On Vector Integral Inequalities

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Abstract. The first author introduced an integration theory of vector functions with respect to an operator-valued measure in complete bornological locally convex topological vector spaces. In this paper some important results behind this Dobrakov-type integration technique in non-metrizable spaces are given.

1 Introduction

A large number of different methods of integration of Banach-space-valued functions have been introduced based on the various possible constructions of the Lebesgue integral. Among them only the Dobrakov integral, cf. [5] and [6], defined in Banach spaces dealing with an operator-valued measure σ-additive in the strong operator topology, is the complete all pervading generalization of the abstract Lebesgue integral. For the reader's convenience let us briefly recall the definition of the Dobrakov integral.

Let $\mathbf{X}$ and $\mathbf{Y}$ be two Banach spaces, Δ be a δ-ring of subsets of a non-void set T, $L(\mathbf{X}, \mathbf{Y})$ be the space of all continuous operators $L : \mathbf{X} \to \mathbf{Y}$, and $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ be an operator-valued measure σ-additive in the strong operator topology of $L(\mathbf{X}, \mathbf{Y})$, i.e. $\mathbf{m}(\cdot)\mathbf{x} : \Delta \to \mathbf{Y}$ is a $\mathbf{Y}$-valued vector measure for every $\mathbf{x} \in \mathbf{X}$. We say that a measurable function $\mathbf{f} : T \to \mathbf{X}$ is *integrable in the sense of Dobrakov* if there exists a sequence of simple functions $\mathbf{f}_n : T \to \mathbf{X}$, $n \in \mathbb{N}$, converging $\mathbf{m}$-a.e. to $\mathbf{f}$ and the integrals $\int_{\cdot} \mathbf{f}_n \,\mathrm{d}\mathbf{m}$ are uniformly σ-additive measures on $\sigma(\Delta)$ (i.e. σ-algebra generated by Δ). The integral of the function $\mathbf{f}$ on $E \in \sigma(\Delta)$ is defined by the equality

$$\int_E \mathbf{f} \,\mathrm{d}\mathbf{m} = \lim_{n\to\infty} \int_E \mathbf{f}_n \,\mathrm{d}\mathbf{m},$$

cf. [5], Definition 2. For an excellent review paper on Dobrakov integral see [20]. It is well-known, cf. [5], that the Dobrakov integral yields a greater class of integrable functions than the also well-known (Lebesgue-type) integral of R. G. Bartle, cf. [1], considering the same measure and set systems.

There is a natural tendency to generalize integrations from Banach spaces to "higher floors". For instance, there is a question how to construct a theory of integration in locally convex spaces which are non-metrizable. The bornological character of the bilinear integration theory developed in [22] shows the fitness of developing a bilinear integration theory in the context of bornological convex vector spaces.

In general, bornological vector spaces, cf. [23], provide an ideal setting for many problems in non-commutative geometry and representation theory, but

[1] *Mathematics Subject Classification (2000):* 46G10
Key words and phrases: Inductive limit of Banach spaces, Dobrakov integral, bornology, vector function, integral inequality.

they are also quite useful for many other purposes. They give rise to a very nice theory of smooth representations of locally compact groups, cf. [18], or they allow to take into account the analytical extra structure on sheaves of smooth or holomorphic functions, cf. [24]. Bornological spaces of vector-valued functions were studied in [9] and of null sequences in [3].

In this paper we deal with the complete bornological locally convex topological vector spaces (C. B. L. C. S., for short) which include:

(i) all Banach spaces, in general non-separable;

(ii) all Fréchet spaces (i.e. the complete metrizable linear spaces);

(iii) a large number of non-metrizable locally convex spaces, e.g. various types of nuclear spaces, Schwartz spaces, DF-spaces and LB-spaces, etc., cf. [17], many of which have their origin in practical needs of theoretical physics, see e.g. [19].

In [11]–[13] the first author developed a new technique for C. B. L. C. S. and operator-valued measure. The specificity of this technique is that we work with lattices. In places where an object appears in the classical theory, e.g. a submeasure, a norm, a metric, a unit sphere, an L_p-space, a σ-ideal of null sets, etc., in this theory we work with lattices of submeasures, norms, etc. So, we can see an interesting union of the measure and integration theory with the lattice theory in the frame of functional analysis. The sense of this theory is that, at the present, this is the only integration theory which completely generalizes the Dobrakov integration to a class of non-metrizable locally convex topological vector spaces.

Some theorems on integrable functions and convergence theorems for such an integral are proved in [14]. The construction and existence of product measures in C. B. L. C. S. in connection with this integration technique is given in [2]. A Fubini-type theorem is also given therein.

In order to state our results we first give a brief development of a theory of integration in C. B. L. C. S. in the following section. In the third section we give some easy, but important results which are useful in this integration technique in many cases, e.g. when studying classes of integrable functions, convergence questions, or generalized L_p-spaces related to a bornological operator-valued measure.

2 Preliminaries

In this section we collect the needed definitions and results from [11], [12], and [13]. The complete description of the theory of C. B. L. C. S. may be found in [15], [16] and [21].

2.1 C. B. L. C. S.

Let $\mathbf{X}, \mathbf{Y}$ be two C. B. L. C. S. over the field $\mathbb{K}$ of real $\mathbb{R}$ or complex numbers $\mathbb{C}$, equipped with the bornologies $\mathfrak{B}_{\mathbf{X}}$, $\mathfrak{B}_{\mathbf{Y}}$, respectively. One of the equivalent definitions of C. B. L. C. S. is to define these spaces as the inductive limit of Banach spaces. Remind that a *(separable) Banach disk* in the space $\mathbf{X}$ is a set U which is closed, absolutely convex and the linear span $\mathbf{X}_U$ of which is a (separable) Banach space. Let us denote by $\mathcal{U}$ the set of all Banach disks U in $\mathbf{X}$ such that $U \in \mathfrak{B}_{\mathbf{X}}$. So, the space $\mathbf{X}$ is an inductive limit of Banach spaces $\mathbf{X}_U$, $U \in \mathcal{U}$, i.e.

$$\mathbf{X} = \operatorname*{injlim}_{U \in \mathcal{U}} \mathbf{X}_U,$$

cf. [16], where the family $\mathcal{U}$ is directed by inclusion and forms the basis of bornology $\mathfrak{B}_{\mathbf{X}}$ (analogously for $\mathbf{Y}$ and $\mathcal{W}$). The basis $\mathcal{U}$ in the inductive limit need not be unambiguous and, in particular, it may be chosen such that $\mathbf{X}_U$, $U \in \mathcal{U}$, are separable, cf. [16], § 13.2, Th. 3.

We say that the basis $\mathcal{U}$ of bornology $\mathfrak{B}_{\mathbf{X}}$ has a *vacuum vector* [2] $U_0 \in \mathcal{U}$, if $U_0 \subset U$ for every $U \in \mathcal{U}$. Let the bases $\mathcal{U}$, $\mathcal{W}$ be chosen to consist of all $\mathfrak{B}_{\mathbf{X}}$-, $\mathfrak{B}_{\mathbf{Y}}$-bounded Banach disks in $\mathbf{X}$, $\mathbf{Y}$, with vacuum vectors $U_0 \in \mathcal{U}$, $U_0 \neq \{0\}$, and $W_0 \in \mathcal{W}$, $W_0 \neq \{0\}$, respectively.

The convergence on C. B. L. C. S. is called the bornological convergence which is also a sequential convergence, cf. [8]. We say that a sequence of elements $\mathbf{x}_n \in \mathbf{X}$, $n \in \mathbb{N}$ (the set of all natural numbers), converges bornologically with respect to the bornology $\mathfrak{B}_{\mathbf{X}}$ with the basis $\mathcal{U}$ (shortly, *$\mathcal{U}$-converges*) to $\mathbf{x} \in \mathbf{X}$, if there exists $U \in \mathcal{U}$ such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $(\mathbf{x}_n - \mathbf{x}) \in U$ for every $n \geq n_0$. We write $\mathbf{x} = \mathcal{U}\text{-}\lim_{n\to\infty} \mathbf{x}_n$. To be more precise we will sometimes call this U-convergence of elements from $\mathbf{X}$ to show explicitly which $U \in \mathcal{U}$ we have in the mind.

Remark 2.1 A classical bornology consists of all sets which are bounded in the von Neumann sense, i.e. for a locally convex topological vector space $\mathbf{X}$ equipped with a family of seminorms Q, the set B is *bounded* (or belongs to the von Neumann bornology) if and only if for every $q \in Q$ there exists a constant C_q such that $q(\mathbf{x}) \leq C_q$ for every $\mathbf{x} \in B$. In this case the bornological convergence implies the topological convergence. On the other hand, we can introduce the von Neumann bornology on an arbitrary complete locally convex space $\mathbf{X}$ and the topological completeness of $\mathbf{X}$ implies the completeness in the sense of bornology, cf. [21].

Note that each vector space $\mathbf{X}$ (or $\mathbf{Y}$) over the field $\mathbb{K}$ can be equipped with various bornological bases of Banach disks (defining this way various C. B. L. C. S.), moreover, with the property such that $U_1 \cap U_2 \neq \{\emptyset\}$ for every $U_1 \in \mathcal{U}$, $U_2 \in \mathcal{U}$.

2.2 Operator structures

On $\mathcal{U}$ *the lattice operations* are defined as follows: for $U_1, U_2 \in \mathcal{U}$ we have $U_1 \wedge U_2 = U_1 \cap U_2$, and $U_1 \vee U_2 = \operatorname{acs}(U_1 \cup U_2)$, where acs denotes the topological closure of the absolutely convex span of the set. Analogously for $\mathcal{W}$. For $(U_1, W_1), (U_2, W_2) \in \mathcal{U} \times \mathcal{W}$, we write $(U_1, W_1) \ll (U_2, W_2)$ if and only if $U_1 \subset U_2$, and $W_1 \supset W_2$.

We use Φ to denote the *class of all functions $\mathcal{U} \to \mathcal{W}$ with order* $<$ defined as follows: for $\varphi, \psi \in \Phi$ we write $\varphi < \psi$ whenever $\varphi(U) \subset \psi(U)$ for every $U \in \mathcal{U}$.

