International Journal of General Systems 31(2002), no.1, 73 –96.

UNCERTAINTY MEASURES OF WELL TEMPERED SYSTEMS

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Abstract

Investigations of tone systems are motivated by mathematical models in acoustics. In the present paper, tone systems are represented by systems of fuzzy numbers. The construction idea of these systems consists of using of approximations of 3/2. A formalization of well tempered systems is given. Uncertainty-based-information measures are considered. The Huygens–Fokker, Opelt, Hába, Sumec, Petzval, and more other tone systems (used in the musical theory and practice and also new ones) are derived as consequences of the theory.
Keywords. Applied harmonic analysis, Uncertainty - based information, Uncertainty measure, Tone systems
Acknowledgement. This paper was supported by Grant VEGA 7192.

1 INTRODUCTION

1.1 A Revival of Interest in Tone Systems

For an overview of the topic, including historical aspects, cf. (Barbour, 1951; Benade, 1976; Feichtinger and Dörfler, 1999; Neuwirth, 1997; Sethares, 1998; Mazzola, 1990).

In the late 20th century we can observe a revival of interest in tone systems among musicians and in the industry in connection with the development of the so called computer music and production of the electronic musical instruments with computer control. At present, there are no real technological boundaries for practical use of arbitrary tone systems within the human aural perceptive abilities and sensitivity. In particular, this concerns the Just Intonation (pure tunings), equal temperaments (freedom in transpositions), and tunings with the uncertain pitch (including noises).

Recall that historically the main motivation for the development of tone systems was a physical limitation on the number of keys of the keyboard which can be used to play tones. Some means had to be found to permit the tonalities demanded by developments in polyphonic music to work within this limitation. In fact, this way the keyboard became the dominant force in Western composition and composers seek it out to explore new tonal palettes.

At present, choices of various discrete tone systems by composers are based on the conviction that the process of limitation is necessary to give the musician the material with which he can work. He has to select the tones he wants to use. The strong emphasis on the necessity of limitation, which has guided these considerations thus far, reflects not a subjective prejudice but a fundamental artistic law. There is no art without limitation. E.g., the sculptor working in marble has set his limit by the choice of this material to the exclusion of all other materials. The point, amply discussed in aesthetics, need not be labored here. Goethe sums it up as follows:

Das ist die Eigenschaft der Dinge:
Natürlichem genügt das Weltall kaum;
Was künstlich ist, verlangt geschlossnen Raum.

(This is the property of things:
the all scarcely suffices for the natural;
the artificial needs a bounded space.)

Faust II, 6882-6884

1.2 Uncertainty and Tone Systems

The 12 tone equal temperament $E_{12} = \{(\sqrt[12]{2})^z; z \in \mathcal{Z}\}$ ($\mathcal{Z}$ denotes the set of all integer numbers) is known since cca 1600 (as authors are quoted: Werckmeister, Stevin, Mersenne, Chu Hoa) and will be our theoretical starting point. The system E_{12} has its advantages and also disadvantages.

The main advantage or merit of E_{12} is that it has lead the way for the full development of harmonic music, and the rich variety of musical styles which has grown up in the last one hundred and fifty years. It took nearly two centuries for E_{12} to find universal acceptance by the musical world: the first pianos to be 12 tone equal tuned were produced by Broadwoods in the middle of the 19th century and by the beginning of the 20th, virtually all pianos were (re)tuned this way, cf. (Jorgenson, 1991). The advantages of E_{12} lead often to its non-critical use. Here is a small list when it is not suitable to play music in E_{12}: music using wolf intervals; music based mainly on melody; counterpoint based essentially on the pure major and minor thirds and sixths, timbre reasons, effects exploiting the building acoustics; non-European music (gamelan, hindu, arabic music, etc.).

