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Weber – Fechner's law, uncertainty, and Pythagorean system

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Abstract

In the paper we explain the notion of geometrical net from the view of coding of music information. A direct, elementary, and very short alternative proof of the assertion that there are no transcendental semitones generating Pythagorean system, is given. This is a conclusion of the negation of the psychological Weber – Fechner's law. Further, we discuss about a kind of uncertainty bounded with the melodic and harmonic structures in music.
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1 Geometrical nets: coding of information

Denote by $\mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{Q}$ the sets of all real, integer, natural, and rational numbers, respectively.

Definition 1 , c.f. [8]. Let T be a set of objects (called *tones*). Let $\omega : T \to (0, \infty)$ be an injective function (called the *pitch function*). Then the *tone system* in a broader sense is the couple (T, ω). The set $\omega(T) = \{\omega(t); t \in T\}$ is called the tone system in a narrower sense.

If we take into the account that tone is a dual object – both spiritual and physical – we need to use a more ingenious mathematical tool to reflect at least the information aspects. The concept of geometrical nets provides this apparatus. Recall that a *net* with values in $\mathcal{L}$ is a function from I to $\mathcal{L}$, where I is a directed partially ordered set, c.f. [5].

So, let $\mathcal{L} = ((0, \infty), \cdot, 1, \leq)$ be the usual multiplicative group with the usual order on $\mathbb{R}$. If $a \leq b, a, b \in (0, \infty)$, then b/a is an $\mathcal{L}$-length of the interval (a, b). Or, borrowing the usual musical terminology, we simply say that b/a is an interval.

The set $P \subset \mathbb{R}$ in the following Definition 2 represents *information coding*, e.g. notes in a score (very often P is assumed to be $\mathbb{Z}$ in the Western music, but also P finite). To each $t \in T$ there exists a unique $p \in P$ (note) but, due to psychological or spiritual aspects, possible more than one physical variants or theoretical decompositions of this tone. This ambiguity will be expressed in the structure of the index set I of the geometrical net in Definition 3.

Definition 2 Let $P \subset (-\infty, +\infty)$ be a set. Let $I = \{\phi = (\alpha_1, \ldots, \alpha_n);\ \alpha_i : P \to (-\infty,\ +\infty),\ \ i = 1,\ 2,\ \ldots, n\}$. We say that the index set I satisfies the P-chain condition if

1. $\alpha_1, \ldots, \alpha_n$ are isotone, i.e. for every $i = 1, 2, \ldots, n$,

$$p \leq q \Rightarrow \alpha_i(p) \leq \alpha_i(q),\ \ p, q \in P,$$

2. $\alpha_1(p) + \ldots + \alpha_n(p) = p,\ p \in P$.

Here are some examples of the set I.

Example 1 The example of a "very poor" index set I.
$P = \{\frac{-1-\sqrt{5}}{2}, 0, \frac{+1-\sqrt{5}}{2}\}, n = 2, I = \{(p^3, p^2); p \in P\}$. There I consists of a chain of 3 elements.

Example 2 The example of a "too rich" index set I.
$P = (0, \infty), n = 2, I = \{(\alpha_1(p), \alpha_2(p)); p \in P\}$, where $\alpha_1(p) = pw_1, \alpha_2(p) = pw_2, w_1 + w_2 = 1$ (weights). Since the real nondecreasing functions α_1, α_2 can be chosen arbitrarily, there is no practical use of such index sets .

Example 3 A discrete index set I.
Let $P = \mathbb{Z}, n = 2$, define $I = \{(\alpha_1(z), \alpha_2(z)); z \in \mathbb{Z}\}$ as follows:

$(\alpha_1(z), \alpha_2(z)) = (z/2, z/2)$ if $z \in \mathbb{Z}$ is even (one couple of integers) and

$(\alpha_1(z), \alpha_2(z)) = (z-1/2, z+1/2), (z+1/2, z-1/2)$ if $z \in \mathbb{Z}$ is odd (two couples of integers).

