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Results in Mathematics

The Fubini Theorem for Bornological Product Measures

Ján Haluška and Ondrej Hutník

Abstract. The goal of this note is to present a construction of bornological product measures and prove a Fubini-type theorem for them.

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1. Introduction and preliminaries

It is well known that the tensor product of two vector measures, cf. [5,6], need not always exist, even in the case of measures ranged in the same Hilbert space and being the linear mapping (used in its definition) the corresponding inner product, cf. [7]. Several authors have given sufficient conditions for the existence of the tensor product measure, including the case of measures valued in locally convex spaces. In [19], a bilinear integral is defined in the context of locally convex spaces which is related to Bartle integral, cf. [1], and which allows to state the existence of the product measures valued in locally convex spaces under certain conditions. The bornological character of the bilinear integration theory in [19] shows the fitness of making a development of bilinear integration theory in the context of the complete bornological locally convex spaces (C. B. L. C. S., for short), cf. [14]. For a list of references to this problem, cf. [2].

The description of the theory of complete bornological locally convex topological vector spaces may be found in [17], and [18]. In what follows we recall some necessary notions from [11–13], and [14].

Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ be Hausdorff C. B. L. C. S. over the field $\mathbb{K}$ of real $\mathbb{R}$ or complex numbers $\mathbb{C}$, equipped with the bornologies $\mathfrak{B}_{\mathbf{X}}$, $\mathfrak{B}_{\mathbf{Y}}$, $\mathfrak{B}_{\mathbf{Z}}$. *A Banach disk* in $\mathbf{X}$ is a set U which is closed, absolutely convex and the linear span $\mathbf{X}_U$ of which is a

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Banach space. Let us denote by $\mathcal{U}$ the set of all Banach disks U in $\mathbf{X}$ such that $U \in \mathfrak{B}_{\mathbf{X}}$. So, the space $\mathbf{X}$ is an inductive limit of Banach spaces $\mathbf{X}_U$, $U \in \mathcal{U}$,

$$\mathbf{X} = \operatorname*{injlim}_{U \in \mathcal{U}} \mathbf{X}_U \,,$$

and the family $\mathcal{U}$ is directed by inclusion and forms the basis of bornology $\mathfrak{B}_{\mathbf{X}}$ (analogously for $\mathbf{Y}$ and $\mathcal{W}$, $\mathbf{Z}$ and $\mathcal{V}$, respectively). We say that the basis $\mathcal{U}$ of the bornology $\mathfrak{B}_{\mathbf{X}}$ has the *vacuum vector* [1] $U_0 \in \mathcal{U}$, if $U_0 \subset U$ for every $U \in \mathcal{U}$. Let the bases $\mathcal{U}$, $\mathcal{W}$, $\mathcal{V}$ be chosen to consist of all $\mathfrak{B}_{\mathbf{X}}$, $\mathfrak{B}_{\mathbf{Y}}$, $\mathfrak{B}_{\mathbf{Z}}$ bounded Banach disks in $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ with vacuum vectors $U_0 \in \mathcal{U}$, $U_0 \neq \{0\}$, $W_0 \in \mathcal{W}$, $W_0 \neq \{0\}$, $V_0 \in \mathcal{V}$, $V_0 \neq \{0\}$, respectively. We say that a sequence of elements $\mathbf{x}_n \in \mathbf{X}$, $n \in \mathbb{N}$ (the set of all natural numbers), $\mathcal{U}$-*converges* to $\mathbf{x} \in \mathbf{X}$, if there exists $U \in \mathcal{U}$ such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $(\mathbf{x}_n - \mathbf{x}) \in U$ for every $n \geq n_0$. We write $\mathbf{x} = \mathcal{U}\text{-}\lim_{n\to\infty} \mathbf{x}_n$.

On $\mathcal{U}$ *the lattice operations* are defined as follows. For $U_1, U_2 \in \mathcal{U}$ we have: $U_1 \wedge U_2 = U_1 \cap U_2$, and $U_1 \vee U_2 = \operatorname{acs}(U_1 \cup U_2)$, where acs denotes the topological closure of the absolutely convex span of the set; analogously for $\mathcal{W}$ and $\mathcal{V}$. For $(U_1, W_1, V_1), (U_2, W_2, V_2) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$, we write $(U_1, W_1, V_1) \ll (U_2, W_2, V_2)$ if and only if $U_1 \subset U_2$, $W_1 \supset W_2$, and $V_1 \supset V_2$.

