not necessarily nets). Let $\mathbf{X}$ and $\mathbf{Y}$ be two Banach spaces in Examples 1.9, 1.10, 1.11. In these examples Σ denotes a σ-algebra of sets generated by a ring $\mathcal{R}$ of sets of a nonempty set T and $L(\mathbf{X}, \mathbf{Y})$ is the set of all continuous linear operators $L : \mathbf{X} \to \mathbf{Y}$.

Example 1.9 A *semivariation* $\hat{\mathbf{m}} : \Sigma \to [0, \infty]$ of a charge (= finitely additive measure) $\mathbf{m} : \mathcal{R} \to L(\mathbf{X}, \mathbf{Y})$ is defined as

$$\hat{\mathbf{m}}(E) = \sup \left\| \sum_{i=1}^{I} \mathbf{m}(E \cap E_i)\mathbf{x}_i \right\|, \quad E \in \Sigma$$

where the supremum is taken over all finite sets $\{\mathbf{x}_i \in \mathbf{X};\ \|\mathbf{x}_i\| \leq 1, i = 1, 2, \ldots, I\}$ and all disjoint sets $\{E_i \in \mathcal{R};\ i = 1, 2, \ldots, I\}$. It is well-known that $\hat{\mathbf{m}}$ is a monotone, subadditive set function with $\hat{\mathbf{m}}(\emptyset) = 0$, but it need not be continuous. ¿From it follows that the Dobrakov integral, [5], is not built on D-submeasures because it solves also the case of non-continuous semivariation.

Example 1.10 A *scalar semivariation* $\|\mathbf{m}\|$ of a charge $\mathbf{m} : \mathcal{R} \to L(\mathbf{X}, \mathbf{Y})$ is given by

$$\|\mathbf{m}\|(E) = \sup \left\| \sum_{i=1}^{I} \lambda_i \, \mathbf{m}(E \cap E_i) \right\|, \quad E \in \Sigma,$$

where $\|L\| = \sup_{\|\mathbf{x}\| \leq 1} \|L(\mathbf{x})\|$ and the supremum is taken over all finite sets of scalars $\{\lambda_i \in \mathbb{K};\ \|\lambda_i\| \leq 1, i = 1, 2, \ldots, I\}$ and all disjoint sets $\{E_i \in \mathcal{R};\ i = 1, 2, \ldots, I\}$.

Example 1.11 Denote by $|\mu| : \Sigma \to [0,\infty]$ *a vector semivariation* of a charge $\mu : \Sigma \to \mathbf{Y}$, where

$$|\mu|(E) = \sup \left\| \sum_{i=1}^{I} \lambda_i \, \mu(E \cap E_i) \right\|, E \in \Sigma,$$

where the supremum is taken over all finite sets of scalars $\{\lambda_i \in \mathbb{K};\ \|\lambda_i\| \leq 1, i = 1,2,\ldots,I\}$ and all disjoint sets $\{E_i \in \mathcal{R};\ i = 1,2,\ldots,I\}$.

The next simple example shows such a set function which is not a D-(net)-submeasure even if the set functions used in its definition are uniform D-(net)-submeasures on a σ-algebra (possibly with some additional properties, e.g. uniform exhaustivity).

Example 1.12 Let $T = [0,1]$, let $\mathcal{B}$ be the Borel σ-algebra of T and $\lambda : \mathcal{B} \to [0,1]$ be the Lebesgue measure. For $n = 1,2,\ldots$ and $F \in \mathcal{B}$ put

$$\mu_n(F) = \lambda(F) \wedge \frac{1}{2} + \left(n \left(\lambda(F) - \frac{1}{2} \right) \wedge \frac{1}{2} \right) \vee 0,$$

where $a \vee b$, resp. $a \wedge b$, means the maximum, resp. the minimum, of the real numbers a, b. Then each $\mu_n : \mathcal{B} \to [0,1]$ is a uniform D-(net)-submeasure. Put

$$\mu(E) = \sup_{n\in\mathbb{N}} \mu_n(E), \quad E \in \mathcal{B}.$$

Let $F_k = [0, 1/2 + 1/(k+1)]$ for $k = 1,2,\ldots$. Then $F_k \searrow [0,1/2] = F$ and $\mu(F_k) = 1$ for each $k = 1,2,\ldots$, but $\mu(F) = 1/2$. By Corollary 1 of Theorem 7 in [3], μ is not a D-(net)-submeasure.