Denote by $L(\mathbf{X}, \mathbf{Y})$ the space of all continuous linear operators $L : \mathbf{X} \to \mathbf{Y}$. We suppose $L(\mathbf{X}, \mathbf{Y}) \subset \Phi$. The bornologies $\mathfrak{B}_{\mathbf{X}}$, $\mathfrak{B}_{\mathbf{Y}}$ are supposed to

[2] in literature we can find also as terms as the *ground state* or *fiducial vector* or *marked element* or *mother wavelet* depending on the context

be stronger than the corresponding von Neumann bornologies, i.e. the vector operations on the space $L(\mathbf{X},\mathbf{Y})$ are compatible with the topologies, and the bornological convergence implies the topological convergence. Note that in the terminology [21] the space $L(\mathbf{X},\mathbf{Y})$ (as an inductive limit of seminormed spaces) is a bornological convex vector space. For a more detailed explanation of the topological and bornological methods of functional analysis in connection with operators, cf. [25].

2.3 Set structures

Let $T \neq \emptyset$ be a set. Denote by $\Delta \subset 2^T$ a δ-ring of subsets of T. If $\mathcal{A}$ is a system of subsets of the set T, then $\sigma(\mathcal{A})$ denotes the σ-algebra generated by the system $\mathcal{A}$. Denote by $\Sigma = \sigma(\Delta)$. We use χ_E to denote the characteristic function of the set E. By $p_U : \mathbf{X} \to [0,\infty]$ we denote the Minkowski functional of the set $U \in \mathcal{U}$, i.e. $p_U(\mathbf{x}) = \inf_{\mathbf{x}\in\lambda U} |\lambda|$ (if U does not absorb $\mathbf{x} \in \mathbf{X}$, we put $p_U(\mathbf{x}) = \infty$). Similarly, p_W denotes the Minkowski functional of the set $W \in \mathcal{W}$.

For every $(U,W) \in \mathcal{U}\times\mathcal{W}$ denote by $\hat{\mathbf{m}}_{U,W} : \Sigma \to [0,\infty]$ a (U,W)-*semivariation* of a charge (= finitely additive measure) $\mathbf{m} : \Delta \to L(\mathbf{X},\mathbf{Y})$ given by

$$\hat{\mathbf{m}}_{U,W}(E) = \sup p_W\left(\sum_{i=1}^{I} \mathbf{m}(E\cap E_i)\mathbf{x}_i\right), \quad E \in \Sigma,$$

where the supremum is taken over all finite sets $\{\mathbf{x}_i \in U, i = 1,2,\ldots,I\}$ and all disjoint sets $\{E_i \in \Delta;\ i = 1,2,\ldots,I\}$. It is well-known that $\hat{\mathbf{m}}_{U,W}$ is a submeasure on Σ, i.e. a monotone, subadditive set function with $\hat{\mathbf{m}}_{U,W}(\emptyset) = 0$ for every $(U,W) \in \mathcal{U}\times\mathcal{W}$. Denote by $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}} = \{\hat{\mathbf{m}}_{U,W};\ (U,W) \in \mathcal{U}\times\mathcal{W}\}$, and by $\Delta_{U,W} \subset \Delta$ the largest δ-ring of sets $E \in \Delta$, such that $\hat{\mathbf{m}}_{U,W}(E) < \infty$.

For every $(U,W) \in \mathcal{U}\times\mathcal{W}$, denote by $\|\mathbf{m}\|_{U,W} : \Sigma \to [0,\infty]$ a *scalar* (U,W)-*semivariation* of a charge $\mathbf{m} : \Delta \to L(\mathbf{X},\mathbf{Y})$ defined as

$$\|\mathbf{m}\|_{U,W}(E) = \sup \left\|\sum_{i=1}^{I} \lambda_i \mathbf{m}(E\cap E_i)\right\|_{U,W}, \quad E \in \Sigma,$$

where $\|L\|_{U,W} = \sup_{\mathbf{x}\in U} p_W(L(\mathbf{x}))$ and the supremum is taken over all finite sets of scalars $\{\lambda_i \in \mathbb{K};\ |\lambda_i| \leq 1, i = 1,2,\ldots,I\}$ and all disjoint sets $\{E_i \in \Delta;\ i = 1,2,\ldots,I\}$. The scalar (U,W)-semivariation $\|\mathbf{m}\|_{U,W}$ is also a submeasure on Σ. Denote by $\|\mathbf{m}\|_{\mathcal{U},\mathcal{W}} = \{\|\mathbf{m}\|_{U,W}; (U,W) \in \mathcal{U}\times\mathcal{W}\}$.

For every $(U,W) \in \mathcal{U}\times\mathcal{W}$, denote by $\mathbf{var}_{U,W}(\mathbf{m},\cdot) : \Sigma \to [0,\infty]$ a (U,W)-*variation* of a charge $\mathbf{m} : \Delta \to L(\mathbf{X},\mathbf{Y})$ given as

$$\mathbf{var}_{U,W}(\mathbf{m},E) = \sup \sum_{i=1}^{I} \|\mathbf{m}(E\cap E_i)\|_{U,W}, \quad E \in \Sigma,$$

where the supremum is taken over all finite collections of disjoint sets $E_i \in \Delta$, $i = 1,2,\ldots,I$. The (U,W)-variation $\mathbf{var}_{U,W}(\mathbf{m},\cdot)$ is also a submeasure on Σ. Clearly, for every set $E \in \Sigma$ the inequality

$$\|\mathbf{m}\|_{U,W}(E) \leq \hat{\mathbf{m}}_{U,W}(E) \leq \mathbf{var}_{U,W}(\mathbf{m},E)$$

holds. It is also not hard to find examples showing the existence of such a measure $\mathbf{m}$ and a set $E \in \Sigma$ with $\hat{\mathbf{m}}_{U,W}(E) < \infty$ but $\mathbf{var}_{U,W}(\mathbf{m},E) = \infty$.

2.4 Basic convergences of functions

For $(U, W) \in \mathcal{U} \times \mathcal{W}$, let $\beta_{\mathcal{U},\mathcal{W}}$ be a lattice of submeasures $\beta_{U,W} : \Sigma \to [0, \infty]$, where lattice operations are defined as

$$\begin{aligned} \beta_{U_2,W_2} \wedge \beta_{U_3,W_3} &= \beta_{U_2 \wedge U_3, W_2 \vee W_3}, \\ \beta_{U_2,W_2} \vee \beta_{U_3,W_3} &= \beta_{U_2 \vee U_3, W_2 \wedge W_3}, \end{aligned}$$

for $(U_2, W_2), (U_3, W_3) \in \mathcal{U} \times \mathcal{W}$, (e.g. $\beta_{\mathcal{U},\mathcal{W}} = \hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$, or $\|\mathbf{m}\|_{\mathcal{U},\mathcal{W}}$).

Denote by $\mathcal{O}(\beta_{U,W}) = \{N \in \Sigma;\ \beta_{U,W}(N) = 0,\ (U, W) \in \mathcal{U} \times \mathcal{W}\}$. The set $N \in \Sigma$ is called $\beta_{\mathcal{U},\mathcal{W}}$*-null* if there exists a couple $(U, W) \in \mathcal{U} \times \mathcal{W}$, such that $\beta_{U,W}(N) = 0$. We say that an assertion holds $\beta_{\mathcal{U},\mathcal{W}}$*-almost everywhere*, shortly $\beta_{\mathcal{U},\mathcal{W}}$*-a.e.*, if it holds everywhere except in a $\beta_{\mathcal{U},\mathcal{W}}$-null set. A set $E \in \Sigma$ is said to be of *finite submeasure* $\beta_{\mathcal{U},\mathcal{W}}$ if there exists a couple $(U, W) \in \mathcal{U} \times \mathcal{W}$, such that $\beta_{U,W}(E) < \infty$.

The following definitions introduce the analogies of the notions of convergence almost everywhere and convergence almost uniform in the case of operator valued charges in C. B. L. C. S.

Definition 2.2 Let $E \in \Sigma$ and $R \in \mathcal{U}, (U, W) \in \mathcal{U} \times \mathcal{W}$. We say that *a sequence* $\mathbf{f}_n : T \to \mathbf{X}$, $n \in \mathbb{N}$, *of functions* (R, E)*-converges* $\beta_{U,W}$-a.e. *to a function* $\mathbf{f} : T \to \mathbf{X}$ if $\lim_{n\to\infty} p_R(\mathbf{f}_n(t) - \mathbf{f}(t)) = 0$ for every $t \in E \setminus N$, where $N \in \mathcal{O}(\beta_{U,W})$.

We say that *a sequence* $\mathbf{f}_n : T \to \mathbf{X}$, $n \in \mathbb{N}$, *of functions* $(\mathcal{U}, E)$*-converges* $\beta_{\mathcal{U},\mathcal{W}}$-a.e. *to a function* $\mathbf{f} : T \to \mathbf{X}$ if there exist $R \in \mathcal{U}$, $(U, W) \in \mathcal{U} \times \mathcal{W}$, such that the sequence $\mathbf{f}_n$, $n \in \mathbb{N}$, of functions (R, E)-converges $\beta_{U,W}$-a.e. to $\mathbf{f}$. We write $\mathbf{f} = \mathcal{U}\text{-}\lim_{n\to\infty} \mathbf{f}_n\ \beta_{\mathcal{U},\mathcal{W}}$-a.e.

If $E = T$, then we will simply say that the sequence R-converges $\beta_{U,W}$-a.e., resp. $\mathcal{U}$-converges $\beta_{\mathcal{U},\mathcal{W}}$-a.e.

Definition 2.3 Let $E \in \Sigma$ and $R \in \mathcal{U}$, $(U, W) \in \mathcal{U} \times \mathcal{W}$. We say that *a sequence* $\mathbf{f}_n : T \to \mathbf{X}$, $n \in \mathbb{N}$, *of functions* (R, E)*-converges uniformly to a function* $\mathbf{f} : T \to \mathbf{X}$, if $\lim_{n\to\infty} \|\mathbf{f}_n - \mathbf{f}\|_{E,R} = 0$, where $\|\mathbf{f}\|_{E,R} = \sup_{t\in E} p_R(\mathbf{f}(t))$.