We use Φ, Ψ, Γ to denote the *classes of all functions* $\mathcal{U} \to \mathcal{W}$, $\mathcal{W} \to \mathcal{V}$, $\mathcal{U} \to \mathcal{V}$ *with orders* $<_\Phi$, $<_\Psi$, $<_\Gamma$ defined as follows: for $\varphi_1, \varphi_1 \in \Phi$ we write $\varphi_1 <_\Phi \varphi_2$ whenever $\varphi_1(U) \subset \varphi_2(U)$ for every $U \in \mathcal{U}$ (analogously for $<_\Psi$, $<_\Gamma$ and $\mathcal{W} \to \mathcal{V}$, $\mathcal{U} \to \mathcal{V}$, respectively). Denote by $L(\mathbf{X}, \mathbf{Y})$ the space of all continuous linear operators $L : \mathbf{X} \to \mathbf{Y}$. We suppose $L(\mathbf{X}, \mathbf{Y}) \subset \Phi$ (analogously, $L(\mathbf{Y}, \mathbf{Z}) \subset \Psi$ and $L(\mathbf{X}, \mathbf{Z}) \subset \Gamma$).

Let T and S be two non-void sets. Let Δ and ∇ be two δ-rings of subsets of sets T and S respectively. If $\mathcal{A}$ is a system of subsets of the set T, then $\sigma(\mathcal{A})$ (resp. $\delta(\mathcal{A})$) denotes the σ-ring (resp. δ-ring) generated by the system $\mathcal{A}$. Set $\Sigma = \sigma(\Delta)$ and $\Xi = \sigma(\nabla)$. By $p_U : \mathbf{X} \to [0, \infty]$ we denote the Minkowski functional of the set $U \in \mathcal{U}$. Similarly, p_W and p_V indicate the Minkowski functionals of the sets $W \in \mathcal{W}$ and $V \in \mathcal{V}$, respectively.

For every $(U, W) \in \mathcal{U} \times \mathcal{W}$, denote by $\hat{\mathbf{m}}_{U,W} : \Sigma \to [0, \infty]$ *a* (U, W)-*semivariation* of a charge (= finitely additive measure) $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$, given by

$$\hat{\mathbf{m}}_{U,W}(E) = \sup p_W \left(\sum_{i=1}^{I} \mathbf{m}(E \cap E_i)\mathbf{x}_i \right) , \quad E \in \Sigma \,,$$

where the supremum is taken over all finite sets $\{\mathbf{x}_i \in U, i = 1, 2, \ldots, I\}$ and all disjoint sets $\{E_i \in \Delta;\ i = 1, 2, \ldots, I\}$. For every $(U, W) \in \mathcal{U} \times \mathcal{W}$, denote by $\|\mathbf{m}\|_{U,W} : \Sigma \to [0, \infty]$ *a scalar* (U, W)-*semivariation* of $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$, defined by

$$\|\mathbf{m}\|_{U,W}(E) = \sup p_W \left\| \sum_{i=1}^{I} \lambda_i \mathbf{m}(E \cap E_i) \right\|_{U,W} , \quad E \in \Sigma \,,$$

[1] In literature we can find also as terms as the *ground state* or *marked element* or *fiducial vector* or *mother wavelet* depending on the context.

where $\|L\|_{U,W} = \sup_{\mathbf{x}\in U} p_W(L(\mathbf{x}))$ and the supremum is taken over all finite sets of scalars $\{\lambda_i \in \mathbb{K};\ |\lambda_i| \le 1, i = 1, 2, \dots, I\}$ and all disjoint sets $\{E_i \in \Delta;\ i = 1, 2, \dots, I\}$. Analogously, we may define a (W, V)-semivariation $\hat{\mathbf{l}}_{W,V}$ and a scalar (W, V)-semivariation $\|\mathbf{l}\|_{W,V}$ of a charge $\mathbf{l} : \nabla \to L(\mathbf{Y}, \mathbf{Z})$.

Definition 1.1. Let $(U, W) \in \mathcal{U} \times \mathcal{W}$. Denote by

(a) $\Delta_{U,W}$ the greatest δ-subring of Δ of subsets of finite (U, W)-semivariation $\hat{\mathbf{m}}_{U,W}$ and $\Delta_{\mathcal{U},\mathcal{W}} = \{\Delta_{U,W}; (U, W) \in \mathcal{U}\times\mathcal{W}\}$ the lattice with the order given with inclusions of $U \in \mathcal{U}$ and $W \in \mathcal{W}$, respectively;