One fact to note is that the figures used to calculate E_{12} are based on the theoretical values and E_{12} is itself a famous psychoacoustic compromise which holds for 3-4 middle octaves, where the ear hears linearly. What about the rest 3-4 octaves from the total 7 ones? In practice, instruments, such as piano, sound flat in their upper octaves when they are tuned in strict accordance with the equal tempered scale. Piano-tuners employ a trick called, “brightening the treble” or “stretched tuning”, cf. (Boomsliter, 1961), which means that the top one and a half to two octaves are sharpened slightly; the low bass octaves are also lowered in a similar fashion. No piano can be principally tuned into E_{12}: when tuning piano into E_{12} "via computer", the upper and lower octaves sound not in E_{12}. On the other hand, when tuning piano into E_{12} "via ear", it is not E_{12} physically. Since these tunings are depending on individual psyches of tuners, the usage of a deterministic modelling of the deviation, in the best case, has a statistical character. So, the second psychoacoustical compromise should be made when constructing tone systems in the whole diapason 16 – 20 000 Hz.

We tend to save the advantages of well temperaments and, in the same time, we take into account individual psyches of composers, interpreters, and listeners. One of ways how to solve the mentioned problems at one stroke is to "densify" sound. This is the idea of the period of tone systems since cca 1880 and which lasts till now although it is not the main stream of investigation. The stream could be called *more than 12* and it is characterized by well-tempered tone systems with more than 12 steps per octave possible, including N-tone equal temperaments E_N, $N \geq 12$. Key persons are: Helmholtz, Petzval, Hába, Sumec, Wilson, and others. Uncertainty (which has its reflection in the notion of temperament) used in these tone systems is coming from psychoacoustics.

Besides psychoacoustics, there are at least two other types of uncertainty which are reflected in the tone system constructions. The second one comes from music itself. Indeed, music cannot be reduced to a study of consonance of intervals. There are sequences of consonant, dissonant and various "semi-consonant" passages of intervals which only together co-create a very musical composition. All used intervals (consonant, "semi-consonant", dissonant) are appropriate for a given composition. There is a class of musical temperaments which are suitable for a given composition.

The third type of uncertainty is bounded with the creativity and psyche of the interpreter and listener of the composition. Perhaps, the idea is the best visible on Indian ragas when the same raga is played in different pitch system depending on the mood, year season, occasion, place, etc.

We see that the choosing and describing of suitable tone systems for concrete musical compositions is a challenge for the uncertainty based information theory. In this paper, we describe a theory of well tempered systems based on the generalized fifths (approximations of 3/2) from this viewpoint.

N. A. Garbuzov (1948), was rather the first who realized the importance of the notion of uncertainty in the theory of tone systems. Garbuzov supported his *theory of zones* with hundreds of statistics.

1.3 Temperature and Mistuning

Let us consider *tones* represented by Fourier series. *Musical interval* of two tones is the ratio of frequencies (say, measured in Hz) of the first partials of these two tones. E.g., the numbers 1/1, 6/5, 5/4, 3/2, 7/4, and 2/1 correspond to the musical intervals which are known as the unison, minor third, major third, perfect fifth, natural seventh, and octave, respectively.

Terms such as musical interval, relative frequency, and positive number are used as synonyms depending on the context. The situation is similar when using terms point, vector, and n-tuple in analytical geometry. This become quite evident when we realize that there is a bijection between the set of all positive numbers with the group operation of multiplication and the set of all musical intervals. We can use all techniques and results from vector spaces in the theory of tone systems, but not vice versa. Musical intervals are often represented on a special logarithm scale, in units called *cents*: if f is a musical interval and c this same frequency but in cents, then $c = 1200 \cdot \log_2 f, f = 2^{c/1200}$, where $c \in \mathcal{R}$ (the set of all real numbers) and $f \in \mathcal{R}^+$ (the set of all positive real numbers).

In this paper, we will understand under tone system S a discrete subset of the set of all fuzzy numbers $\mathcal{F}$, where

$$\mathcal{F} = \{F : \mathcal{R}^+ \to [0,1]; \exists f \in \mathcal{R}^+, F(f) = 1\},$$

i.e. *tone system* $S \subset \mathcal{F}$ is a discrete set of fuzzy numbers F which “approximate in the psychological sense” musical intervals f. Denote the class of all tone systems by $\mathfrak{S}$.

Of course, the kernel of a fuzzy number F, i.e. $ker(F) = \{f \in \mathcal{R}^+; F(f) = 1\}$ need not be a unique value (e.g., when $C_\sharp \neq D_\flat$ in music). Denote by $ker(S) = \{f \in \mathcal{R}^+; F(f) = 1, F \in S\} = \bigcup_{F\in S} ker(F)$ and call the *kernel of the tone system* S.