We say that *a sequence* $\mathbf{f}_n : T \to \mathbf{X}$, $n \in \mathbb{N}$, *of functions* (R, E)*-converges* $\beta_{U,W}$*-almost uniformly to a function* $\mathbf{f} : T \to \mathbf{X}$ if for every $\varepsilon > 0$ there exists a set $N \in \Sigma$, such that $\beta_{U,W}(N) < \varepsilon$ and the sequence $\mathbf{f}_n$, $n \in \mathbb{N}$, of functions $(R, E \setminus N)$-converges uniformly to $\mathbf{f}$.

We say that *a sequence* $\mathbf{f}_n : T \to \mathbf{X}$, $n \in \mathbb{N}$, *of functions* $(\mathcal{U}, E)$*-converges* $\beta_{\mathcal{U},\mathcal{W}}$*-almost uniformly to a function* $\mathbf{f} : T \to \mathbf{X}$, if there exist $R \in \mathcal{U}$, $(U, W) \in \mathcal{U} \times \mathcal{W}$, such that the sequence $\mathbf{f}_n$, $n \in \mathbb{N}$, of functions (R, E)-converges $\beta_{U,W}$-almost uniformly to $\mathbf{f}$.

If $E = T$, then we will simply say that the sequence of functions R-converges uniformly, resp. R-converges $\beta_{U,W}$-almost uniformly, resp. $\mathcal{U}$-converges $\beta_{\mathcal{U},\mathcal{W}}$-almost uniformly.

For a more detailed explanation of described convergences of functions in C. B. L. C. S., cf. [12].

2.5 Measure structures

The Dobrakov integral is defined in Banach spaces. Since $\mathbf{X}$ and $\mathbf{Y}$ are inductive limits of Banach spaces, the question is whether an integral in C. B. L. C. S. may be defined as a finite sum of Dobrakov integrals in various Banach spaces the choice of which may depend on the function which we integrate. A suitable class of operator measures in C. B. L. C. S. which allow such a generalization is a class of all σ-additive bornological measures. The basic idea consists in additional condition about σ-finiteness of measure which enables the generalization of the whole Dobrakov integration to C. B. L. C. S.

Let $(U, W) \in \mathcal{U} \times \mathcal{W}$. We say that a charge $\mathbf{m}$ is of *σ-finite (U, W)-semivariation* if there exist sets $E_n \in \Delta_{U,W}$, $n \in \mathbb{N}$, such that $T = \bigcup_{n=1}^{\infty} E_n$. For $\varphi \in \Phi$, we say that a charge $\mathbf{m}$ is of *σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation* if for every $U \in \mathcal{U}$, the charge $\mathbf{m}$ is of *σ-finite $(U, \varphi(U))$-semivariation*.

Definition 2.4 We say that a charge $\mathbf{m}$ is *of σ-finite $(\mathcal{U}, \mathcal{W})$-semivariation* if there exists a function $\varphi \in \Phi$ such that $\mathbf{m}$ is of σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation.

In what follows we suppose the charge $\mathbf{m}$ is of σ-finite $(\mathcal{U}, \mathcal{W})$-semivariation. To be more exact we will sometimes specify that a charge $\mathbf{m}$ is of σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation to indicate that φ is that function in Definition 2.4 which provides the σ-finiteness of the $(\mathcal{U}, \mathcal{W})$-semivariation of $\mathbf{m}$.

Let $W \in \mathcal{W}$. We say that a charge $\mu : \Sigma \to \mathbf{Y}$ is a (W, σ)-additive vector measure, if μ is a $\mathbf{Y}_W$-valued (countable additive) vector measure.

Definition 2.5 We say that a charge $\mu : \Sigma \to \mathbf{Y}$ is a *$(\mathcal{W}, \sigma)$-additive vector measure*, if there exists $W \in \mathcal{W}$ such that μ is a (W, σ)-additive vector measure.

Note that if $W \in \mathcal{W}$ and $\mu : \Sigma \to \mathbf{Y}$ is a (W, σ)-additive vector measure, then μ is a (W_1, σ)-additive vector measure whenever $W \subset W_1$, $W_1 \in \mathcal{W}$.

Let $W \in \mathcal{W}$ and let $\nu_n : \Sigma \to \mathbf{Y}$, $n \in \mathbb{N}$, be a sequence of (W, σ)-additive vector measures. If for every $\varepsilon > 0$, $E \in \Sigma$ with $p_W(\nu_n(E)) < \infty$, and $E_i \in \Sigma$, $E_i \cap E_j = \emptyset$, $i \neq j$, $i, j \in \mathbb{N}$, there exists $J_0 \in \mathbb{N}$ such that for every $J \geq J_0$,

$$p_W\left(\nu_n\left(\bigcup_{i=J+1}^{\infty} E_i \cap E\right)\right) < \varepsilon$$

uniformly for every $n \in \mathbb{N}$, then we say that the sequence of measures ν_n, $n \in \mathbb{N}$, is *uniformly (W, σ)-additive on* Σ.

Note that if a sequence ν_n, $n \in \mathbb{N}$, of measures is uniformly (W, σ)-additive on Σ for $W \in \mathcal{W}$, then the sequence ν_n, $n \in \mathbb{N}$, of measures is uniformly (W_1, σ)-additive on Σ whenever $W_1 \supset W$, $W_1 \in \mathcal{W}$.

Definition 2.6 We say that the family of measures $\nu_n : \Sigma \to \mathbf{Y}$, $n \in \mathbb{N}$, is *uniformly $(\mathcal{W}, \sigma)$-additive on* Σ, if there exists $W \in \mathcal{W}$ such that the family of measures ν_n, $n \in \mathbb{N}$, is uniformly (W, σ)-additive on Σ.

The following definition generalizes the notion of the σ-additivity of an operator valued measure in the strong operator topology in Banach spaces, cf. [5], to C. B. L. C. S.

Definition 2.7 Let $\varphi \in \Phi$. We say that a charge $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ is *a σ_φ-additive measure* if $\mathbf{m}$ is of σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation, and for every $A \in \Delta_{U,\varphi(U)}$, the charge $\mathbf{m}(A \cap \cdot)\mathbf{x} : \Sigma \to \mathbf{Y}$ is a $(\varphi(U), \sigma)$-additive measure for every $\mathbf{x} \in \mathbf{X}_U$, $U \in \mathcal{U}$. We say that a charge $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ is a *σ-additive (bornological) measure* if there exists $\varphi \in \Phi$, such that $\mathbf{m}$ is a σ_φ-additive measure.

In what follows a charge $\mathbf{m}$ is supposed to be a σ-additive bornological measure. Note that if $\varphi < \psi$, $\varphi, \psi \in \Phi$, and a charge $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ is a σ_φ-additive measure, then $\mathbf{m}$ is a σ_ψ-additive measure.

2.6 A generalized Dobrakov integral in C. B. L. C. S.

We use $\mathcal{M}_{\Delta,\mathcal{U}}$ to denote *the space of all* $(\Delta,\mathcal{U})$*-measurable functions*, i.e. the largest vector space of functions $\mathbf{f} : T \to \mathbf{X}$ with the property: there exists $R \in \mathcal{U}$, such that for every $U \in \mathcal{U}$, $U \supset R$, and $\delta > 0$ the set $\{t \in T;\, p_U(\mathbf{f}(t)) \geq \delta\} \in \Sigma$. In what follows we deal only with functions which are $(\Delta,\mathcal{U})$-measurable.

Definition 2.8 A function $\mathbf{f} : T \to \mathbf{X}$ is called Δ-*simple* if $\mathbf{f}(T)$ is a finite set and $\mathbf{f}^{-1}(\mathbf{x}) \in \Delta$ for every $\mathbf{x} \in \mathbf{X} \setminus \{0\}$. Let $\mathcal{S}$ denote the space of all Δ-simple functions.

For $(U, W) \in \mathcal{U} \times \mathcal{W}$, a function $\mathbf{f} : T \to \mathbf{X}$ is said to be $\Delta_{U,W}$-*simple* if $\mathbf{f} = \sum_{i=1}^{I} \mathbf{x}_i \chi_{E_i}$, where $\mathbf{x}_i \in \mathbf{X}_U$, $E_i \in \Delta_{U,W}$, such that $E_i \cap E_j = \emptyset$, for $i \neq j$, $i, j = 1, 2, \ldots, I$. The space of all $\Delta_{U,W}$-simple functions is denoted by $\mathcal{S}_{U,W}$.

A function $\mathbf{f} \in \mathcal{S}$ is said to be $\Delta_{\mathcal{U},\mathcal{W}}$-*simple* if there exists a couple $(U, W) \in \mathcal{U} \times \mathcal{W}$, such that $\mathbf{f} \in \mathcal{S}_{U,W}$. The space of all $\Delta_{\mathcal{U},\mathcal{W}}$-simple functions is denoted by $\mathcal{S}_{\mathcal{U},\mathcal{W}}$.

For every $E \in \Sigma$ and $\mathbf{f} \in \mathcal{S}_{U,W}$, $(U, W) \in \mathcal{U} \times \mathcal{W}$, we define the integral by the formula $\int_E \mathbf{f}\, d\mathbf{m} = \sum_{i=1}^{I} \mathbf{m}(E \cap E_i)\mathbf{x}_i$, where $\mathbf{f} = \sum_{i=1}^{I} \mathbf{x}_i \chi_{E_i}$, $\mathbf{x}_i \in \mathbf{X}_U$, $E_i \in \Delta_{U,W}$, $E_i \cap E_j = \emptyset$, $i \neq j$, $i, j = 1, 2, \ldots, I$. Note that for the function $\mathbf{f} \in \mathcal{S}_{U,W}$ the integral $\int_\cdot \mathbf{f}\, d\mathbf{m}$ is a (W, σ)-additive measure on Σ.

The following result is a version of the classical Vitali-Hahn-Saks-Nikodym theorem in our setting, cf. [4].