(b) $\Delta^u_{U,W}$ the greatest δ-subring of Δ on which the restriction $\mathbf{m}_{U,W} : \Delta^u_{U,W} \to L(\mathbf{X}_U, \mathbf{Y}_W)$ of the measure $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ is uniformly countable additive, with $\mathbf{m}_{U,W}(E) = \mathbf{m}(E)$, for $E \in \Delta^u_{U,W}$ and $\Delta^u_{\mathcal{U},\mathcal{W}} = \{\Delta^u_{U,W}; (U, W) \in \mathcal{U}\times\mathcal{W}\}$ the lattice with the order given with inclusions of $U \in \mathcal{U}$ and $W \in \mathcal{W}$, respectively;

(c) $\Delta^c_{U,W}$ the greatest δ-subring of Δ where $\hat{\mathbf{m}}_{U,W}$ is continuous and $\Delta^c_{\mathcal{U},\mathcal{W}} = \{\Delta^c_{U,W}; (U, W) \in \mathcal{U} \times \mathcal{W}\}$ the lattice with the order given with inclusions of $U \in \mathcal{U}$ and $W \in \mathcal{W}$, respectively.

Analogously, we may define $\nabla_{W,V}$, $\nabla^u_{W,V}$, $\nabla^c_{W,V}$, where $(W, V) \in \mathcal{W}\times\mathcal{V}$, and $\nabla_{\mathcal{W},\mathcal{V}}$, $\nabla^u_{\mathcal{W},\mathcal{V}}$, $\nabla^c_{\mathcal{W},\mathcal{V}}$. We denote by $\Delta_{U,W}\otimes\nabla_{W,V}$ the smallest δ-ring containing all rectangles $A\times B$, $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, where $(U, W) \in \mathcal{U}\times\mathcal{W}$, $(W, V) \in \mathcal{W}\times\mathcal{V}$. If $\mathcal{D}_1$, $\mathcal{D}_2$ are two δ-rings of subsets of T, S, respectively, then clearly $\sigma(\mathcal{D}_1\otimes\mathcal{D}_2) = \sigma(\mathcal{D}_1) \otimes \sigma(\mathcal{D}_2)$. For every $E \in \delta(\mathcal{D}_1 \otimes \mathcal{D}_2)$ there exist $A \in \mathcal{D}_1$, $B \in \mathcal{D}_2$, such that $E \subset A \times B$. For $E \subset T \times S$, $s \in S$, put $E^s = \{t \in T;\ (t, s) \in E\}$.

The sense of the theory of integration developed in [14] is that it is the integration theory which completely generalizes the Dobrakov integration, cf. [3], to a class of non-metrizable locally convex topological vector spaces. A suitable class of operator measures in C. B. L. C. S. which allows such a generalization is a class of all σ_Φ-additive measures.

For $(U, W) \in \mathcal{U}\times\mathcal{W}$ we say that a charge $\mathbf{m}$ is of *σ-finite (U, W)-semivariation* if there exist sets $E_n \in \Delta_{U,W}$, $n \in \mathbb{N}$, such that $T = \bigcup_{n=1}^{\infty} E_n$. For $\varphi \in \Phi$, we say that a charge $\mathbf{m}$ is of *σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation* if for every $U \in \mathcal{U}$, the charge $\mathbf{m}$ is of *σ-finite $(U, \varphi(U))$-semivariation*.

Definition 1.2. We say that a charge $\mathbf{m}$ is *of σ_Φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation* if there exists a function $\varphi \in \Phi$ such that $\mathbf{m}$ is of σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation.

Let $W \in \mathcal{W}$. We say that a charge $\mu : \Sigma \to \mathbf{Y}$ is a *(W, σ)-additive vector measure*, if μ is a $\mathbf{Y}_W$-valued (countable additive) vector measure.

Definition 1.3. We say that a charge $\mu : \Sigma \to \mathbf{Y}$ is a *$(\mathcal{W}, \sigma)$-additive vector measure*, if there exists $W \in \mathcal{W}$ such that μ is a (W, σ)-additive vector measure.

The following definition generalizes the notion of σ-additivity of an operator-valued measure in the strong operator topology in Banach spaces, cf. [3], to C. B. L. C. S.

Definition 1.4. Let $\varphi \in \Phi$. We say that a charge $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ is a σ_φ-*additive measure* if $\mathbf{m}$ is of σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation, and for every $A \in \Delta_{U,\varphi(U)}$ the charge $\mathbf{m}(A \cap \cdot)\mathbf{x} : \Sigma \to \mathbf{Y}$ is a $(\varphi(U), \sigma)$-additive measure for every $\mathbf{x} \in \mathbf{X}_U$, $U \in \mathcal{U}$. We say that a charge $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ is a σ_Φ-*additive measure* if there exists $\varphi \in \Phi$ such that $\mathbf{m}$ is a σ_φ-additive measure.