The following lemma shows a limit process of creating new D-net-submeasures. Its proof is easy and therefore omitted. The second statement follows immediately from the monotonicity of the considered set functions. However we do not solve the question on existence of a limit on this place. A sufficient condition for the existence of a limit is given in Theorem 3.2.

Lemma 1.13 *Let $\mu_{(\omega)}$, $\omega \in \Omega$, be a net of D-net-submeasures. If a limit $\mu(E) = \lim_{\omega\in\Omega} \mu_{(\omega)}(E)$ exists for each $E \subset \mathbb{R}$, then μ is a D-net-submeasure, and moreover, $\mu_{(\omega)}$, $\omega \in \Omega$, are uniformly continuous.*

In the following two sections we bring a more sophisticated method of creating new D-net-submeasures.

2 Some classes of D-net-submeasures

Let $(\mathcal{F}, \|\cdot\|)$ be an (additive) *l-group*, cf. [1], of real functions on $\mathbb{R}$ equipped with the following system of gauges

$$\|f\|_E = \sup_{t\in E} |f(t)|, \quad E \subset \mathbb{R},\ f \in \mathcal{F},$$

such that $f, g \in \mathcal{F}$, $E \subset \mathbb{R}$ and

$$|f| \leq |g| \Rightarrow \|f\|_E \leq \|g\|_E.$$

Shortly, we say that $\mathcal{F}$ is an $(l, \|\cdot\|)$-group.

Definition 2.1 We say that a class $\mathcal{D}_{\mathcal{F}} = \{\mu_f; f \in \mathcal{F}\}$ of D-net-submeasures is an *$\mathcal{F}$-class of D-net-submeasures* if it satisfies the following conditions:

(a) $\mu_f \in \mathcal{D}_{\mathcal{F}}$ implies $\mu_{-f} \in \mathcal{D}_{\mathcal{F}}$ and $\mu_f(E) = \mu_{-f}(E)$,

(b) $\mu_f \in \mathcal{D}_{\mathcal{F}}$ and $\mu_g \in \mathcal{D}_{\mathcal{F}}$ implies $\mu_{f+g} \in \mathcal{D}_{\mathcal{F}}$ and

$$\mu_{f+g}(E) \leq \mu_f(E) + \mu_g(E)$$

for every $f, g \in \mathcal{F}$ and $E \subset \mathbb{R}$.

If, moreover, there exists a D-net-submeasure α on $2^{\mathbb{R}}$ such that

(c) $\mu_f(E) \leq \alpha(E) \cdot \|f\|_E$

for every finite interval $E \subset \mathbb{R}$, then we say that the $\mathcal{F}$-class of D-net-submeasures is *α-dominated.* For an α-dominated $\mathcal{F}$-class of D-net-submeasures we write $\mathcal{D}^{\alpha}_{\mathcal{F}}$.

Remark 2.2 Note that although both α and $\|f\|_.$ are D-(net)-submeasures, their product need not be a D-(net)-submeasure in general.

Definition 2.3 Let α be a D-net-submeasure on $2^{\mathbb{R}}$. *A net of D-net-submeasures* $\mu_{(\omega)}$, $\omega \in \Omega$, is *α-equicontinuous* if for every $\varepsilon > 0$ there exist a finite $E \in 2^{\mathbb{R}}$ and $\kappa > 0$, such that $\alpha(E) < \kappa$ and the net $\mu_{(\omega)}(\mathbb{R} \setminus E), \omega \in \Omega$, is eventually in the interval $[0; \varepsilon)$.