Theorem 2.9 *Let* $\gamma_n : \Sigma \to \mathbf{Y}$, $n \in \mathbb{N}$, *be* $(\mathcal{W}, \sigma)$*-additive measures and let* $\mathcal{W}\text{-}\lim_{n\to\infty} \gamma_n(E) = \gamma(E)$ *exist in* $\mathbf{Y}$ *for each* $E \in \Sigma$. *Then* γ_n, $n \in \mathbb{N}$, *are uniformly* $(\mathcal{W}, \sigma)$*-additive measures on* Σ, *and consequently,* γ *is a* $(\mathcal{W}, \sigma)$*-additive measure on* Σ.

The following theorem gives a construction of the integral.

Theorem 2.10 [cf. [13], Theorem 3.8] *Let* $\mathbf{m}$ *be a* σ*-additive measure and* $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$. *If there exists a sequence* $\mathbf{f}_n \in \mathcal{S}_{\mathcal{U},\mathcal{W}}$, $n \in \mathbb{N}$, *of functions, such that*

(a) $\mathcal{U}\text{-}\lim_{n\to\infty} \mathbf{f}_n = \mathbf{f}$ $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$*-a.e.,*

(b) the integrals $\int_\cdot \mathbf{f}_n\, d\mathbf{m}$, $n \in \mathbb{N}$, *are uniformly* $(\mathcal{W}, \sigma)$*-additive measures on* Σ,

then the limit $\nu(E, \mathbf{f}) = \mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n\, d\mathbf{m}$ *exists uniformly in* $E \in \Sigma$.

Definition 2.11 A function $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$ is said to be $\Delta_{\mathcal{U},\mathcal{W}}$-*integrable* if there exists a sequence $\mathbf{f}_n \in \mathcal{S}_{\mathcal{U},\mathcal{W}}$, $n \in \mathbb{N}$, of functions, such that

(a) $\mathcal{U}\text{-}\lim_{n\to\infty} \mathbf{f}_n = \mathbf{f}$ $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e.,

(b) $\int_E \mathbf{f}_n\, d\mathbf{m}$, $n \in \mathbb{N}$, are uniformly $(\mathcal{W}, \sigma)$-additive measures on Σ.

Let $\mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$ denote *the family of all $\Delta_{\mathcal{U},\mathcal{W}}$-integrable functions*. Then *the integral* of the function $\mathbf{f} \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$ on a set $E \in \Sigma$ is defined by the equality

$$\mathbf{y}_E = \int_E \mathbf{f}\,\mathrm{d}\mathbf{m} = \mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n\,\mathrm{d}\mathbf{m}.$$

Some basic properties of the above defined integral are summarized in the following theorems, cf. [13].

Theorem 2.12 [cf. [13], Theorem 4.2] *Let $\nu(E, \mathbf{f}) = \int_E \mathbf{f}\,\mathrm{d}\mathbf{m}$, $E \in \Sigma$ and $\mathbf{f} \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$. Then $\nu(\cdot, \mathbf{f}) : \Sigma \to \mathbf{Y}$ is a $(\mathcal{W}, \sigma)$-additive measure.*

Theorem 2.13 [cf. [13], Theorem 4.3] *A function $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$ is $\Delta_{\mathcal{U},\mathcal{W}}$-integrable if and only if there exists a sequence $\mathbf{f}_n \in \mathcal{S}_{\mathcal{U},\mathcal{W}}$, $n \in \mathbb{N}$, of functions such that*

(a) $(\mathcal{U}, E)$-converges $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e. to $\mathbf{f}$, and

(b) the limit $\mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n\,\mathrm{d}\mathbf{m} = \nu(E)$ exists

for every $E \in \Sigma$. In this case $\int_E \mathbf{f}\,\mathrm{d}\mathbf{m} = \mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n\,\mathrm{d}\mathbf{m}$ for every set $E \in \Sigma$ and this limit is uniform on Σ.

Combining Theorems 2.10 and 2.13 for $\Delta_{\mathcal{U},\mathcal{W}}$-simple functions yields

Theorem 2.14 *Let $\mathbf{f}_n$, $n \in \mathbb{N}$, be a sequence of $\Delta_{\mathcal{U},\mathcal{W}}$-simple functions $\mathcal{U}$-converging $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e. to $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$. Then the following statements are equivalent:*

(i) for every set $E \in \Sigma$ there exists a limit

$$\mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n\,\mathrm{d}\mathbf{m} = \nu(E) \in \mathbf{Y};$$

(ii) the integrals $\int_\cdot \mathbf{f}_n\,\mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, are uniformly $(\mathcal{W}, \sigma)$-additive measures on Σ;

(iii) the limit $\mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n\,\mathrm{d}\mathbf{m} = \nu(E)$ exists in $\mathbf{Y}$ uniformly with respect to $E \in \Sigma$.

Moreover, ν is a $(\mathcal{W}, \sigma)$-additive measure on Σ.

Proof. Since $\int_\cdot \mathbf{f}_n\,\mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$ are $(\mathcal{W}, \sigma)$-additive measures on Σ, then by Theorem 2.9 (i)⇒(ii). By Theorem 2.10 (ii)⇒(iii) and obviously (iii)⇒(i). The last assertion follows directly from Theorem 2.9. □

An example of a Dobrakov-type integral in C. B. L. C. S. is given in [14].

3 Vector integral inequalities

In this section we present some useful inequalities which are important tools and play a key role in this integration technique in C. B. L. C. S. In order to state our results, we need the following notions and notations from [10]: for a function $\mathbf{f} : T \to \mathbf{X}$ denote $N(\mathbf{f}) = \{t \in T; \mathbf{f}(t) \neq 0\}$. Recall also that for $(U, W) \in \mathcal{U} \times \mathcal{W}$ a scalar function $\mathbf{f} : T \to \mathbb{K}$ is said to be *measurable in the sense of Halmos* if $N(\mathbf{f}) \cap \mathbf{f}^{-1}(B) \in \sigma(\Delta_{U,W})$ for each Borel set $B \in \mathbb{K}$.

Proposition 3.1 *Let $(U,W) \in \mathcal{U} \times \mathcal{W}$ and $\mathbf{f}$ be a $\Delta_{U,W}$-measurable function. Then there exists a sequence $\mathbf{f}_n$, $n = 1, 2, \ldots$, of $\Delta_{U,W}$-simple functions U-converging on the whole T to $\mathbf{f}$ such that*

$$p_U(\mathbf{f}_n(t)) \le p_U(\mathbf{f}(t)),$$

for each $t \in T$ and $n \in \mathbb{N}$.

Proof. Let $\mathbf{k}_n$, $n = 1, 2, \ldots$, be a sequence of $\Delta_{U,W}$-simple functions U-converging on the whole T to $\mathbf{f}$. Then $p_U(\mathbf{k}_n(\cdot))$ converges to $p_U(\mathbf{f}(\cdot))$ in T and $p_U(\mathbf{f}(\cdot))$ is a measurable function (in the sense of Halmos [10]). Therefore, by Theorem 20.B therein, there exists a non-decreasing sequence $\mathbf{h}_n$, $n = 1, 2, \ldots$, of non-negative $\sigma(\Delta_{U,W})$-simple functions such that $\mathbf{h}_n(t) \le p_U(\mathbf{f}(t))$ for $t \in T$. Since $N(\mathbf{f}) \in \sigma(\Delta_{U,W})$, there exists $A_n \in \Delta_{U,W}$, $n = 1, 2, \ldots$, such that $A_n \nearrow N(\mathbf{f})$. Then $\mathbf{g}_n = \mathbf{h}_n \chi_{A_n}$, $n = 1, 2, \ldots$, are $\Delta_{U,W}$-simple functions and $\mathbf{g}_n(t) \le p_U(\mathbf{f}(t))$ for $t \in T$. Putting

$$\mathbf{f}_n(t) = \begin{cases} \frac{\mathbf{k}_n(t)\mathbf{g}_n(t)}{p_U(\mathbf{k}_n(t))}, & t \in N(\mathbf{f}) \cap N(\mathbf{k}_n), \\ 0, & \text{otherwise} \end{cases},$$

the sequence $\mathbf{f}_n$, $n = 1, 2, \ldots$, satisfies the conditions of the proposition. □

Let us recall two basic results of our interest from [5], Lemma 1, and [13], Lemma 3.7, collected in the following lemma.

Lemma 3.2 *Let $(U,W) \in \mathcal{U} \times \mathcal{W}$. If $\mathbf{f} \in \mathcal{S}_{U,W}$ and $E \in \sigma(\Delta_{U,W})$, then*

$$\hat{\mathbf{m}}_{U,W}(E) = \sup \left\{ p_W \left(\int_E \mathbf{f} \, \mathrm{d}\mathbf{m} \right); \, \mathbf{f} \in \mathcal{S}_{U,W}, \|\mathbf{f}\|_{E,U} \le 1 \right\}$$

and

$$p_W \left(\int_E \mathbf{f} \, \mathrm{d}\mathbf{m} \right) \le \|\mathbf{f}\|_{E,U} \cdot \hat{\mathbf{m}}_{U,W}(E).$$

For $(U,W) \in \mathcal{U} \times \mathcal{W}$ denote by $\mathcal{I}_{U,W}$ the set of all Dobrakov's integrable functions with respect to Banach spaces $\mathbf{X}_U$, $\mathbf{Y}_W$. Our aim is to prove that the assertion of Lemma 3.2 holds also when replacing $\Delta_{U,W}$-simple by $\Delta_{U,W}$-integrable functions.

Lemma 3.3 *Let $(U,W) \in \mathcal{U} \times \mathcal{W}$. Then*

$$\hat{\mathbf{m}}_{U,W}(E) = \sup \left\{ p_W \left(\int_E \mathbf{f} \, \mathrm{d}\mathbf{m} \right); \, \mathbf{f} \in \mathcal{I}_{U,W}, \|\mathbf{f}\|_{E,U} \le 1 \right\}$$

for every set $E \in \Sigma$. Hence for every $\mathbf{f} \in \mathcal{I}_{U,W}$ and every set $E \in \Sigma$ the inequality

$$p_W \left(\int_E \mathbf{f} \, \mathrm{d}\mathbf{m} \right) \le \|\mathbf{f}\|_{E,U} \cdot \hat{\mathbf{m}}_{U,W}(E) \tag{1}$$

holds.

