ON A DIOPHANTINE EQUATION DERIVED FROM THE THEORY OF MEANTONE SYSTEMS

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1. Introduction

The overview of the general mathematical theory of tone systems we can find in [1]. For the excellent historical retrospective of meantone systems from the viewpoint of the present musicology, c.f. [3]. In this preliminary section, we recall some facts and formalize the problem which we solve in the rest sections.

Denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ and by $\mathbb{Q}, \mathbb{P}$ the sets of all natural, rational, and prime numbers, respectively.

Rational and irrational numbers have their counterparts in music, harmonic and inharmonic music intervals, respectively. It is known that Pythagorean System can be constructed using only two rational numbers $X = 256/243$ (the minor Pythagorean semitone, limma) and $Y = (9/8) : X = 2187/2048$ (the major Pythagorean semitone, apotome). It is also well known that this couple $(X, Y)$ is unique. Another tone system, Equal Temperament, is based on one number, the equal tempered semitone $X = 12\sqrt[12]{2}$, which is irrational number (i.e., it is inharmonic music interval) and it is unique. The only rational numbers (harmonic intervals) in this tone system are numbers $12\sqrt[12]{2}i, i \in \mathbb{N}$, i.e., octaves in music. The third famous meantone system is the Praetorius tone system, also known as the $1/4$-comma meantone. It contains both harmonic (octaves, major thirds, augmented fifths) and inharmonic music intervals.

Note that words ”number” and ”(relative) musical interval” (or simply – interval) are used as synonyms in the tone systems theory and the mathematical theory of music. In this paper, we will do this as well. This will not lead to any misunderstanding because the notion of interval is only used in this sense. More precisely, if we denote by $\mathcal{L} = ((0, \infty), \cdot, 1, \leq)$ the usual multiplicative group on real line with the usual order, then $b/a$ is called the $\mathcal{L}$-length of $(a, b), 0 < a \leq b < \infty$ is called the interval.

Having a motivation the meantone algorithm (“the spiral of generalized fifths”), c.f. [1], we can formulate the problem we solve in this paper as follows.

**Problem.** Let $p, q$ be given prime numbers, let $a, b$ be given natural relatively prime numbers. To find all couples $(X, Y)$ in rational, such that there exist natural numbers $\alpha, \beta, \gamma, \delta$ such that

$$X^\alpha Y^\beta = p, \quad X^\gamma Y^\delta = q,$$

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and
\[ \alpha + \beta = a, \quad \gamma + \delta = b. \]

Such numbers \( p, q, a, b, X, Y \) exist. For instance, for Pythagorean system we have: \( p = 2, q = 3, a = 12, b = 19, X = 256/243, Y = 2187/2048. \)

2. Results

For the sake of simplicity and without loss of generality, we will consider \( p < q \) in this paper on.

**Theorem 1** Let \( a \in \mathbb{N} \) and \( b \in \mathbb{N} \) be two fixed relatively prime positive integer numbers. Then there exists unique positive integer solution of the following Diophantine equation
\[
\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = 1, \quad \begin{cases} \alpha + \beta = a, \\ \gamma + \delta = b. \end{cases}
\]
on the rectangle \( I = \langle 0, b \rangle \times \langle 0, a \rangle \).

**Proof.** We split the proof into two steps:

**Step 1 (Existence).** The Diophantine equation (1) can be rewritten as \( a\delta - b\beta = 1 \), where \( \delta \in \{0, 1, \ldots, b\} \), and \( \beta \in \{0, 1, \ldots, a\} \). Since \( a, b \) are relatively prime numbers, there exists a solution of Diophantine equation, c.f. [2].

**Step 2 (Uniqueness).** Let \((\delta_0, \beta_0)\) and \((\delta_1, \beta_1)\) be two positive integer solutions of Diophantine equation \( a\delta - b\beta = 1 \) on the rectangle \( I = \langle 0, b \rangle \times \langle 0, a \rangle \). Then
\[
a\delta_0 - b\beta_0 = 1, \quad \delta_0 \in \{0, 1, \ldots, b\}, \beta_0 \in \{0, 1, \ldots, a\}
\]
and
\[
a\delta_1 - b\beta_1 = 1, \quad \delta_1 \in \{0, 1, \ldots, b\}, \beta_1 \in \{0, 1, \ldots, a\}.
\]
So
\[
a(\delta_0 - \delta_1) = b(\beta_0 - \beta_1).
\]
That implies that \( a \) divides \( b(\beta_0 - \beta_1) \) and \( (a, b) = 1 \) (the greatest common divisor), so \( a \) divides \( (\beta_0 - \beta_1) \). Since \( \beta_0 - \beta_1 < a \), this relation holds if and only if \( \beta_0 = \beta_1 \).

Analogously for \( \delta_0 \) and \( \delta_1 \). \( \square \)

We use Theorem 1 for proving of the following main theorem.

**Theorem 2** Let \( A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) be a matrix of nonnegative integers with \( \det A \neq 0 \).
Then for all relatively prime \( a, b \in \mathbb{N} \) such that \( \alpha + \beta = a, \gamma + \delta = b \), there exist a unique and positive \( (X,Y) \in \mathbb{L}^2 \), such that
\[
X^\alpha Y^\beta = p \quad \text{(2)}
\]
\[
X^\gamma Y^\delta = q \quad \text{(3)}
\]
for \( p \in \mathbb{P} \) and \( q \in \mathbb{P} \) and the following statements are equivalent:

(a) \((X,Y) \in \mathbb{Q}^2\), (b) \( \det A = 1 \).