Definition 2.4 Let β be a D-net-submeasure on $2^{\mathbb{R}}$. *A net of D-net-submeasures* $\mu_{(\omega)}$, $\omega \in \Omega$, is *uniformly absolutely β-continuous* if for every $\varepsilon > 0$ there exists $\eta > 0$, such that for every $A \in 2^{\mathbb{R}}$ with $\beta(A) < \eta$, the net $\mu_{(\omega)}(A)$, $\omega \in \Omega$, is eventually in the interval $[0; \varepsilon)$.

Example 2.5 Let $(\mathbb{R}, \Sigma, \lambda)$ be the Lebesgue measure space. If $\mathcal{F}_1$ is the space of all integrable functions, then the following D-net-submeasure

$$\mu_f(E) = \inf_{A \in \Sigma, E \subset A} \int_A |f| \, d\lambda \leq \alpha(E) \cdot \|f\|_E, \quad E \in 2^{\mathbb{R}},\ f \in \mathcal{F}_1,$$

is α-dominated, where $\alpha(E) = \inf_{A \in \Sigma, E \subset A} \lambda(A)$.

Example 2.6 Let $\mathcal{F}_1$ be the space of all real measurable functions, and λ be a Borel measure. Then the D-net-submeasure

$$\mu_f(E) = \alpha(\{t \in E; |f(t)| \geq \delta\}) \leq \alpha(E) \cdot \|f\|_E,$$

is α-dominated, where α is the same as in previous example.

Example 2.7 The D-net-submeasure μ_f is not λ-dominated in general in Example 1.7, because the condition (c) in Definition 2.1 does not hold e.g. for $f(x) = x$, $x > 0$, and λ the Lebesgue measure.

3 Constructing new D-net-submeasures

It is obvious from definition that the D-(net)-submeasures are not subadditive in general. But according to the results in [6] it is, in fact, inessential, because every D-(net)-submeasure μ is equivalent to a subadditive D-(net)-submeasure η such that, in addition, μ is absolutely η-continuous. Therefore, in the sequel of this paper, we reduce our considerations to the case of the subadditive D-net-submeasures even if it is not explicitly stated.

Definition 3.1 Let $\mathcal{F}_1, \mathcal{F}_2$, be two $(l, \|\cdot\|)$-groups of functions and let β be a D-net-submeasure. *A net $f_\omega \in \mathcal{F}_1$, $\omega \in \Omega$, of functions β-converges to a function $f \in \mathcal{F}_2$* if for every $\delta > 0$,

$$\lim_{\omega \in \Omega} \beta(\{t \in \mathbb{R}; |f_\omega(t) - f(t)| \geq \delta\}) = 0.$$

Theorem 3.2 *Let α, β be D-net-submeasures on $2^{\mathbb{R}}$. Let $\mathcal{F}_1, \mathcal{F}_2$, be two $(l, \|\cdot\|)$-groups of functions and let a net $f_\omega \in \mathcal{F}_1$, $\omega \in \Omega$, of functions β-converge to a function $f \in \mathcal{F}_2$. If $\mu_{f_\omega}(\cdot) \in \mathcal{D}^{\alpha}_{\mathcal{F}_1}$, $\omega \in \Omega$, is a net of D-net-submeasures, such that it is*

(i) *uniformly absolutely β-continuous, and*

(ii) *α-equicontinuous,*

then the limit

$$\overline{\mu}_f(F) = \lim_{\omega \in \Omega} \mu_{f_\omega}(F), \tag{1}$$

exists for every $F \subset \mathbb{R}$ and $\overline{\mu}_f(\cdot)$ is a D-net-submeasure.

Proof. Let $F \subset \mathbb{R}$. If the limit $\overline{\mu}_f(F)$ exists for every $F \subset \mathbb{R}$, then it is a D-net-submeasure by Lemma 1.13. Show that $\overline{\mu}_f(F)$ exists.