Definition 3
Let $P \subset (-\infty, +\infty)$. Let $X_1 > 0, \ldots, X_n > 0$. Let $I = \{\phi\}$ be an index set satisfying the P-chain condition, c.f. Definition 2. The *geometrical net* is

$$\phi(p) = (\alpha_1(p), \cdots, \alpha_n(p)) \mapsto X_1^{\alpha_1(p)} \cdots X_n^{\alpha_n(p)}, p \in P, \qquad (1)$$

(the form of a many valued map) or equivalently,

$$\{(X_1^{\alpha_1(p)} \cdots X_n^{\alpha_n(p)}); \phi(p) \in I, p \in P\}$$

(the form of the net defined on the lattice).

Example 4 It is easy to see that the notion of geometrical net generalized the elementary notion of *geometrical progression* (equivalently, equal temperament in music). In this case $X_1 = \ldots = X_n$ and $I = \mathbb{Z}$. According to this origin, we use the term *quotient* for objects $X_1, \ldots, X_n$ in Definition 1.

Example 5 For equal temperaments, $n = 1$. For meantones, $n = 2$. For diatonic tone systems, $n = 3$.

Remark 1 The partial order in $(I, \leq)$ is induced with the linear order in $(0, \infty)$, i.e.

$$p_1 \leq p_2 \Rightarrow \phi(p_1) \leq \phi(p_2), p_1, p_1 \in P.$$

Since $(0, \infty)$ is linearly ordered upward and backward, too, the direction of the lattice I is also upward and backward.

The usefulness of the notion of geometrical net [1] was examined in various connections in the mathematical theory of tone systems, c.f. e.g. [1], [2], [3], [4]. The following, rather trivial, theorem says that every tone system in the sense of Definition 1 can be represented in the form (2), i.e. as a geometrical net. The key is the equation (3) which is, in fact, an expression of the Weber-Fechner's law: on different sides of this equation there are separated the psychological and natural scientific objects (pitch and decomposition of the sound) and the relation between them is exponential.

[1] Musicians use also the term "lattice", which is confusing in any mathematical context and not correct

Theorem 1 *Let T be a set. Let $\omega : T \to (0, \infty)$ be an injective function. The couple (T, ω) is a a tone system if and only if there exist $X_1 > 0, \dots, X_n > 0$, and a $P \subset (-\infty, +\infty)$ such that for every $t \in T$, there exists a unique $p = p(t) \in P$ and an index set I of n-tuples $\phi = (\alpha_1, \dots, \alpha_n) \in I$ of real functions satisfying the P-chain condition and*

$$\omega(t) = X_1^{\alpha_1(p(t))} \cdots X_n^{\alpha_n(p(t))}. \tag{2}$$

Proof. If the function $\omega : T \to (0, \infty)$ is expressed in the form (2), then (T, ω) is the tone system trivially.

If (T, ω) is a tone system, order the set $\{\omega(t); t \in T\} \subset \mathbb{R}$ in the natural way. Define $P = \{p = p(t) = \ln(\omega(t)); t \in T\}, X = e, \alpha(p) = p$. Trivially, the P-condition is satisfied and

$$\omega(t) = e^{\alpha(p(t))}. \tag{3}$$

□

Depending on the additional conditions on the set P, number n, functions $\alpha_1, \dots, \alpha_n$, or values $X_1 > 0, \dots, X_n > 0$, the consideration of tone system (T, ω) with elements $\omega(T)$ in the form (2) get substantial and non-trivial. In any case, geometrical nets are a powerful and universal tool for the analytical descriptions and consideration of tone systems.

2 Pythagorean system and geometrical nets

It is commonly known that Pythagorean System (the 5th cent. B. C.) is defined as the set of all numbers of the form $2^\alpha 3^\beta$, where $\alpha, \beta \in \mathbb{Z}$. Using geometrical nets, we implant information aspects into the consideration of this system.