Proof. The fact that the supremum does not increase when replacing $\Delta_{U,W}$-simple functions by $\Delta_{U,W}$-integrable functions is obvious from the definition of (U,W)-semivariation $\hat{\mathbf{m}}_{U,W}$.

Let us consider $E \in \Sigma$ and $\mathbf{f} \in \mathcal{I}_{U,W}$ with $\|\mathbf{f}\|_{E,U} \leq 1$. Since $\mathbf{f}$ is $\Delta_{U,W}$-measurable, then by Proposition 3.1 there exists a sequence $\mathbf{f}_n$, $n = 1, 2, \ldots$, of $\Delta_{U,W}$-simple functions U-converging on the whole T to $\mathbf{f}$ such that for each $t \in T$, $p_U(\mathbf{f}_n(t)) \leq p_U(\mathbf{f}(t))$ for each $n \in \mathbb{N}$. Using notation of Theorem 1 in [5] put

$$F = \bigcup_{n=1}^{\infty} \Big\{ t \in T;\, p_U(\mathbf{f}_n(t)) > 0 \Big\}.$$

Then there is a set $N \in \sigma(\Delta_{U,W})$, $N \subset F$, and a nondecreasing sequence of sets $F_{j,k} \in \Delta_{U,W}$, $j,k \in \mathbb{N}$, with $\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} F_{j,k} = F \setminus N$, such that $\int_E \mathbf{f}_n \chi_N \,\mathrm{d}\mathbf{m} = 0$ for every $E \in \Sigma$ and $n \in \mathbb{N}$, and, moreover, the sequence $\mathbf{f}_n$, $n = 1, 2, \ldots$, uniformly U-converges to the function $\mathbf{f}$ on every set $F_{j,k}$, $j,k \in \mathbb{N}$. Put $G = F \setminus N$. Consequently, for each n and each couple (j,k) we have

$$\begin{aligned} p_W\left(\int_E \mathbf{f}\,\mathrm{d}\mathbf{m}\right) \leq{}& p_W\left(\int_{E\cap(G\setminus F_{j,k})} \mathbf{f}\,\mathrm{d}\mathbf{m}\right) + p_W\left(\int_{E\cap F_{j,k}} (\mathbf{f}-\mathbf{f}_n)\,\mathrm{d}\mathbf{m}\right) \\ &+ p_W\left(\int_{E\cap F_{j,k}} \mathbf{f}_n\,\mathrm{d}\mathbf{m}\right). \end{aligned}$$

Let $\varepsilon > 0$ be chosen arbitrarily. Since the integral $\int_{\cdot} \mathbf{f}\,\mathrm{d}\mathbf{m}$ is a uniformly (W,σ)-additive measure on Σ, then

$$p_W\left(\int_{E\cap(G\setminus F_{j,k})} \mathbf{f}\,\mathrm{d}\mathbf{m}\right) < \frac{\varepsilon}{2}$$

for sufficiently large $j = j_0$ and $k = k_0$. Since the sequence $\mathbf{f}_n$, $n = 1, 2, \ldots$, uniformly U-converges to $\mathbf{f}$ on each $F_{j,k}$, $j,k \in \mathbb{N}$, then by Lemma 3.2

$$p_W\left(\int_{E\cap F_{j,k}} \mathbf{f}_n\,\mathrm{d}\mathbf{m} - \int_{E\cap F_{j,k}} \mathbf{f}_r\,\mathrm{d}\mathbf{m}\right) \leq \|\mathbf{f}_n - \mathbf{f}_r\|_{F_{j,k},U} \cdot \hat{\mathbf{m}}_{U,W}(F_{j,k}),$$

and hence the sequence $\int_{E\cap F_{j,k}} \mathbf{f}_n\,\mathrm{d}\mathbf{m}$, $n = 1, 2, \ldots$, is uniformly Cauchy (in $\mathbf{Y}_W$) with respect to $E \in \Sigma$. Thus

$$\int_{E\cap F_{j,k}} \mathbf{f}\,\mathrm{d}\mathbf{m} = W\text{-}\lim_{r\to\infty} \int_{E\cap F_{j,k}} \mathbf{f}_r\,\mathrm{d}\mathbf{m}.$$

Consequently,

$$\begin{aligned} p_W\left(\int_{E\cap F_{j,k}} (\mathbf{f}-\mathbf{f}_n)\,\mathrm{d}\mathbf{m}\right) &= \lim_{r\to\infty} p_W\left(\int_{E\cap F_{j,k}} (\mathbf{f}_r-\mathbf{f}_n)\,\mathrm{d}\mathbf{m}\right) \\ &\leq \lim_{r\to\infty} \|\mathbf{f}_r - \mathbf{f}_n\|_{F_{j,k},U} \cdot \hat{\mathbf{m}}_{U,W}(F_{j,k}) \\ &= \|\mathbf{f} - \mathbf{f}_n\|_{F_{j,k},U} \cdot \hat{\mathbf{m}}_{U,W}(F_{j,k}) < \frac{\varepsilon}{2} \end{aligned}$$

for sufficiently large $n = n_0$. The fact that $\mathbf{f}_{n_0}$ is $\Delta_{U,W}$-simple with $\|\mathbf{f}_{n_0}\|_{E,U} \leq 1$ completes the proof. $\square$

The inequality (1) plays a fundamental role showing the importance of the (U,W)-semivariation $\hat{\mathbf{m}}_{U,W}$ in this integration theory. We only demonstrate its using when proving the following basic result on integrable functions. Recall that a charge $\mathbf{m} : \Delta \to L(\mathbf{X},\mathbf{Y})$ is of *continuous* (U,W)*-semivariation* on $\Delta_{U,W}$ if $E_n \in \Delta_{U,W}$, $n = 1,2,\ldots$ such that $E_n \searrow \emptyset$ implies $\hat{\mathbf{m}}_{U,W}(E_n) \to 0$. Further theorems on integrable functions and convergence theorems are proved in [14].

Theorem 3.4 *Let $(U,W) \in \mathcal{U}\times\mathcal{W}$ and let a charge $\mathbf{m}$ be of continuous (U,W)-semivariation on $\Delta_{U,W}$. If $A \in \Delta_{U,W}$ and $\mathbf{f}$ is a U-bounded $\Delta_{U,W}$-measurable function on T, then $\mathbf{f}\chi_A \in \mathcal{I}_{U,W}$.*

Proof. Let us consider a sequence $\mathbf{f}_n$, $n = 1,2,\ldots$, of $\Delta_{U,W}$-simple functions, such that the sequence $\mathbf{f}_n\chi_A$, $n = 1,2,\ldots$, U-converges on the whole space T to the function $\mathbf{f}\chi_A$ and $\|\mathbf{f}_n\|_{A,U} \le \|\mathbf{f}\|_{A,U}$ for every $n \in \mathbb{N}$. Then the inequality

$$p_W\left(\int_E \mathbf{f}_n\chi_A \,\mathrm{d}\mathbf{m}\right) \le \|\mathbf{f}\|_{A,U}\cdot\hat{\mathbf{m}}_{U,W}(A\cap E), \quad n\in\mathbb{N},\ E\in\Sigma,\ A\in\Delta_{U,W},$$

and the fact that $\mathbf{m}$ is of continuous (U,W)-semivariation on $\Delta_{U,W}$ imply the uniform (W,σ)-additivity of the integrals $\int_{\cdot} \mathbf{f}_n\chi_A \,\mathrm{d}\mathbf{m}$, $n\in\mathbb{N}$, on Σ. This proves the theorem. □

Note that without the assumption $\mathbf{m}$ is of continuous (U,W)-semivariation on $\Delta_{U,W}$ the preceding theorem does not hold even in the case when a charge $\mathbf{m} : \Delta \to L(\mathbf{X},\mathbf{Y})$ is σ-additive in the equibornology of the space $L(\mathbf{X},\mathbf{Y})$, i.e., if for every $(U,W)\in\mathcal{U}\times\mathcal{W}$ the restriction $\mathbf{m}_{U,W}(E)\mathbf{x} = \mathbf{m}(E)\mathbf{x}$, $E\in\Delta_{U,W}$, of a charge $\mathbf{m}$ to the set system $\Delta_{U,W}$ is a σ-additive vector measure in the uniform topology of the space $L(\mathbf{X}_U,\mathbf{Y}_W)$, cf. [14].

Similarly as Theorem 3.4 we may prove

Theorem 3.5 *Let $(U,W)\in\mathcal{U}\times\mathcal{W}$. Let $\mathbf{f}$ be a $\Delta_{U,W}$-measurable function and*

$$\int_F p_U(\mathbf{f})\,\mathrm{d}\mathbf{var}_{U,W}(\mathbf{m},\cdot) < \infty,$$

where $F = \{t\in T;\, p_U(\mathbf{f}(t)) > 0\}$. Then $\mathbf{f}$ is a $\Delta_{U,W}$-integrable function and the inequality

$$\mathbf{var}_{U,W}\left(\int_{\cdot}\mathbf{f}\,\mathrm{d}\mathbf{m}, E\right) \le \int_E p_U(\mathbf{f})\,\mathrm{d}\mathbf{var}_{U,W}(\mathbf{m},E)$$

holds for every set $E\in\Sigma$.

Note that there are functions $\mathbf{f}$ such that $\mathbf{var}_{U,W}(\int_{\cdot}\mathbf{f}\,\mathrm{d}\mathbf{m},E) = 0$ for every set $E\in\Sigma$, but in the same time it may happen $\int_T p_U(\mathbf{f})\,\mathrm{d}\mathbf{var}_{U,W}(\mathbf{m},\cdot) = \infty$.