In what follows, $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ and $\mathbf{l} : \nabla \to L(\mathbf{Y}, \mathbf{Z})$ are supposed to be operator valued σ_Φ- and σ_Ψ-additive measures respectively. For the construction of (Dobrakov) integral in C.B.L.C.S. see [14].

2. A construction of bornological product measure

Definition 2.1. We say that a (bornological) *product measure* of a σ_Φ-additive measure $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ and σ_Ψ-additive measure $\mathbf{l} : \nabla \to L(\mathbf{Y}, \mathbf{Z})$ exists on $\Delta \otimes \nabla$ (we write $\mathbf{m} \otimes \mathbf{l} : \Delta \otimes \nabla \to L(\mathbf{X}, \mathbf{Z})$), if there exists one and only one σ_Γ-additive measure $\mathbf{m} \otimes \mathbf{l} : \Delta \otimes \nabla \to L(\mathbf{X}, \mathbf{Z})$ such that

$$(\mathbf{m} \otimes \mathbf{l})(A \times B)\mathbf{x} = \mathbf{l}(B)\mathbf{m}(A)\mathbf{x}$$

for every $\mathbf{x} \in \mathbf{X}_U$, $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, where there exists $\gamma \in \Gamma$, $\varphi \in \Phi$, $\psi \in \Psi$, such that $\gamma = \psi \circ \varphi$ and $V \subseteq \psi(W)$, $W \subseteq \varphi(U)$, $\gamma(U) \subset \psi(\varphi(U))$.

The Hahn–Banach theorem and the uniqueness of enlarging of the finite scalar measure from the ring to the generated σ-ring imply that if $\mathbf{n}_1, \mathbf{n}_2 : \Delta_{U,W} \otimes \nabla_{W,V} \to L(\mathbf{X}_U, \mathbf{Z}_V)$, are two σ_γ-additive measures $(\gamma \in \Gamma)$ such that $\mathbf{n}_1(A \times B) = \mathbf{n}_2(A \times B)$ for every $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, then $\mathbf{n}_1 = \mathbf{n}_2$ on $\Delta_{U,W} \otimes \nabla_{W,V}$.

Remark 2.2. Definition 2.1 differs from that of Dobrakov [4], Definition 1, in reduction to Banach spaces. Instead of the general $\Delta \otimes \nabla$ we deal only with $\Delta_{U,W} \otimes \nabla_{W,V}$, $V \subseteq \psi(W)$, $W \subseteq \varphi(U)$, $\gamma(U) \subset \psi(\varphi(U))$. In fact, only our case is needed for proving the Fubini theorem in [4].

Remark 2.3. The bornological product measure is a complicated object from the reason of the following implications: if $(U_1, W_1, V_1), (U_2, W_2, V_2) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$, then

$$(U_1, W_1) \ll (U_2, W_2) \Rightarrow \Delta_{U_2,W_2} \subset \Delta_{U_1,W_1},$$

and

$$(W_1, V_1) \ll (W_2, V_2) \Rightarrow \nabla_{W_2,V_2} \subset \nabla_{W_1,V_1}.$$

In general, for a fixed $W \in \mathcal{W}$,

$$(U_1, V_1) \ll (U_2, V_2) \Rightarrow \Delta_{U_2,W} \otimes \nabla_{W,V_2} \subset \Delta_{U_1,W} \otimes \nabla_{W,V_1}$$

and we may say nothing about the uniqueness, the existence, etc. of $W \in \mathcal{W}$. However, we guarantee the uniqueness of the measure in the case if it exists.

Theorem 2.4. *If there exists $W \in \mathcal{W}$ such that for every $(U,V) \in \mathcal{U} \times \mathcal{V}$, every $E \in \Delta_{U,W} \otimes \nabla_{W,V}$ and every $\mathbf{x} \in \mathbf{X}_U$, the function $s \mapsto \mathbf{m}(E^s)\mathbf{x}$, $s \in S$, is $\nabla_{W,V}$-integrable, then the product measure $\mathbf{m} \otimes \mathbf{l} : \Delta \otimes \nabla \to L(\mathbf{X}, \mathbf{Z})$ exists on $\Delta \otimes \nabla$. In this case, for every $E \in \Delta_{U,W} \otimes \nabla_{W,V}$ and every $\mathbf{x} \in \mathbf{X}_U$ holds*

$$(\mathbf{m}_{U,W} \otimes \mathbf{l}_{W,V})(E)\mathbf{x} = \int_S \mathbf{m}(E^s)\mathbf{x}\,\mathrm{d}\mathbf{l}\,. \tag{2.1}$$