Without loss of generality of conclusions and to prove Theorem 2, let us switch the explanation from the general level to important special case of Pythagorean System. It can be described as a geometrical net generated with two quotients $X > 0$ and $Y > 0$ and defined on discrete lattices. More precisely,

$$\Pi_0 = \{X^{\alpha_z} Y^{\beta_z}; \alpha_z, \beta_z, z \in \mathbb{Z}\}. \tag{4}$$

such that for every $z \in \mathbb{Z}$,

$$\alpha_z + \beta_z = z, \tag{5}$$

$$\alpha_z \leq \alpha_{z+1}, \beta_z \leq \beta_{z+1}. \tag{6}$$

Remark 2 Likely the sculptor chooses marble or glass from numerous materials to create his sculpture, so the music composer chooses his tone system for his composition. From the musical viewpoint, the whole set Π_0 is not suitable to serve as tone system [2] (although it is by the definition), because in the case of Pythagorean System, this set is a dense subset of $(0, \infty)$. From the artificial reasons there is needed a restriction of Π_0 (with the large entropy) to a sub-geometrical-net $\Pi \subset \Pi_0$ (with the small or no entropy). In Figure 1 we see two: $\Pi = \Pi_{17}, \Pi_{31}$, the 17-valued and 31-valued Pythagorean system, respectively. It is reasonable also to ask the condition $X^0Y^0 = 1 \in \Pi$ (a reference frequency in music). We will not introduce this complication, which is substantial from the view of music but rather formal for the aim of this paper.

We will obtain Pythagorean system choosing some $X > 0$ and $Y > 0$ in (4). What is important, this can be done ambiguously but not arbitrarily. Many couples (X, Y) are suitable. In Figure 1, there are the lattices I for nets of the 17-valued Pythagorean System Π_{17} (circles) and 31-valued Pythagorean System Π_{31} (both circles and squares) for $X = 253/243$ (limma) and $Y = 2187/2048$ (apotome), e.g. $f = X^3Y^2 = 4/3$, etc. Pythagorean systems $\Pi_{17} \subset \Pi_{31} \subset \Pi_0$.

Clearly, different couples (X, Y) yield different geometrical nets, i.e. nets with different directions I of lattices.

To a given set of numbers Π_0, there may exist more geometrical nets because of different domain lattices. For instance, the following $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ are four geometrical nets which are identical as the sets, they are various representations of the Pythagorean System:

$$\begin{aligned} \Pi_1 &= \{(256/243)^{\alpha_z}(2187/2048)^{\beta_z}; \alpha_z, \beta_z, z \in \mathbb{Z}\}, \\ \Pi_2 &= \{2^{\gamma_z}3^{\delta_z}; \gamma_z, \delta_z, z \in \mathbb{Z}\}, \\ \Pi_3 &= \{2^{\varepsilon_z}(3/2)^{\eta_z}; \varepsilon_z, \eta_z, z \in \mathbb{Z}\}, \\ \Pi_4 &= \{(9/8)^{\theta_z}(256/243)^{\varkappa_z}; \theta_z, \varkappa_z, z \in \mathbb{Z}\}. \end{aligned} \tag{7}$$

[2]This is the property of things: the all scarcely suffices for the natural; the artificial needs a bounded space. J. W. Goethe: Faust II, 6882-6884