The order on $\mathcal{F}$ we define as follows: for $F_a, F_b \in \mathcal{F}$,

$$F_a \leq_{\mathcal{F}} F_b \Leftrightarrow \forall f_a \in ker(F_a), \forall f_b \in ker(F_b); f_a \leq f_b.$$

In general, the order $\leq_{\mathcal{F}}$ is not linear since elements $F \in \mathcal{F}$ with non-disjoint kernels need not be comparable.

Let $K > 1$ be a real number called *comma*. Mathematically, the concrete comma value does not play any principal role in our theory. Usually, it is bounded with the size of the fuzzy number support. Analogously to $\epsilon > 0$ in calculus, which has to be close to 0, it is reasonable to deal with commas $K > 1$ which are close to 1 in some sense. The examples of commas with historical names: Comma of Dydimus (81/80), schizma (32805/32768), Pythagorean comma (531441/524288). For more abour historical commas and they structure, cf. (Hellegouarch, 1982).

Let $K > 1$ be a comma. A tone system $\Sigma \in \mathfrak{S}$ is called the *K-temperament* of the tone system $S \in \mathfrak{S}$, if there exists a constant $k \in \mathcal{R}^+$ (called the *shift*) and for every $f \in ker(S)$ there exists $\tau_f \in \mathcal{R}^+$, $1/K \leq \tau_f \leq K$, such that $\phi = k \cdot \tau_f \cdot f \in ker(\Sigma)$, where $ker(\Sigma) = \{\phi \in \mathcal{R}^+; \Phi(\phi) = 1, \Phi \in \Sigma\}$. Without loss of generality, we will suppose $k = 1$. Denote by $\Theta_{f,\tau_f} = \tau_f \cdot f$.

Let $K > 1$. The kernel $ker(\Sigma)$ is said the set of *K-tempered values* of S. The set $\Gamma = \{\tau_f; f \in ker(S), \Theta_{f,\tau_f} \in ker(\Sigma), 1/K \leq \tau_f \leq K\}$ is called the *K-temperature* of S and the set $\Xi = \{\mu_f; \mu_f = \tau_f - 1, \tau_f \in \Gamma\}$ is called the *K-mistuning* of S.

For $f \in ker(S)$ (one single element), we will simply say that τ_f, $1/K \leq \tau_f \leq K$, is a *K-temperature* of f and μ_f is a *K-mistuning* of f.

Clearly, if Σ is a K-temperament of S, then S is a K-temperament of Σ. The relation “to be K-temperament” is reflexive, symmetric, but not transitive (because of the condition $1/K \leq \tau_f \leq K$).

Everywhere in the ongoing text we will suppose the given fixed comma $K > 1$ and will say simply about temperament instead of K-temperament, similarly: mistuning, tempered value, etc. For concrete tone systems, we will put everywhere $K = 81/80$ (comma of Dydimus).

It is technologically reasonable to give S as a theoretical crisp set (e.g. the 12-tone equal temperament E_{12} mentioned before is given as a set of positive real numbers), cf. (Romanowska, 1979). However this never really occurs. For this reason and also for the sake of symmetry, we consider both tone systems S and its temperament Σ as systems of fuzzy numbers.

1.4 Three Types of Temperaments – Examples

To a given tone system $S \in \mathfrak{S}$, there are distinguished three types of temperaments: the psychoacoustic, musical, and variable ones. We may consider also the psychoacoustic, musical, and variable sides of one temperament S. According to the classification of uncertainty types in (Klir and Wierman, 1997) (i.e. quantifying the real world object), the uncertainty of the psychoacoustic temperament is *fuzziness*, of the musical temperament is *strife* and of the variable temperament is *nonspecificity*.

In practice, temperatures depends on the aural sensitivity of a considered individual human and is of psychoacoustic nature. Especially, a temperature τ_f^* (similarly, mistuning) is *virtual* if a temperature (resp. mistuning) of the musical interval $\tau_f^* = \Theta_{f,\tau_f}^*/f$ is not "hearable". Dealing with *psychoacoustic* temperaments (or, the psychoacoustic side of a temperament), we are interested e.g. in the supports of fuzzy numbers $\Phi \in \Sigma$.