Since $\mathbb{R}$ is complete, it is enough to show that for every $\varepsilon > 0$ there exists $\omega_\varepsilon \in \Omega$, such that for every $\omega, \omega' \geq \omega_\varepsilon$, there is $|\mu_{f_\omega}(F) - \mu_{f_{\omega'}}(F)| < \varepsilon$.

By (ii) the net $\mu_{f_\omega}(\cdot)$, $\omega \in \Omega$, is α-equicontinuous. So, for a given $\varepsilon > 0$ there exist $E \subset \mathbb{R}$, $\kappa > 0$, and $\omega_2 \in \Omega$, such that $\alpha(E) < \kappa$ and for every $\omega \geq \omega_2$, with $\omega \in \Omega$, there is

$$\mu_{f_\omega}(\mathbb{R} \setminus E) < \varepsilon. \tag{2}$$

By Definition 2.1(b) we have that

$$\mu_{f_\omega}(E \cap F) \leq \mu_{f_\omega - f_{\omega'}}(E \cap F) + \mu_{f_{\omega'}}(E \cap F).$$

This implies

$$|\mu_{f_\omega}(E \cap F) - \mu_{f_{\omega'}}(E \cap F)| \leq \mu_{f_\omega - f_{\omega'}}(E \cap F). \tag{3}$$

By (3), monotonicity, and subadditivity of $\mu_{f_\omega}(\cdot)$ and $\mu_{f_{\omega'}}(\cdot)$, we get

$$\begin{aligned}
&|\mu_{f_\omega}(F) - \mu_{f_{\omega'}}(F)| \\
&\leq |\mu_{f_\omega}(F \cap (\mathbb{R} \setminus E)) + \mu_{f_\omega}(F \cap E) + \mu_{f_{\omega'}}(F \cap (\mathbb{R} \setminus E)) - \mu_{f_{\omega'}}(F \cap E)| \\
&\leq |\mu_{f_\omega}(F \cap (\mathbb{R} \setminus E))| + |\mu_{f_{\omega'}}(F \cap (\mathbb{R} \setminus E))| + |\mu_{f_\omega - f_{\omega'}}(E \cap F)|.
\end{aligned}$$

Clearly, $F \cap (\mathbb{R} \setminus E) \subset \mathbb{R} \setminus E$. By (2),

$$|\mu_{f_\omega}(F) - \mu_{f_{\omega'}}(F)| \leq 2\varepsilon + \mu_{f_\omega - f_{\omega'}}(E \cap F)$$

for every $\omega, \omega' \geq \omega_2$. By Definition 2.1(c) we obtain

$$\mu_{f_\omega - f_{\omega'}}(E \cap F) \leq \mu_{f_\omega - f_{\omega'}}(E) \leq \alpha(E) \cdot \|f_\omega - f_{\omega'}\|_E < \kappa \cdot \|f_\omega - f_{\omega'}\|_E.$$

Then for a given $\varepsilon > 0$ there exists $\delta = \varepsilon/\kappa > 0$, such that

$$\|f_\omega - f_{\omega'}\|_E < \delta \Rightarrow \mu_{f_\omega - f_{\omega'}}(E \cap F) < \varepsilon. \tag{4}$$

Put $G = \{t \in \mathbb{R}; |f_\omega(t) - f_{\omega'}(t)| < \delta\}$. From subadditivity of $\mu_{f_\omega - f_{\omega'}}(\cdot)$ we have

$$\mu_{f_\omega - f_{\omega'}}(F \cap E) \leq \mu_{f_\omega - f_{\omega'}}(F \cap E \cap G) + \mu_{f_\omega - f_{\omega'}}((F \cap E) \setminus G). \tag{5}$$

By (4) and (5) we get

$$|\mu_{f_\omega}(F) - \mu_{f_{\omega'}}(F)| \leq 3\varepsilon + \mu_{f_\omega - f_{\omega'}}((E \cap F) \setminus G). \tag{6}$$