The values are as follows:
\[
X = p^\delta q^{-\beta}, \quad Y = p^{-\gamma} q^\alpha. \quad \text{(4)}
\]
Proof. If \( X, Y \in \mathbb{Q} \), then there exist \( r \in \mathbb{P} \) and \( E_{2,X}, \ldots, E_{r,Y} \in \mathbb{Z} \) such that
\[
X = 2^{E_{2,X}} \ldots p^{E_{p,X}} \ldots q^{E_{q,X}} \ldots r^{E_{r,X}}, \\
Y = 2^{E_{2,Y}} \ldots p^{E_{p,Y}} \ldots q^{E_{q,Y}} \ldots r^{E_{r,Y}}.
\]
Combining (2) and (3),
\[
\begin{pmatrix}
E_{2,X} & E_{2,Y} \\
\vdots & \vdots \\
E_{p,X} & E_{p,Y} \\
\vdots & \vdots \\
E_{q,X} & E_{q,Y} \\
\vdots & \vdots \\
E_{r,X} & E_{r,Y}
\end{pmatrix}
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 \\
\vdots & \vdots \\
1 & 0 \\
\vdots & \vdots \\
0 & 1 \\
\vdots & \vdots \\
0 & 0
\end{pmatrix}.
\]
Since \( \det A = 1 \),
\[
\begin{pmatrix}
E_{2,X} & E_{2,Y} \\
\vdots & \vdots \\
E_{p,X} & E_{p,Y} \\
\vdots & \vdots \\
E_{q,X} & E_{q,Y} \\
\vdots & \vdots \\
E_{r,X} & E_{r,Y}
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 \\
\vdots & \vdots \\
1 & 0 \\
\vdots & \vdots \\
0 & 1 \\
\vdots & \vdots \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\delta & -\gamma \\
-\beta & \alpha
\end{pmatrix}
= 
\begin{pmatrix}
\delta & -\gamma \\
-\beta & \alpha
\end{pmatrix}.
\]
Equivalently we can obtain values of \( X, Y \) as follows: system (2) and (3) could be rewritten into
\[
\alpha + \beta \log_X Y = \log_X p \\
\gamma + \delta \log_X Y = \log_X q.
\]
Multiplying both sides of the first equation by \( \delta \), multiplying both sides of the second equation by \( -\beta \) and adding the resulting equations one side by side we get
\[
\alpha \delta - \beta \gamma = \log_X \frac{p^\delta}{q^\beta}.
\]
Since \( \det A \neq 0 \), we have \( X = (p^\delta q^{-\beta})^\frac{1}{\alpha - \beta} \) and \( Y = (p^{-\gamma} q^\alpha)^\frac{1}{\alpha - \beta} \). Then \( X, Y \) are rational if and only if \( \alpha \delta - \beta \gamma = 1 \), i.e. \( \det A = 1 \).

Remark 1 If \( p = q \) then \( X = p^{\delta-\beta}, Y = p^{\alpha-\gamma} \).

The proof of the following lemma is easy and therefore can be omitted.

Lemma 1 Let \( p, q \neq 1 \). Then \( X \neq Y \) in the equation (4).

3. Applications

We bring examples and some corollaries of the Problem solution.
### Example 1
Consider $p = 2$ and $q = 3$. Let $\alpha + \beta = 12$, $\gamma + \delta = b$, such that $\det A = 1$. The rational semitones for some appropriate values of $b \in \mathbb{N}$ (12 < $b$ < 32) are collected in Table 1. As we can see, the bigger natural $b$ is, the bigger difference between rational interval $X, Y$ we can observe. When we establish some other conditions for intervals $X, Y$, e.g. upper and lower bound for them, we obtain the following well-known results.

### Corollary 1
The unique (up to symmetry) pair of rational intervals $X, Y$ satisfying $X^\alpha Y^\beta = 2$ and $X^\gamma Y^\delta = 3$ for $\alpha + \beta = 12$, $\gamma + \delta = b$ and $\alpha \delta - \gamma \beta = 1$, such that $1 < X < Y \leq \frac{9}{8}$ are $X = 2^3 - 5$ and $Y = 2^{-1}3^7$ for $b = 19$.

### Corollary 2
For Praetorius tone system, $p = 2$, $q = 5$ and $a = 12$ there is no $b$ such that there exist rational solutions $X, Y$ of our problem such that $1 < X < Y \leq \frac{16}{15}$ (musically, there are no rational semitones for this tone system).

### References

### Zusammenfassung
Wir stellen und lösen das folgende Problem, das mit der mathematischen Theorie des mittentönigen Temperatursystems inspiriert war.

Sei $p, q$ zwei gegebene Primzahlen und $a, b$ zwei gegebene natürliche teilerfremde Zahlen. Wir suchen alle rationalen Zahlenpaare $(X, Y)$, für die solche natürliche Zahlen $\alpha, \beta, \gamma, \delta$ existieren, dass $X^\alpha Y^\beta = p$, $X^\gamma Y^\delta = q$, $\alpha + \beta = a$, $\gamma + \delta = b$ gilt. Dieses Gleichungssystem ist lösbar und wir werden zeigen, dass die Lösung eindeutig ist. Wir führen einige Verwendungen in der Mittentönigkeitstheorie ein.