Proof. Suppose that the product measure $\mathbf{m} \otimes \mathbf{l} : \Delta \otimes \nabla \to L(\mathbf{X}, \mathbf{Z})$ exists on $\Delta \otimes \nabla$. Let it hold for the set $W \in \mathcal{W}$ and let $\mathbf{x} \in \mathbf{X}_U$, $(U,V) \in \mathcal{U} \times \mathcal{V}$. Denote by $\mathcal{D}$ the class of all sets $G \in \Delta_{U,W} \otimes \nabla_{W,V}$ for which the function $s \mapsto \mathbf{m}(G^s)\mathbf{x}$, $s \in S$, is $\nabla_{W,V}$-integrable and for which (2.1) holds. Then clearly $\mathcal{D}$ is a subring of $\Delta_{U,W} \otimes \nabla_{W,V}$ which consists of all rectangles of the type $A \times B$, where $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$ and it may be easily shown that $\mathcal{D}$ is a δ-ring, cf. [10], Theorem E, § 33. Since $\mathbf{x} \in \mathbf{X}_U$ is an arbitrary vector, the second assertion of the theorem is proved.

Next we suppose that there exists $W \in \mathcal{W}$ such that for a given set $E \in \Delta_{U,W} \otimes \nabla_{W,V}$, every $(U,V) \in \mathcal{U} \times \mathcal{V}$ and $\mathbf{x} \in \mathbf{X}_U$, the function $s \mapsto \mathbf{m}(E^s)\mathbf{x}$, $s \in S$, is $\nabla_{W,V}$-integrable. For $\mathbf{x} \in \mathbf{X}_U$ and $E \in \Delta_{U,W} \otimes \nabla_{W,V}$, put $\mathbf{n_x}(E) = \int_S \mathbf{m}(E^s)\mathbf{x}\,\mathrm{d}\mathbf{l}$. Since $\mathbf{n_x}(A \times B) = \mathbf{l}_{W,V}(B)\mathbf{m}_{U,W}(A)\mathbf{x}$ for every $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, clearly $\mathbf{n_x} : \Delta_{U,W} \otimes \nabla_{W,V} \to \mathbf{Z}_V$ is a σ-additive measure. Let $\mathbf{x} \in \mathbf{X}_U$ and suppose that $E_n \in \Delta_{U,W} \otimes \nabla_{W,V}$, $n \in \mathbb{N}$, are pairwise disjoint sets with their union $E = \bigcup_{n=1}^{\infty} E_n \in \Delta_{U,W} \otimes \nabla_{W,V}$. We have to show that $\mathbf{n_x}(E) = \sum_{n=1}^{\infty} \mathbf{n_x}(E_n)$, where the series unconditionally V-bornologically converges. Therefore we take $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$ such that $E \subset A \times B$ and consider the σ-ring $\Delta_{U,W} \otimes \nabla_{W,V} \cap (A \times B)$. Since the measure $\mathbf{n_x} : \Delta_{U,W} \otimes \nabla_{W,V} \cap (A \times B) \to \mathbf{Z}_V$ is additive, by the Orlicz-Pettis theorem, see [8], IV.10.1, it is sufficient to prove that

$$\left\langle \mathbf{n_x}(E), z' \right\rangle = \sum_{n=1}^{\infty} \left\langle \mathbf{n_x}(E_n), z' \right\rangle$$

for each $z' \in V^0$ (here V^0 is the polar set of $V \in \mathcal{V}$), where the series unconditionally V-bornologically converges. If E_n^*, $n \in \mathbb{N}$, is some rearrangement of the sequence E_n, and $z' \in V^0$, then the inequality

$$\left| \left\langle \mathbf{n_x}(E) - \sum_{n=1}^{\infty} \mathbf{n_x}(E_n^*), z' \right\rangle \right| \le \int_S \|\mathbf{m}(\,\cdot\,)\mathbf{x}\|_{U,W} \left(\left(\bigcup_{i=n+1}^{\infty} E_i^* \right)^s \right) \mathrm{d}\mathbf{var}_{\mathrm{W}}(\mathrm{z}'\mathbf{l}, \,\cdot\,)\,,$$

σ-additivity of the vector measure $\mathbf{m}_{U,W}(\,\cdot\,)\mathbf{x} : \Delta_{U,W} \to \mathbf{Y}_W$, and the Lebesgue dominated convergence theorem yields

$$\sum_{n=1}^{\infty} \left\langle \mathbf{n_x}(E_i^*), z' \right\rangle \to \left\langle \mathbf{n_x}(E), z' \right\rangle,$$

which proves the theorem. □

Remark 2.5. For Fréchet spaces Theorem 2.4 holds also in the inverse direction, i.e. it gives a necessary and sufficient condition for the existence of the bornological product measure $\mathbf{m} \otimes \mathbf{l}$.