The net f_ω, $\omega \in \Omega$, of functions β-converges to f. Denote by χ_A the characteristic function of the set $A \subset \mathbb{R}$. Since β is a monotone set function, the net $f_\omega \chi_A$, $\omega \in \Omega$, of functions β-converges to $f_\omega \chi_A$, $\omega \in \Omega$ as well, where $A \subset \mathbb{R}$. Therefore, for every $\eta > 0$ there exists $\omega_1 \in \Omega$, such that for every $\omega \geq \omega_1$ with $\omega \in \Omega$,

$$\beta\{t \in A;\, |f_\omega(t) - f_{\omega'}(t)| \geq \delta\} < \eta. \tag{7}$$

¿From uniform absolute β-continuity of $\mu_{f_\omega}(\cdot)$, $\omega \in \Omega$, we obtain that for every $\varepsilon > 0$ there exist $\eta > 0$ and $\omega_3 \in \Omega$ such that for every $\omega \geq \omega_3$ with $\omega \in \Omega$,

$$A \subset \mathbb{R}, \quad \beta(A) < \eta \Rightarrow \mu_{f_\omega}(A) < \varepsilon. \tag{8}$$

Further, if $\mu_{f_\omega}(A) < \varepsilon$, $\omega \in \Omega$, $A \subset \mathbb{R}$, then

$$\mu_{f_\omega - f_{\omega'}}(A) \leq \mu_{f_\omega}(A) + \mu_{f_{\omega'}}(A) < 2\varepsilon \tag{9}$$

for every $\omega, \omega' \geq \omega_3$.

Put $A = (E \cap F) \setminus G$ and take $\omega_\varepsilon \in \Omega$, such that $\omega_\varepsilon \geq \omega_1$, $\omega_\varepsilon \geq \omega_2$ and $\omega_\varepsilon \geq \omega_3$. Then (6), (7), (8), and (9) imply that for every $F \subset \mathbb{R}$ and $\varepsilon > 0$ there exists $\omega_\varepsilon = \omega_1 \in \Omega$ such that for every $\omega \geq \omega_\varepsilon$ there is $|\mu_{f_\omega}(F) - \mu_{f_{\omega'}}(F)| < 5\varepsilon$. Hence the result. □

Remark 3.3 It is clear that the family $\{\overline{\mu}_f(\cdot)\} \cup \{\mu_{f_\omega}(\cdot);\, \omega \in \Omega\}$ is uniformly absolutely β-continuous and α-equicontinuous. Also, it may be easily verified that for a fixed directed set Ω, the limit (1) does not depend on the choice of the net of functions $f_\omega \in \mathcal{F}_1$, $\omega \in \Omega$.

For β a D-net-submeasure, the following concept of β-approximate continuity is a generalization of the notion of approximate continuity, cf. e.g. [9].

Definition 3.4 Let $\beta : 2^{\mathbb{R}} \to [0, \infty)$ be a D-net-submeasure. A *β-density* of a set $F \subset \mathbb{R}$ at $t \in \mathbb{R}$, written $\mathfrak{D}_F^\beta(t)$, is $\lim \beta(E \cap F)/\beta(E)$ provided the limit exists, where the limit is taken over $E, t \in E$, and $\beta(E)$ approaching 0. A point t is a *point of β-density of F* if $\mathfrak{D}_F^\beta(t) = 1$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *β-approximately continuous at t* if t is a point of β-density of a set F and f is continuous at t with respect to F. A function f is *β-approximately continuous in* (a, b), where $a, b \in \mathbb{R}$, $a < b$, if f is β-approximately continuous at each $t \in (a, b)$.

For our next result we need the following theorem which generalizes the result from [14], Theorem 1.