Let $\mathbf{Z}$ be a complete bornological locally convex topological vector space with the bornology $\mathfrak{B}_{\mathbf{Z}}$ equipped with the basis of Banach disks $\mathcal{V}$. Let $(U,W,V)\in\mathcal{U}\times\mathcal{W}\times\mathcal{V}$, $K\in L(\mathbf{Y},\mathbf{Z})$ and $\mathbf{m} : \Delta\to L(\mathbf{X},\mathbf{Y})$ be a σ-additive bornological measure. Then clearly the set function $K\mathbf{m} : \Delta\to L(\mathbf{X},\mathbf{Z})$ given by $K\mathbf{m}(E) = K[\mathbf{m}(E)]$ for $E\in\Delta$, is a σ-additive bornological measure. Since

$$(\widehat{K\mathbf{m}})_{U,V}(E) \le \|K\|_{W,V}\cdot\hat{\mathbf{m}}_{U,W}(E)$$

for every set $E \in \Sigma$, then every $\Delta_{U,W}$-simple function $\mathbf{f}$ is also $\Delta_{U,V}$-simple (with respect to the measure $K\mathbf{m}$) and there holds

$$K\left(\int_E \mathbf{f}\, \mathrm{d}\mathbf{m}\right) = \int_E \mathbf{f}\, \mathrm{d}(K\mathbf{m})$$

for every set $E \in \Sigma$. From this fact, using Theorem 2.13 we get directly that every $\Delta_{U,W}$-integrable function $\mathbf{f}$ is also $\Delta_{U,V}$-integrable (with respect to the measure $K\mathbf{m}$) and for every set $E \in \Sigma$ there holds

$$K\left(\int_E \mathbf{f}\, \mathrm{d}\mathbf{m}\right) = \int_E \mathbf{f}\, \mathrm{d}(K\mathbf{m}).$$

Let $\gamma : \Delta \to \mathbf{Y}$ be a vector measure. For every $W \in \mathcal{W}$, denote by $\overline{\gamma}_W : \Sigma \to [0, \infty]$ a W-*supremation* of γ given by

$$\overline{\gamma}_W(E) = \sup\{p_W(\gamma(F));\ F \subset E, F \in \Delta\}, \quad E \in \Sigma.$$

We put $\overline{\gamma}_W(T) = \sup\{\overline{\gamma}_W(E); E \in \Sigma\}$. Recall that if $\gamma : \Sigma \to \mathbf{Y}$ is a (W, σ)-additive measure on Σ, then according to [4] (Proposition I.1.11 and Theorem I.2.4) the set function $\overline{\gamma}_W$ is a (W-bounded) continuous submeasure on Σ, see also [20], Theorem 3.

For every $W \in \mathcal{W}$, denote by $|\mu|_W : \Sigma \to [0, \infty]$ a W-*semivariation* of a charge $\mu : \Sigma \to \mathbf{Y}$ given by

$$|\mu|_W(E) = \sup p_W\left(\sum_{i=1}^{I} \lambda_i \mu(E \cap E_i)\right), \quad E \in \Sigma,$$

where the supremum is taken over all finite sets of scalars $\{\lambda_i \in \mathbb{K};\ |\lambda_i| \le 1, i = 1, 2, \ldots, I\}$ and all disjoint sets $\{E_i \in \Delta;\ i = 1, 2, \ldots, I\}$. The W-semivariation $|\mu|_W$ is a submeasure on Σ.

Now, the result of Lemma 3.3 may be extended as follows:

Lemma 3.6 *Let $(U, W) \in \mathcal{U} \times \mathcal{W}$ and $\mathbf{f} \in \mathcal{I}_{U,W}$. Then there exist a sequence $\mathbf{f}_n$, $n = 1, 2, \ldots$ of $\Delta_{U,W}$-simple functions and a set $M \in \Sigma$, $M \in \mathcal{O}(\hat{\mathbf{m}}_{U,W})$, such that*

(i) the sequence $\mathbf{f}_n$ U-converges on $T \setminus M$ to $\mathbf{f}$;

(ii) $p_U(\mathbf{f}_n(t)) \le p_U(\mathbf{f}(t))$ for $t \in T \setminus M$; and

(iii) W-$\lim_{n\to\infty} \int_E \mathbf{f}_n\, \mathrm{d}\mathbf{m} = \int_E \mathbf{f}\, \mathrm{d}\mathbf{m}$ for $E \in \Sigma$, the limit being uniform with respect to $E \in \Sigma$.

Consequently,

$$\hat{\mathbf{m}}_{U,W}(E) = \sup\left\{p_W\left(\int_E \mathbf{f}\, \mathrm{d}\mathbf{m}\right);\ \mathbf{f} \in \mathcal{I}_{U,W}, \|\mathbf{f}\|_{E,U} \le 1\right\}$$

for every set $E \in \Sigma$ and hence for every $\mathbf{f} \in \mathcal{I}_{U,W}$ and every set $E \in \Sigma$ the inequality (1) holds.

Proof. From Proposition 3.1 and Definition 2.11 there exist two sequences $\mathbf{k}_n$ and $\mathbf{h}_n$, $n = 1, 2, \ldots$, of $\Delta_{U,W}$-simple functions and a set $M \in \Sigma$, $M \in \mathcal{O}(\hat{\mathbf{m}}_{U,W})$, such that $\mathbf{k}_n$ U-converges on $T \setminus M$ to $\mathbf{f}$ with $p_U(\mathbf{k}_n(t)) \le p_U(\mathbf{f}(t))$ for all $t \in T \setminus M$, and $\mathbf{h}_n$ U-converges on $T \setminus M$ to $\mathbf{f}$ such that $\gamma_n(\cdot) = \int_\cdot \mathbf{h}_n \, d\mathbf{m}$, $n = 1, 2, \ldots$, are uniformly (W, σ)-additive measures on Σ with W-$\lim_{n\to\infty} \gamma_n(E) = \int_E \mathbf{f} \, d\mathbf{m}$ for $E \in \Sigma$.

Put $\eta_n(\cdot) = \int_\cdot \mathbf{k}_n \, d\mathbf{m}$ for $n = 1, 2, \ldots$ and $\nu(E) = \int_E \mathbf{f} \, d\mathbf{m}$ for $E \in \Sigma$. Let

$$F = \bigcup_{n=1}^{\infty} \Big\{ t \in T \setminus M;\, p_U(\mathbf{h}_n) + p_U(\mathbf{k}_n) > 0 \Big\},$$

and define

$$\lambda_W(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{\overline{\gamma}_{W,n}(E)}{1 + \overline{\gamma}_{W,n}(T)} + \frac{\overline{\eta}_{W,n}(E)}{1 + \overline{\eta}_{W,n}(T)} \right), \quad E \in \Sigma.$$

Since γ_n, η_n, $n = 1, 2, \ldots$, are (W, σ)-additive measures on Σ, then $\overline{\gamma}_{W,n}$, $\overline{\eta}_{W,n}$ are (W-bounded) continuous submeasures on Σ for all $n \in \mathbb{N}$. Clearly, λ_W is a (W-bounded) submeasure on Σ. To show continuity, for a given $\varepsilon > 0$ choose n_0 such that $\frac{1}{2^{n_0}} < \frac{\varepsilon}{2}$ and consider sets $E_r \in \Sigma$, $r = 1, 2, \ldots$, such that $E_r \searrow \emptyset$. From continuity of $\overline{\gamma}_{W,n}$, $\overline{\eta}_{W,n}$, $n = 1, 2, \ldots, n_0$, there exists r_0 such that $(\overline{\gamma}_{W,n}(E_r) + \overline{\eta}_{W,n}(E_r)) < \frac{\varepsilon}{2}$ for $r \ge r_0$ and $n = 1, 2, \ldots, n_0$, which implies $\lambda_W(E_r) < \varepsilon$. Thus λ_W is a continuous submeasure on Σ.

Consider the sequence $\mathcal{K} = \{\mathbf{h}_1, \mathbf{k}_1, \mathbf{h}_2, \mathbf{k}_2, \ldots, \mathbf{h}_n, \mathbf{k}_n, \ldots\}$. Clearly, $\mathcal{K}$ is a sequence of $\Delta_{U,W}$-simple functions U-converging to $\mathbf{f}$ in $T \setminus M$. Then by Egoroff-Luzin theorem, cf. [20], Theorem 5, there exist a set $N \in F \cap \sigma(\Delta_{U,W})$ with $\lambda_W(N) = 0$ and a sequence of sets $F_{j,k} \in \Delta_{U,W}$, $j, k = 1, 2, \ldots$, with $\bigcup_{j=1}^\infty \bigcup_{k=1}^\infty F_{j,k} = F \setminus N$, such that $\mathcal{K}$ uniformly U-converges to $\mathbf{f}$ on each $F_{j,k}$, $j, k \in \mathbb{N}$. Thus we may choose a subsequence $n_{j,k}$, $j, k \in \mathbb{N}$, such that for every couple (j, k) there is

$$\|\mathbf{h}_{n_{j,k}} - \mathbf{k}_{n_{j,k}}\|_{F_{j,k},U} \cdot \hat{\mathbf{m}}_{U,W}(F_{j,k}) < \frac{1}{2^j k}.$$

Put $\mathbf{f}_{j,k} = \mathbf{k}_{n_{j,k}} \chi_N + \mathbf{k}_{n_{j,k}} \chi_{F_{j,k}}$ for every (j, k). Then $\mathbf{f}_{j,k}$, $j, k \in \mathbb{N}$, is a sequence of $\Delta_{U,W}$-simple functions U-converging to the function $\mathbf{f}$ in $T \setminus M$ with $p_U(\mathbf{f}_{j,k}(t)) \le p_U(\mathbf{f}(t))$ for $t \in T \setminus M$.