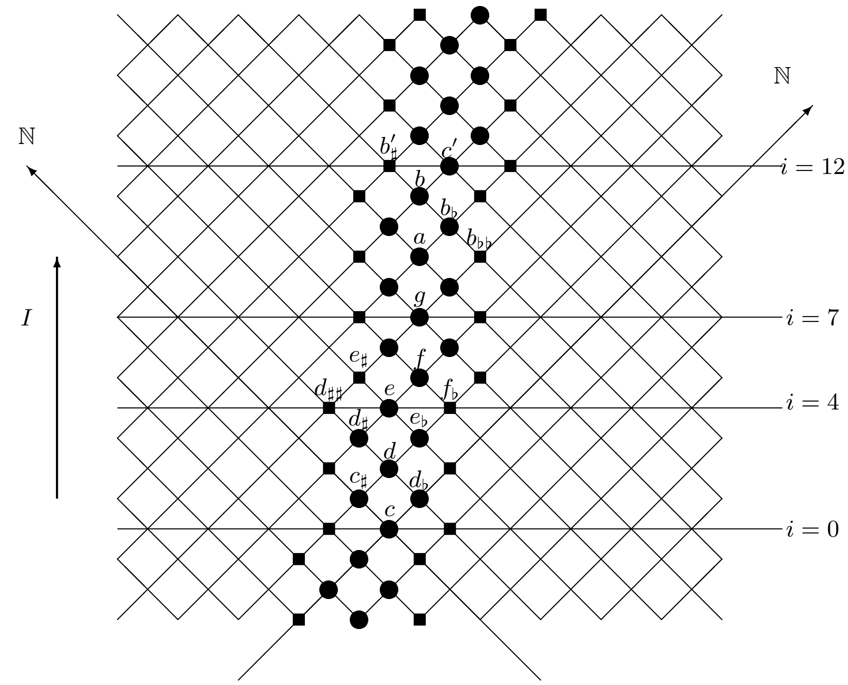

Figure 1: Directed sets I for Pythagorean 17- and 31- valued systems Π_{17} and Π_{31}, respectively; $\Pi_{17} \subset \Pi_{31} \subset \Pi_0$

The directions of the nets (i.e. lattices of integer numbers in our case) are given in all four cases with the equations (5)(6) and are important. Musically, Π_1 represents the set Π_0 by semitones, Π_2 by overtones, Π_3 by the perfect fifths and octaves, Π_4 by the whole tones and minor semitones. We can observe an very interesting psychological information moment. The geometrical net Π_1 gives the melodic structure of music (pitches of tones), Π_2 – overtone structure, Π_3 – harmonic structure ("the spiral of fifths"), Π_4 – the diatonic structure. All these structures are present (in the same time, of course) in every musical composition. Clearly, these structures are not isomorphic (we cannot transform melody into harmony, etc.). In other words, transforms of the supporting lattices change the geometrical net substantially.

3 Weber – Fechner's law and its negation

There are some transforms of geometrical nets which do not change the domain lattices. One of them is exceptional and is known as the Weber – Fechner's psychological law. Grubby spoken, Weber – Fechner's law says that we do not hear frequencies of tones expressed by values of the set Π_0, but we hear their logarithms. According to this Weber – Fechner's law, we hear:

$$\begin{aligned} \Pi_1' &= \{\alpha_z \ln(256/243) + \beta_z \ln(2187/2048); \alpha_z, \beta_z, z \in \mathbb{Z}\}, \\ \Pi_2' &= \{\gamma_z \ln 2 + \delta_z \ln 3; \gamma_z, \delta_z, z \in \mathbb{Z}\}, \\ \Pi_3' &= \{\varepsilon_z \ln 2 + \eta_z \ln(3/2); \varepsilon_z, \eta_z, z \in \mathbb{Z}\}, \\ \Pi_4' &= \{\theta_z \ln(9/8) + \varkappa_z \ln(256/243); \theta_z, \varkappa_z, z \in \mathbb{Z}\}. \end{aligned} \tag{8}$$

This way, we hear *linearly*. E. g., octaves 2^z, $z \in \mathbb{Z}$, are equidistant in our psyché, $\log_2 2^z = z, z \in \mathbb{Z}$. It is clear that $\Pi_1', \Pi_2', \Pi_3', \Pi_4'$ are pairwise isomorphic with respect the vector operations (the scalar multiplication and vector addition) as $\mathbb{Z}$-submodules of the corresponding four different 2-dimensional vector spaces over $\mathbb{Q}$. These isomorphisms are given by transforms of the following form:

$$(\ln X, \ln Y) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (\ln Z, \ln V) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix}, \tag{9}$$

where

$$\left(\begin{array}{cc} a & b \\ c & d \end{array}\right), \ \det\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \neq 0,$$

is a transform matrix, and X, Y, Z, V are some positive reals.