Theorem 3.5 *Let β be a D-net-submeasure and $\mathcal{F}$ be a space of all β-approximately continuous real functions on $\mathbb{R}$. If a net $f_\omega : \mathbb{R} \to \mathbb{R}$, $\omega \in \Omega$, of monotone functions β-converges to $f \in \mathcal{F}$ on a finite interval (a, b), $a < b$, then a net $\{f_\omega\}_{\omega\in\Omega}$, of functions β-converges to f in each point of the β-approximate continuity of f.*

Proof. Let $\{f_\omega\}_{\omega\in\Omega}$ be a net of nondecreasing functions and $t_0 \in (a, b)$ be a point of the β-approximate continuity of f. Suppose the contrary, i.e. that a net $\{f_\omega(t_0)\}_{\omega\in\Omega}$ does not β-converge to $f(t_0)$. Then there exists $\eta > 0$ such that

$$\limsup_{\omega' :\, \omega \geq \omega'} |f_\omega(t_0) - f(t_0)| \geq \eta.$$

Let us define a set

$$\overline{\Omega} = \{\omega \in \Omega;\ |f_\omega(t_0) - f(t_0)| \geq \eta\}.$$

Clearly, the sets Ω and $\overline{\Omega}$ are cofinal. We define sets

$$\Omega' = \{\omega \in \overline{\Omega};\ f_\omega(t_0) \geq f(t_0) + \eta\}$$

and

$$\Omega'' = \{\omega \in \overline{\Omega};\ f_\omega(t_0) \leq f(t_0) - \eta\}.$$

Since $\overline{\Omega} = \Omega' \cup \Omega''$, there are two possible cases:

(i) the sets Ω' and Ω are cofinal, or

(ii) the sets Ω'' and Ω are cofinal.

Let us suppose that the case (i) is true. The net $\{f_\omega\}_{\omega\in\Omega'}$ is the subnet of $\{f_\omega\}_{\omega\in\Omega}$ and for every $\omega \in \Omega'$ we have

$$f_\omega(t_0) \geq f(t_0) + \eta.$$

Since t_0 is the point of the β-approximate continuity of f, there exists a measurable subset F of (a, b) such that t_0 is the point of its β-density and $f|_F$ is β-continuous at t_0. There exists $\delta > 0$ such that for every $t' \in F$ we have $|f(t') - f(t_0)| < \eta/2$ whenever $0 \leq t' - t_0 < \delta$ and so

$$f_\omega(t') - f(t') \geq f_\omega(t_0) - f(t') \geq f(t_0) + \eta - f(t') > \frac{\eta}{2}$$

for arbitrary $\omega \in \Omega'$. It follows that

$$(t_0, t_0 + \delta) \cap F \subset \bigcap_{\omega\in\Omega'} \left\{t;\ f_\omega(t) - f(t) > \frac{\eta}{2}\right\}.$$

Since t_0 is the point of β-density of F, then

$$\mu\Big((t_0, t_0 + \delta) \cap F\Big) > 0.$$

Hence

$$\inf_{\omega\in\Omega'} \mu\left(\left\{t;\ f_\omega(t)-f(t) > \frac{\eta}{2}\right\}\right) \geq \mu\Big((t_0, t_0+\delta)\cap F\Big) > 0,$$

but it denies the β-convergence in measure of the net $\{f_\omega\}_{\omega\in\Omega}$ to the limit f. Analogously we proceed in the case (ii). This proves the theorem. $\square$

Using the fact that a measurable function is β-a.e. approximately continuous, cf. [9], and from Theorem 3.5 we get the following corollary.

Corollary 3.6 Let β be a D-net-submeasure. Let $\mathcal{F}$ be a space of all β-approximately continuous real functions on $\mathbb{R}$. If a net $f_\omega : \mathbb{R} \to \mathbb{R}$, $\omega \in \Omega$, of monotone functions β-converges to $f \in \mathcal{F}$ on a finite interval (a, b), $a < b$, then the net f_ω, $\omega \in \Omega$, of functions β-a.e. converges to f on (a, b).

Now we are able to prove the following main result of this section.