Let $\mathbf{g} : S \to \mathbf{Y}_W$ be a $\nabla_{W,V}$-measurable function and define the submeasure $\hat{\mathbf{l}}_{W,V}(\mathbf{g}, B)$ for $B \in \sigma(\nabla_{W,V})$ as follows:

$$\hat{\mathbf{l}}_{W,V}(\mathbf{g}, B) = \sup \left\{ p_V \left(\int_B \mathbf{h} \, \mathrm{d}\mathbf{l} \right) \right\},$$

where the supremum is taken over all $\mathbf{h} \in \sigma(\nabla_{W,V}, \mathbf{Y}_W)$, and $s \in S$ such that $p_W(\mathbf{h}(s)) \leq p_W(\mathbf{g}(s))$. Let us denote by $L^1_{W,V}(\mathbf{l})$ the space of all $\nabla_{W,V}$-integrable functions with the bounded and continuous seminorm $\hat{\mathbf{l}}_{W,V}(\,\cdot\,, B)$. Analogously we define $\hat{\mathbf{m}}_{U,W}(\,\cdot\,, A)$ and the space $L^1_{U,W}(\mathbf{m})$. This immediately implies the following result.

Theorem 2.6. *Let $(U, W, V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$ and let the product measure $\mathbf{m}_{U,W} \otimes \mathbf{l}_{W,V} : \Delta_{U,W} \otimes \nabla_{W,V} \to L(\mathbf{X}_U, \mathbf{Z}_V)$ exist. Let $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ and $\mathbf{f} : T \otimes S \to \mathbf{X}_U$ be a $\Delta_{U,W} \otimes \nabla_{W,V}$-measurable function. Then*

$$\|\mathbf{m} \otimes \mathbf{l}\|_{U,V}(E) \leq \hat{\mathbf{l}}_{W,V}\big(\|\mathbf{m}\|_{U,W}(E^s), S\big),$$

and

$$(\widehat{\mathbf{m} \otimes \mathbf{l}})_{U,V}(\mathbf{f}, E) \leq \hat{\mathbf{l}}_{W,V}\Big(\hat{\mathbf{m}}_{U,W}\big(\mathbf{f}(\,\cdot\,, s), E^s\big), S\Big).$$

In the special case of $E = A \times B$, $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, we have

$$\|\mathbf{m} \otimes \mathbf{l}\|_{U,V}(A \times B) \leq \|\mathbf{m}\|_{U,W}(A) \cdot \hat{\mathbf{l}}_{W,V}(B) < \infty,$$

and

$$(\widehat{\mathbf{m} \otimes \mathbf{l}})_{U,V}(A \times B) \leq \hat{\mathbf{m}}_{U,W}(A) \cdot \hat{\mathbf{l}}_{W,V}(B).$$

Thus (U, V)-semivariation $(\widehat{\mathbf{m} \otimes \mathbf{l}})_{U,V}$ is a finite set function on $\Delta_{U,W} \otimes \nabla_{W,V}$.

3. A Fubini-type theorem

Let $(U, W, V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$. Denote by $\widetilde{\sigma}(\Delta_{U,W} \otimes \nabla_{W,V}, \mathbf{X})$ the closure of the set $\sigma(\Delta_{U,W} \otimes \nabla_{W,V}, \mathbf{X})$ of all $\Delta_{U,W} \otimes \nabla_{W,V}$-simple functions on $T \times S$ with values in $\mathbf{X}$ with respect to the seminorm $\|\cdot\|_{T \times S, U}$ in the Banach space of all U-bounded functions on $T \times S$. For elements from $\widetilde{\sigma}(\Delta_{U,W} \otimes \nabla_{W,V}, \mathbf{X})$ the following Fubini-type theorem holds.