Let $G = F \setminus N$. Since $M \in \mathcal{O}(\hat{\mathbf{m}}_{U,W})$, then by Lemma 3.2 we get $\overline{\gamma}_{W,n}(M) = \overline{\eta}_{W,n}(M) = 0$ for all $n \in \mathbb{N}$ and hence $\lambda_W(M) = 0$. Moreover, as $\lambda_W(M) = \lambda_W(N) = 0$, and $\eta_n(E \cap N) = \eta_n(E \cap M) = \gamma_n(E \cap N) = \gamma_n(E \cap M) = 0$ for all $n \in \mathbb{N}$, then clearly, $\mathbf{f}_{j,k}(t) = 0$ for $t \in E \cap (G \setminus F_{j,k})$. Therefore

$$\begin{aligned} p_W\left(\nu(E) - \int_E \sum_{j=1}^J \mathbf{f}_{j,k} \, d\mathbf{m}\right) \le\ & p_W\left(\sum_{j=1}^J \int_{E \cap F_{j,k}} (\mathbf{f}_{j,k} - \mathbf{h}_{n_{j,k}}) \, d\mathbf{m}\right) \\ &+ p_W\left(\sum_{j=1}^J \int_{E \cap (G \setminus F_{j,k})} \mathbf{f}_{n_{j,k}} \, d\mathbf{m}\right) \\ &+ p_W\left(\nu(E) - \int_E \mathbf{h}_{n_{j,k}} \, d\mathbf{m}\right). \end{aligned}$$

Using Lemma 3.2 and the definition of W-variation we get

$$\begin{aligned} p_W\left(\nu(E) - \int_E \sum_{j=1}^J \mathbf{f}_{j,k}\,\mathrm{d}\mathbf{m}\right) \leq & \sum_{j=1}^J \|\mathbf{k}_{n_{j,k}} - \mathbf{h}_{n_{j,k}}\|_{F_{j,k},U} \cdot \hat{\mathbf{m}}_{U,W}(F_{j,k}) \\ & + \sum_{j=1}^J |\gamma_{n_{j,k}}|_W(G \setminus F_{j,k}) \\ & + p_W\left(\nu(E) - \gamma_{n_{j,k}}(E)\right). \end{aligned}$$

Let $\varepsilon > 0$ be chosen arbitrarily and choose k_0 such that $\frac{1}{2^j k_0} < \frac{\varepsilon}{3}$, $j = 1, 2, \dots, J$. Since $\nu(E) = W\text{-}\lim_{k\to\infty} \gamma_{n_{j,k}}(E)$ uniformly with respect to $E \in \Sigma$ for every $j = 1, 2, \dots, J$, we may choose $k_1 \geq k_0$ such that

$$p_W\left(\nu(E) - \gamma_{n_{j,k}}(E)\right) < \frac{\varepsilon}{3}, \qquad j = 1, 2, \dots, J,$$

for all $k \geq k_1$ and for all $E \in \Sigma$. Thus choosing $k \geq k_1$ we have

$$\|\mathbf{k}_{n_{j,k}} - \mathbf{h}_{n_{j,k}}\|_{F_{j,k},U} \cdot \hat{\mathbf{m}}_{U,W}(F_{j,k}) < \frac{\varepsilon}{3}, \qquad j = 1, 2, \dots, J, \tag{2}$$

and

$$p_W\left(\nu(E) - \gamma_{n_{j,k}}(E)\right) < \frac{\varepsilon}{3}, \qquad j = 1, 2, \dots, J, \tag{3}$$

for all $E \in \Sigma$. Since γ_n, $n = 1, 2, \dots$, are uniformly (W, σ)-additive measures on Σ, then its W-semivariations $|\gamma_n|_W(\cdot)$, $n = 1, 2, \dots$, are uniformly continuous on Σ, and since $(G \setminus F_{j,k}) \searrow \emptyset$, then there exists $k_2 \geq k_1$ such that $|\gamma_n|_W(G \setminus F_{j,k}) < \frac{\varepsilon}{3}$ for all $k \geq k_2$ and for all $n \in \mathbb{N}$. Thus, in particular,

$$|\gamma_{n_{j,k}}|_W(G \setminus F_{j,k}) < \frac{\varepsilon}{3}, \qquad j = 1, 2, \dots, J, \tag{4}$$

for $k \geq k_2$. Consequently, from (2), (3) and (4) we have

$$p_W\left(\nu(E) - \int_E \sum_{j=1}^J \mathbf{f}_{j,k}\,\mathrm{d}\mathbf{m}\right) < \varepsilon$$

for $k \geq k_2$ and $E \in \Sigma$. Since ε is an arbitrary positive number, E is an arbitrary element in Σ and $\mathbf{Y}_W$ is a complete space, the existence and the uniformity in $E \in \Sigma$ of the limit $W\text{-}\lim_{k\to\infty} \int_E \mathbf{f}_k\,\mathrm{d}\mathbf{m} = \nu(E)$ is proved. The remaining parts are immediate from (i), (ii), (iii) and the definition of $\hat{\mathbf{m}}_{U,W}$. □

A possible way how to improve the inequality (1) given in Lemma 3.6 is to consider the notion of $L^1_{U,W}$-gauge of a $\Delta_{U,W}$-measurable function which is a generalization of the classical L_1-norm and also of the notion of the (U, W)-semivariation $\hat{\mathbf{m}}_{U,W}$ (see Corollary 3.17 and Remark 3.10, respectively), similarly as in [6].

Definition 3.7 Let $(U, W) \in \mathcal{U} \times \mathcal{W}$. Let $\mathbf{g}$ be a $\Delta_{U,W}$-measurable function and $E \in \Sigma$. Then the *$L^1_{U,W}$-gauge of the function $\mathbf{g}$ on the set E*, denoted by $\hat{\mathbf{m}}_{U,W}(\mathbf{g}, E)$, is a non-negative not necessarily finite number defined by the equality

$$\hat{\mathbf{m}}_{U,W}(\mathbf{g}, E) = \sup \left\{ p_W \left(\int_E \mathbf{f} \, \mathrm{d}\mathbf{m} \right) \right\},$$

where the supremum is taken over all $\mathbf{f} \in \mathcal{S}_{U,W}$ such that $p_U(\mathbf{f}(t)) \leq p_U(\mathbf{g}(t))$ for each $t \in E$. The $L^1_{U,W}$-gauge of the function $\mathbf{g}$ is then defined by

$$\hat{\mathbf{m}}_{U,W}(\mathbf{g}, T) = \sup_{E \in \Sigma} \hat{\mathbf{m}}_{U,W}(\mathbf{g}, E).$$

From this definition we immediately get the following theorem summarizing basic properties of $L^1_{U,W}$-gauge $\hat{\mathbf{m}}_{U,W}(\cdot, \cdot)$.

Theorem 3.8 *Let $(U, W) \in \mathcal{U} \times \mathcal{W}$. Let $\mathbf{g}$ be a $\Delta_{U,W}$-measurable function and let $E \in \Sigma$. Then*

(a) $\hat{\mathbf{m}}_{U,W}(\mathbf{g}, \cdot)$ *is a monotone and countably subadditive set function on Σ with* $\hat{\mathbf{m}}_{U,W}(\mathbf{g}, \emptyset) = 0$*;*

(b) $\hat{\mathbf{m}}_{U,W}(\alpha \cdot \mathbf{g}, E) = |\alpha| \cdot \hat{\mathbf{m}}_{U,W}(\mathbf{g}, E)$ *for each scalar α;*

(c) $\inf_{t \in E} p_U(\mathbf{g}(t)) \cdot \hat{\mathbf{m}}_{U,W}(E) \leq \hat{\mathbf{m}}_{U,W}(\mathbf{g}, E) \leq \|\mathbf{g}\|_{E,U} \cdot \hat{\mathbf{m}}_{U,W}(E)$*;*

(d) if $\mathbf{h}$ is a $\Delta_{U,W}$-measurable function with $p_U(\mathbf{h}(t)) \leq p_U(\mathbf{g}(t))$ $\hat{\mathbf{m}}_{U,W}$-a.e. on E, then $\hat{\mathbf{m}}_{U,W}(\mathbf{h}, E) \leq \hat{\mathbf{m}}_{U,W}(\mathbf{g}, E)$*;*

(e) $\hat{\mathbf{m}}_{U,W}(\mathbf{g}, E) = \hat{\mathbf{m}}_{U,W}(\mathbf{g}, \{t \in E;\, p_U(\mathbf{g}(t)) > 0\})$*;*

(f) $\hat{\mathbf{m}}_{U,W}(\mathbf{g}, E) = 0$ *if and only if* $\hat{\mathbf{m}}_{U,W}(\{t \in E;\, p_U(\mathbf{g}(t)) > 0\}) = 0$.

Observe that according to (a) $\hat{\mathbf{m}}_{U,W}(\mathbf{g}, \cdot)$ is a σ-subadditive submeasure on Σ. Also, the assertion (f) of the previous theorem implies

Corollary 3.9 *Let $(U, W) \in \mathcal{U} \times \mathcal{W}$ and $\mathbf{g}$ be a $\Delta_{U,W}$-measurable function. Then $\hat{\mathbf{m}}_{U,W}(\mathbf{g}, T) = 0$ if and only if $\mathbf{g} = 0$ $\hat{\mathbf{m}}_{U,W}$-a.e.*

Remark 3.10 From assertion (c) of Theorem 3.8 it is obvious that if $p_U(\mathbf{x}) = 1$, then $\hat{\mathbf{m}}_{U,W}(E) = \hat{\mathbf{m}}_{U,W}(\mathbf{x} \cdot \chi_E, E)$ for each set $E \in \Sigma$. Therefore the $L^1_{U,W}$-gauge $\hat{\mathbf{m}}_{U,W}(\cdot, \cdot)$ generalizes the notion of the (U, W)-semivariation $\hat{\mathbf{m}}_{U,W}$.

From the assertion of the remark we get the following form of the *Čebyšev's inequality* in our setting.