What will happen when we substantially negate Weber – Fechner's law? This will be in the case when one coordinate of the 2-dimensional space will change logarithmically (according to Weber – Fechner's law) and the second one remains the same. One coordinate will be considered psychologically and the second one – physically (and vice versa), i.e.

$$(\ln X, \ln Y)\left(\begin{array}{c} \alpha \\ \beta \end{array}\right) = (\ln Z, V)\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\left(\begin{array}{c} \gamma \\ \delta \end{array}\right). \qquad (10)$$

This way, we artificially produced a nonlinear transform, mathematically regular, which is dilemmatic from the physical or psychological viewpoint. We will manifest the usefulness of this "strange" transform. The idea is to proof alternatively the fact that there are no transfinite semitones generating Pythagorean tuning: clearly no linear transform (isomorphism among Π'_1, Π'_2, Π'_3, Π'_4) can help us when solving the question whether X, Y may happen to be transfinite numbers for Pythagorean system. We knew the answer which we obtained earlier finding all possible concrete values of X and Y. In [2] we proved that there are 23 pairs of semitones generating Pythagorean system. Among them, the only rational couple is (256/243, 2187/2048), the minor and major Pythagorean semitones. The rest 22 pairs are algebraic irrationals and hence not transcendentals.

As we will see, the new proof in Section 4 is elementary and rather elegant.

Many speculations and quotations about Tone apperception and Weber – Fechner's law we can find in [7].

4 The alternative proof

The value of this proof of the following theorem consists of the fact that it is based on the (dialectical) negation of Weber-Fechner's law, a nonlinear transform.

Theorem 2 *There are no transcendental $X > 0$, $Y > 0$ such that*

$$\begin{aligned} X^{\alpha_{12}} Y^{\beta_{12}} &= 2, \\ X^{\alpha_7} Y^{\beta_7} &= 3/2, \end{aligned} \tag{11}$$

where

$$\begin{aligned} \alpha_{12} + \beta_{12} &= 12, \\ \alpha_7 + \beta_7 &= 7, \end{aligned} \tag{12}$$

and $\alpha_{12}, \beta_{12}, \alpha_7, \beta_7 \in \mathbb{Z}$.

Proof. Such $X > 0$, $Y > 0$ exist (e.g. X= 256/243, Y = 2187/2048). Consider the transform

$$\begin{aligned} X = &\ Ze^V \\ Y = &\ Z \end{aligned} \tag{13}$$

Then by (11) and (12),

$$\begin{aligned} 2 &= (Z^{\alpha_{12}} e^{V\alpha_{12}}) Z^{\beta_{12}} = Z^{12} e^{V\alpha_{12}}, \\ 3/2 &= (Z^{\alpha_7} e^{V\alpha_7}) Z^{\beta_7} = Z^7 e^{V\alpha_7}. \end{aligned}$$

Then

$$e^{dV} = K,$$

where $d = 12\alpha_7 - 7\alpha_{12}$ and $K = 3^{12}/2^{19}$ (Pythagorean comma, c.f. [1]). We see that $2 = Y^{12}\sqrt[d]{K}$, so Y is algebraic. By (11), both X, Y are both transcendental or both algebraic in the same time. □

Also, the result can be obtained also as a conclusion of some general (non-elementary) assertions in the number theory. Further, many construction details of Pythagorean system are not easy visible from this proof. For instance, we do not see easy how to find concrete X, Y (suppose additionally $0 \leq \alpha_7 \leq \alpha_{12}, 0 \leq \beta_7 \leq \beta_{12}$).

5 The harmony – melody uncertainty

In [4] are studied uncertainty types of tone systems which were: fuzziness, strife, and non-specificity. Authors of [6] claimed that these types are rather all possible types of uncertainty. In this paper we discussed the fourth type of uncertainty which seems us different with respect to those in [6] – it is a decomposition (analysis) of one integer object according at least two qualitatively different criteria. These criterions should be principally, i.e. quantitatively incomparable. The couple harmony-melody is a model manifestation of this uncertainty. For a musical composition, we cannot express harmony via melody or vice versa. They cannot exist independently and they are ever present as a couple. Melody is a manifestation of harmony and vice versa. There is no hierarchy between harmony and melody. On the other hand, both structures are relatively independent. If we specially avoid places of fuzziness, strife and non-specificity in any composition, we can ever find the harmony-melody couple in the rest music. We suggest to call this join type of uncertainty as *harmony-melody uncertainty*.

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