Theorem 3.7 *Let α, β be D-net-submeasures. Let $\mathcal{F}_1$ be an $(l, \|\cdot\|)$-group of functions and $\mathcal{F}_2$ be an $(l, \|\cdot\|)$-group of functions β-approximately continuous on each open finite interval, such that each $f \in \mathcal{F}_2$ is a β-limit of a net of monotone functions from $\mathcal{F}_1$. If $\overline{\mu}_{\cdot}(\cdot)$ is defined as in Theorem 3.2, then $\{\overline{\mu}_f(\cdot); f \in \mathcal{F}_2\}$ is an α-dominated $\mathcal{F}_2$-class of D-net-submeasures, i.e.*

$$\mathcal{D}^{\alpha}_{\mathcal{F}_2} = \{\overline{\mu}_f(\cdot); f \in \mathcal{F}_2\}.$$

Proof. Let $F \subset \mathbb{R}$. We have to verify conditions of Definition 2.1.

(a) Clearly, $\overline{\mu}_f(F) = \overline{\mu}_{-f}(F)$.

(b) If a net $g_\omega \in \mathcal{F}_1$, $\omega \in \Omega$, of functions β-converges to $g \in \mathcal{F}_2$, and $\overline{\mu}_g(F) = \lim_{\omega\in\Omega} \mu_{g_\omega}(F)$ exists, then $\overline{\mu}_{f+g}(F)$ exists and $\overline{\mu}_{f+g}(F) = \overline{\mu}_f(F) + \overline{\mu}_g(F)$. This yields from the equality

$$\overline{\mu}_{f+g}(F) = \lim_{\omega\in\Omega} \mu_{f_\omega + g_\omega}(F)$$

and the obvious inclusion

$$\begin{aligned}&\left\{t \in F;\ \Big|[f_\omega(t) + g_\omega(t)] - [f(t) + g(t)]\Big| \geq \frac{\delta}{2}\right\}\\ &\subset \{t \in F;\ |f_\omega(t) - f(t)| \geq \delta\} \cup \{t \in F; |g_\omega(t) - g(t)| \geq \delta\}, \quad \delta > 0.\end{aligned}$$

(c) Let a net $f_\omega \in \mathcal{F}_1$, $\omega \in \Omega$, of monotone functions β-converge to a function $f \in \mathcal{F}_2$. Let $\mu_{f_\omega}(\cdot)$, $\omega \in \Omega$, be a net of D-net-submeasures, such that it is uniformly absolutely β-continuous and α-equicontinuous.

Let us show that for $\overline{\mu}_f(F)$ given by (1) the inequality

$$\overline{\mu}_f(F) \leq \alpha(F) \cdot \|f\|_F$$

holds, where $F = (a, b)$, for $a, b \in \mathbb{R}$ with $a < b$.

By Theorem 3.5 and Corollary 3.6 the net f_ω, $\omega \in \Omega$, of functions β-a.e. converges to f on F. Hence, there exists $H \subset \mathbb{R}$, such that $\|f_\omega\|_{F\setminus H}$ converges to $\|f\|_{F\setminus H}$ and $\beta(H) = 0$. Then

$$\lim_{\omega\in\Omega} \mu_{f_\omega}(F \setminus H) \leq \alpha(F) \cdot \lim_{\omega\in\Omega} \|f_\omega\|_{F\setminus H},$$

i.e.

$$\overline{\mu}_{f_\omega}(F \setminus H) \leq \alpha(F) \cdot \|f\|_{F\setminus H}.$$

By uniform absolute β-continuity of $\mu_{f_\omega}(\cdot)$, $\omega \in \Omega$, we have that $\beta(H) = 0$ and $\omega \in \Omega$ imply $\mu_{f_\omega}(H) = 0$. Thus,

$$\overline{\mu}_f(H) = \lim_{\omega\in\Omega} \mu_{f_\omega}(H) = 0.$$

So,

$$\begin{aligned} \overline{\mu}_f(F) &\leq \overline{\mu}_f(F \setminus H) + \overline{\mu}_f(H) = \overline{\mu}_f(F \setminus H) \\ &\leq \alpha(F) \cdot \|f\|_{F\setminus H} \leq \alpha(F) \cdot \|f\|_F. \end{aligned}$$

This completes the proof. □

Corollary 3.8 Combining Theorems 3.2 and 3.7, we see that we have described a recursive procedure how to create new classes of D-net-submeasures from given ones.

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