Theorem 3.1. *Let $(U, W, V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$. Let the product measure $\mathbf{m}_{U,W} \otimes \mathbf{l}_{W,V}$ exist on $\Delta_{U,W} \otimes \nabla_{W,V}$. Let $\mathbf{f} \in \widetilde{\sigma}(\Delta_{U,W} \otimes \nabla_{W,V}, \mathbf{X})$ and $F \in \Delta_{U,W} \otimes \nabla_{W,V}$ (if $\hat{\mathbf{m}}_{U,W}(T) \cdot \hat{\mathbf{l}}_{W,V}(S) < \infty$, then let $F \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$). Then*

(a) *$\mathbf{f}\chi_F$ is a $\Delta_{U,W} \otimes \nabla_{W,V}$-integrable function;*
(b) *for every $s \in S$ the function $\mathbf{f}(\,\cdot\,, s)\chi_F(\,\cdot\,, s)$ is $\Delta_{U,W}$-integrable;*
(c) *for every $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ the function $s \mapsto \int_{E^s} \mathbf{f}(\,\cdot\,, s)\chi_F(\,\cdot\,, s) \, \mathrm{d}\mathbf{m}$, $s \in S$, is $\nabla_{W,V}$-integrable and for every $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ holds*

$$\int_{E^s} \mathbf{f}\chi_F \, \mathrm{d}(\mathbf{m} \otimes \mathbf{l}) = \int_S \int_{E^s} \mathbf{f}(\,\cdot\,, s)\chi_F(\,\cdot\,, s) \, \mathrm{d}\mathbf{m} \, \mathrm{d}\mathbf{l}.$$

Proof. Let $\mathbf{f}_n \in \widetilde{\sigma}(\Delta_{U,W} \otimes \nabla_{W,V}, \mathbf{X})$, $n \in \mathbb{N}$, be a sequence of functions such that $\|\mathbf{f}_n - \mathbf{f}\|_{T\times S,U} \to 0$ as $n \to \infty$. Take $A_0 \in \Delta_{U,W}$ and $B_0 \in \nabla_{W,V}$, such that $F \subset A_0 \times B_0$ (if $\hat{\mathbf{m}}_{U,W}(T) \cdot \hat{\mathbf{l}}_{W,V}(S) = \infty$, take $A_0 \in \sigma(\Delta_{U,W})$ and $B_0 \in \sigma(\nabla_{W,V})$). Then $\mathbf{f}_n(t,s) \to \mathbf{f}(t,s)$ for every $(t,s) \in T \times S$. If $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$, then $\mathbf{f}_n \chi_E \in \widetilde{\sigma}(\Delta_{U,W} \otimes \nabla_{W,V})$ for every $n \in \mathbb{N}$.

(a) From the definition of the (U,V)-semivariation $(\widehat{\mathbf{m} \otimes \mathbf{l}})_{U,V}$ and Theorem 2.6 we have

$$\begin{aligned} p_V\left(\int_E \mathbf{f}_n\chi_F \,\mathrm{d}(\mathbf{m}\otimes\mathbf{l}) - \int_E \mathbf{f}_k\chi_F\,\mathrm{d}(\mathbf{m}\otimes\mathbf{l})\right) &= p_V\left(\int_{B\cap F} (\mathbf{f}_n - \mathbf{f}_k)\,\mathrm{d}(\mathbf{m}\otimes\mathbf{l})\right) \\ &\le \|\mathbf{f}_n - \mathbf{f}_k\|_{T\times S,U} \cdot (\widehat{\mathbf{m}\otimes\mathbf{l}})_{U,V}(F) \\ &\le \|\mathbf{f}_n - \mathbf{f}_k\|_{T\times S,U} \\ &\quad \cdot \hat{\mathbf{m}}_{U,W}(A_0)\cdot \hat{\mathbf{l}}_{W,V}(B_0) \end{aligned}$$

for every $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ and every $n,k \in \mathbb{N}$. Since $\hat{\mathbf{m}}_{U,W}(A_0)\cdot\hat{\mathbf{l}}_{W,V}(B_0) < \infty$, then by Theorem 4.3 in [14] the function $\mathbf{f}\chi_F$ is $\Delta_{U,W} \otimes \nabla_{W,V}$-integrable and

$$\int_E \mathbf{f}_n\chi_F\,\mathrm{d}(\mathbf{m}\otimes\mathbf{l}) \to \int_E \mathbf{f}\chi_F\,\mathrm{d}(\mathbf{m}\otimes\mathbf{l})$$

for every $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$.