Theorem 3.11 *Let $(U, W) \in \mathcal{U} \times \mathcal{W}$ and $\mathbf{g}$ be a $\Delta_{U,W}$-measurable function. Let $E \in \Sigma$ and $\alpha > 0$. Then*

$$\hat{\mathbf{m}}_{U,W}\big(\{t \in E;\, p_U(\mathbf{g}(t)) \geq \alpha\}\big) \leq \frac{\hat{\mathbf{m}}_{U,W}(\mathbf{g}, E)}{\alpha}.$$

Proof. Let $F = \{t \in E;\, p_U(\mathbf{g}(t)) \geq \alpha\}$. Then clearly $F \in \Sigma$. By (a) and (c) of Theorem 3.8 we get

$$\alpha \cdot \hat{\mathbf{m}}_{U,W}(F) \leq \inf_{t \in F} p_U(\mathbf{g}(t)) \cdot \hat{\mathbf{m}}_{U,W}(F) \leq \hat{\mathbf{m}}_{U,W}(\mathbf{g}, F) \leq \hat{\mathbf{m}}_{U,W}(\mathbf{g}, E).$$

Hence we have the result. □

Corollary 3.12 *Let $(U,W) \in \mathcal{U} \times \mathcal{W}$, $\mathbf{g}$ be a $\Delta_{U,W}$-measurable function and $E \in \Sigma$. If $\hat{\mathbf{m}}_{U,W}(\mathbf{g}, E) < \infty$, then $\mathbf{g}$ is finite $\hat{\mathbf{m}}_{U,W}$-a.e. in E. Consequently, if $\hat{\mathbf{m}}_{U,W}(\mathbf{g}, T) < \infty$, then $\mathbf{g}$ is finite $\hat{\mathbf{m}}_{U,W}$-a.e. in T.*

Proof. Put $F = \{t \in E;\, p_U(\mathbf{g}(t)) = \infty\}$ and $F_n = \{t \in E;\, p_U(\mathbf{g}(t)) \geq n\}$, $n = 1, 2, \ldots$. Then $F \subset F_n$ and $F, F_n \in \Sigma$ for each $n \in \mathbb{N}$. By (a) of Theorem 3.8 and by Theorem 3.11 we have

$$\hat{\mathbf{m}}_{U,W}(F) \leq \hat{\mathbf{m}}_{U,W}(F_n) \leq \frac{\hat{\mathbf{m}}_{U,W}(\mathbf{g}, E)}{n}.$$

Assuming that $\hat{\mathbf{m}}_{U,W}(\mathbf{g}, E) < \infty$ the last term tends to 0 as $n \to \infty$. Hence $\mathbf{g}$ is finite $\hat{\mathbf{m}}_{U,W}$-a.e. in E. The last assertion follows from the fact $\hat{\mathbf{m}}_{U,W}(\mathbf{g}, T) = \hat{\mathbf{m}}_{U,W}(\mathbf{g}, N(\mathbf{g}))$. □

Consequently, we may state the following lemma which is a generalization of Lemma 3.3. It may be proved in the same way. Observe that by (c) of Theorem 3.8 the obtained inequality (5) is much better than (1).

Lemma 3.13 *Let $(U,W) \in \mathcal{U} \times \mathcal{W}$, $\mathbf{g}$ be a $\Delta_{U,W}$-measurable function and let $E \in \Sigma$. Then*

$$\hat{\mathbf{m}}_{U,W}(\mathbf{g}, E) = \sup \left\{ p_W \left(\int_E \mathbf{f} \, \mathrm{d}\mathbf{m} \right);\, \mathbf{f} \in \mathcal{I}_{U,W},\, p_U(\mathbf{f}(t)) \leq p_U(\mathbf{g}(t)),\, t \in E \right\}.$$

Hence for every $\mathbf{f} \in \mathcal{I}_{U,W}$ and every set $E \in \Sigma$ the inequality

$$p_W \left(\int_E \mathbf{f} \, \mathrm{d}\mathbf{m} \right) \leq \hat{\mathbf{m}}_{U,W}(\mathbf{f}, E) \tag{5}$$

holds.

Using these results we may prove the triangle inequality:

Theorem 3.14 *Let $(U,W) \in \mathcal{U} \times \mathcal{W}$, let $\mathbf{f}$, $\mathbf{g}$ be $\Delta_{U,W}$-measurable functions and $E \in \Sigma$. Then*

$$\hat{\mathbf{m}}_{U,W}(\mathbf{f} + \mathbf{g}, E) \leq \hat{\mathbf{m}}_{U,W}(\mathbf{f}, E) + \hat{\mathbf{m}}_{U,W}(\mathbf{g}, E).$$

Proof. By assertion (e) of Theorem 3.8 we have

$$\hat{\mathbf{m}}_{U,W}(\mathbf{f} + \mathbf{g}, E) = \hat{\mathbf{m}}_{U,W}(\mathbf{f} + \mathbf{g}, E'),$$

where $E' = \{t \in E,\, p_U(\mathbf{f}(t)) + p_U(\mathbf{g}(t)) > 0\}$. Let $\mathbf{h}$ be a $\Delta_{U,W}$-simple function such that $p_U(\mathbf{h}(t)) \leq p_U(\mathbf{f}(t) + \mathbf{g}(t))$ for each $t \in E'$. Then for each $t \in E'$

$$\mathbf{h}(t) = \frac{\mathbf{h}(t) \cdot p_U(\mathbf{f}(t))}{p_U(\mathbf{f}(t)) + p_U(\mathbf{g}(t))} + \frac{\mathbf{h}(t) \cdot p_U(\mathbf{g}(t))}{p_U(\mathbf{f}(t)) + p_U(\mathbf{g}(t))}.$$

Since by Theorem 3.2 in [14] both summands are $\Delta_{U,W}$-integrable functions, using Lemma 3.13 we get

$$\begin{aligned}
p_W \left(\int_{E'} \mathbf{h} \, \mathrm{d}\mathbf{m} \right) &\leq p_W \left(\int_{E'} \frac{\mathbf{h}(t) \cdot p_U(\mathbf{f}(t))}{p_U(\mathbf{f}(t)) + p_U(\mathbf{g}(t))} \, \mathrm{d}\mathbf{m} \right) \\
&\quad + p_W \left(\int_{E'} \frac{\mathbf{h}(t) \cdot p_U(\mathbf{f}(t))}{p_U(\mathbf{g}(t)) + p_U(\mathbf{g}(t))} \, \mathrm{d}\mathbf{m} \right) \\
&\leq \hat{\mathbf{m}}_{U,W}(\mathbf{f}, E) + \hat{\mathbf{m}}_{U,W}(\mathbf{g}, E),
\end{aligned}$$

which completes the proof. $\square$

Consequently, we have

$$\begin{aligned}\hat{\mathbf{m}}_{U,W}(\mathbf{f}+\mathbf{g},T) &= \sup_{E\in\Sigma}\hat{\mathbf{m}}_{U,W}(\mathbf{f}+\mathbf{g},E)\\ &\le \sup_{E\in\Sigma}\hat{\mathbf{m}}_{U,W}(\mathbf{f},E)+\sup_{E\in\Sigma}\hat{\mathbf{m}}_{U,W}(\mathbf{g},E)\\ &= \hat{\mathbf{m}}_{U,W}(\mathbf{f},T)+\hat{\mathbf{m}}_{U,W}(\mathbf{g},T).\end{aligned}$$

This implies that $\{\mathbf{g};\ \mathbf{g}$ is a $\Delta_{U,W}$-measurable function: $\hat{\mathbf{m}}_{U,W}(\mathbf{g},T)<\infty\}$ is a pseudo-normed space. Hence, $\hat{\mathbf{m}}_{U,W}(\mathbf{g},T)$ may be called as an $L^1_{U,W}$-pseudo-norm of $\mathbf{g}$.

From the definition of $L^1_{U,W}$-gauge it is clear that it depends only on $p_U(\mathbf{g})$ and E. Hence the results about $L^1_{U,W}$-gauges for vector-valued functions remain valid for scalar functions. The next result is a version of the Fatou lemma. Observe that this theorem has no meaning for vector-valued functions in general.

Theorem 3.15 *Let $(U,W)\in\mathcal{U}\times\mathcal{W}$. If $\mathbf{m}:\Delta\to L(\mathbf{X},\mathbf{Y})$ is a σ-additive bornological measure and $\mathbf{f}_n:T\to[0,\infty)$, $n=1,2,\ldots,$ are $\Delta_{U,W}$-measurable functions, then*

$$\hat{\mathbf{m}}_{U,W}(\liminf_n \mathbf{f}_n,E)\le\liminf_n\hat{\mathbf{m}}_{U,W}(\mathbf{f}_n,E)$$

for each $E\in\Sigma$.

Proof. The assertion of theorem immediately follows from Theorem 4 in [6] and the classical Fatou lemma, cf. [10]. $\square$

As a natural generalization of Theorem 3.4 in this setting we get the following theorem. It may be proved in just the same way as Theorem 3.4.

Theorem 3.16 *Let $(U,W)\in\mathcal{U}\times\mathcal{W}$. Let $\mathbf{g}$ be a $\Delta_{U,W}$-measurable function and let its $L^1_{U,W}$-gauge $\hat{\mathbf{m}}_{U,W}(\mathbf{g},\cdot)$ be continuous on Σ. Then $\mathbf{g}\in\mathcal{I}_{U,W}$.*

Note that for $(U,W)\in\mathcal{U}\times\mathcal{W}$ the inequality

$$\hat{\mathbf{m}}_{U,W}(\mathbf{g},E)\le\int_E p_U(\mathbf{g})\,\mathrm{d}\mathbf{var}_{U,W}(\mathbf{m},\cdot) \tag{6}$$

holds in general and it may happens that $\int_E p_U(\mathbf{g})\,\mathrm{d}\mathbf{var}_{U,W}(\mathbf{m},\cdot)=\infty$ and $\hat{\mathbf{m}}_{U,W}(\mathbf{g},E)<\infty$. If $\int_T p_U(\mathbf{g})\,\mathrm{d}\mathbf{var}_{U,W}(\mathbf{m},\cdot)<\infty$, then clearly the $L^1_{U,W}$-gauge $\hat{\mathbf{m}}_{U,W}(\mathbf{g},\cdot)$ is continuous on Σ and, in that case, $\hat{\mathbf{m}}_{U,W}(\mathbf{g},T)<\infty$. From this a natural question arises when the equality in (6) holds. This is answered in the following result from which it is clear that the $L^1_{U,W}$-gauge $\hat{\mathbf{m}}_{U,W}(\cdot,\cdot)$ generalizes the classical L_1-norm. It is a corollary of Lemma 1 in [7].

Corollary 3.17 *Let $(U,W)\in\mathcal{U}\times\mathcal{W}$. If $\mathbf{Y}_W$ is the space of scalars of $\mathbf{X}_U$, or $\mathbf{m}$ is a scalar measure such that $\mathbf{m}(E)\mathbf{x}=\mathbf{m}(E)\cdot\mathbf{x}$, then*

$$\hat{\mathbf{m}}_{U,W}(\mathbf{g},E)=\int_E p_U(\mathbf{g})\,\mathrm{d}\mathbf{var}_{U,W}(\mathbf{m},\cdot)$$

for each $\Delta_{U,W}$-measurable function $\mathbf{g}$ and for each set $E\in\Sigma$.

Acknowledgement. This paper was supported by Grants VEGA 2/5065/05 and APVT-51-006904.

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