(b) Let $s \in S$. Then

$$\begin{aligned} p_W\left(\int_A \mathbf{f}_n(\,\cdot\,,s)\chi_F(\,\cdot\,,s)\,\mathrm{d}\mathbf{m} - \int_A \mathbf{f}_k(\,\cdot\,,s)\chi_F(\,\cdot\,,s)\,\mathrm{d}\mathbf{m}\right)& \\ \le \|\mathbf{f}_n - \mathbf{f}_k\|_{T\times S,U}\cdot \hat{\mathbf{m}}_{U,W}(A_0)& \end{aligned}$$

for every $A \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ and $n,k \in \mathbb{N}$. Since $\hat{\mathbf{m}}_{U,W}(A_0) < \infty$, then by Theorem 4.3 in [14] the function $\mathbf{f}(\,\cdot\,,s)\chi_F(\,\cdot\,,s)$ is $\Delta_{U,W}$-integrable and we have

$$\int_A \mathbf{f}_n(\,\cdot\,,s)\chi_F(\,\cdot\,,s)\,\mathrm{d}\mathbf{m} \to \int_A \mathbf{f}(\,\cdot\,,s)\chi_F(\,\cdot\,,s)\,\mathrm{d}\mathbf{m}$$

for every $A \in \sigma(\Delta_{U,W})$. In particular,

$$\int_{E^s} \mathbf{f}_n(\,\cdot\,,s)\chi_F(\,\cdot\,,s)\,\mathrm{d}\mathbf{m} \to \int_{E^s} \mathbf{f}(\,\cdot\,,s)\chi_F(\,\cdot\,,s)\,\mathrm{d}\mathbf{m}$$

for every $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$.

(c) Let $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$. Then using Lemma 3.3 in [16], we get

$$\begin{aligned} p_V\left(\int_B\int_{E^s} \mathbf{f}_n(\,\cdot\,,s)\chi_F(\,\cdot\,,s)\,\mathrm{d}\mathbf{m}\,\mathrm{d}\mathbf{l} - \int_B\int_{E^s} \mathbf{f}_k(\,\cdot\,,s)\chi_F(\,\cdot\,,s)\,\mathrm{d}\mathbf{m}\,\mathrm{d}\mathbf{l}\right)& \\ \le \sup_{s\in B_0} p_W\left(\int_{E^s}\big(\mathbf{f}_n(\,\cdot\,,s) - \mathbf{f}_k(\,\cdot\,,s)\big)\,\mathrm{d}\mathbf{m}\right)\cdot \hat{\mathbf{l}}_{W,V}(B_0)& \\ \le \|\mathbf{f}_n - \mathbf{f}_k\|_{T\times S,U}\cdot\hat{\mathbf{m}}_{U,W}(A_0)\cdot\hat{\mathbf{l}}_{W,V}(B_0)& \end{aligned}$$

for every $B_0 \in \sigma(\nabla_{W,V})$ and $n, k \in \mathbb{N}$. Since $\hat{\mathbf{m}}_{U,W}(A_0) \cdot \hat{\mathbf{l}}_{W,V}(B_0) < \infty$ and $\|\mathbf{f}_n - \mathbf{f}_k\|_{T\times S,U} \to 0$ as $n, k \to \infty$, according to Theorem on interchange the limit and integral in [15] the relations (a) and (b) imply that the function $s \mapsto \int_{E^s} \mathbf{f}(\,\cdot\,, s)\chi_F(\,\cdot\,, s)\,\mathrm{d}\mathbf{m}$, $s \in S$, is $\nabla_{W,V}$-integrable and, therefore,

$$\int_S \int_{E^s} \mathbf{f}_n(\,\cdot\,, s)\chi_F(\,\cdot\,, s)\,\mathrm{d}\mathbf{m}\,\mathrm{d}\mathbf{l} \to \int_S \int_{E^s} \mathbf{f}(\,\cdot\,, s)\chi_F(\,\cdot\,, s)\,\mathrm{d}\mathbf{m}\,\mathrm{d}\mathbf{l}$$

as $n \to \infty$. It is enough to note that by Theorem 2.4 there holds

$$\int_E \mathbf{f}_n\chi_F\,\mathrm{d}(\mathbf{m}\otimes\mathbf{l}) = \int_S \int_{E^s} \mathbf{f}_n(\,\cdot\,, s)\chi_F(\,\cdot\,, s)\,\mathrm{d}\mathbf{m}\,\mathrm{d}\mathbf{l}$$

for every $E \in \sigma(\Delta_{U,W} \otimes \nabla_{W,V})$ and $n \in \mathbb{N}$. □

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Ján Haluška
Mathematical Institute
Slovak Academy of Science
Grešákova 6
SK-040 01 Košice
Slovakia
e-mail: `jhaluska@saske.sk`

Ondrej Hutník
Institute of Mathematics
Faculty of Science
P. J. Šafárik University
Jesenná 5
SK-041 54 Košice
Slovakia
e-mail: `ondrej.hutnik@upjs.sk`

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