Example 1
(a) The Petzval's conventional suggestion about virtual temperature, cf. (Erményi, 1904):

$$239/240 < \tau^* < 240/239.$$

(b) Average sensitivity of human ear is about 10 cents, tuners do 5-6 cents, the boundary is 2-3 cents. The aural sensitivity depends on many factors, e.g. on the age or health of an individual, cf. (Benade, 1976).

Music is not an acoustic consideration of musical intervals. In each musical composition there are consonant passages, passages with dissonances (tensions) and this together makes music interesting. So, every musical composition uses and requires its own structure of used intervals, the relations among possible consonances and dissonances from the harmonic and melodic musical context. This side of temperament, the structure of played tones, considered in a concrete situation and time depends on the individual human musical education, given musical style, construction of musical instruments, building acoustics, concrete musical composition, etc. Dealing with the *musical* temperaments (or, the musical sides of a temperament), we take notice of the function shape of fuzzy numbers $\Phi \in \Sigma$.

Example 2 *In Example 12, the musical mistuning*

$$\mu_{3/2}{}^{(m')} = \mu_{6/5}{}^{(m')} = \mu_{5/4}{}^{(m')} = \mu_{7/4}{}^{(m')}.$$

In Example 13, the musical mistuning

$$\mu_{3/2}^{(m'')} = \mu_{5/4}^{(m'')}/5 = \mu_{6/5}^{(m'')}/5 = \mu_{7/4}^{(m'')}/5.$$

In Example 4, the weights of $\tau_{3/2}, \tau_{5/4}, \tau_{7/4}$ are 1/9, 1/25, 1/49, respectively. In Example 5, the weights of $\tau_{3/2}, \tau_{5/4}, \tau_{7/4}$ are the same as in Example 4, but the temperament depends proportionally on the number of keys per octave.

Temperature can be mentioned also as a union of equivalent (equally probable) variants, alternatives, or possibilities of a tone system. Dealing with the *variable* temperament (or, the variable side of a temperament) we are interested e.g. in cardinality of $ker(\Sigma)$.

Example 3 *Tone systems consisting of values of 12 major and 12 minor scales (not specifying here this notions) are defined ambiguously and also the cardinality of these systems may vary. Observe that the sets involving the values of 12 major and minor scales in Lemma 6 and Theorem 2, and in Theorem 5 may have different numerical expressions and also cardinalities.*

2 UNCERTAINTY TEMPERAMENT MEASURES

2.1 Harmonic Mean Based Uncertainty Measures

Given a set $\mathcal{F}$ of all fuzzy numbers, let $\mathfrak{S}$ be an algebra of tone systems $S \subset \mathcal{F}$, such that $\emptyset \in \mathfrak{S}$ and the operations $\cap, \cup$ we define on $\mathfrak{S}$ as follows: let $S_a, S_b \in \mathfrak{S}$, then

$$S_a \cup S_b = \{F \in \mathcal{F}; F = F_a \vee F_b, F_a \in S_a, F_b \in S_b\} \in \mathfrak{S},$$
$$S_a \cap S_b = \{F \in \mathcal{F}; F = F_a \wedge F_b, F_a \in S_a, F_b \in S_b\} \in \mathfrak{S},$$

where for every $t \in \mathcal{R}^+$,

$$(F_a \vee F_b)(t) = \max(F_a(t), F_b(t)),$$

and

$$(F_a \wedge F_b)(t) = \begin{cases} \min(F_a(t), F_b(t)), & ker(F_a) \cap ker(F_b) \neq \emptyset, \\ \emptyset, & ker(F_a) \cap ker(F_b) = \emptyset. \end{cases}$$

Let $S_1, S_2 \in \mathfrak{S}$. We say that $S_1 \subset S_2$ if $S_1 \cap S_2 = S_1$.

An *uncertainty measure* $\sigma : \mathfrak{S} \to [0, \infty]$ (the adjective "uncertainty" we will omit) is a non negative extended real valued set function with the properties:

1. $\sigma(\emptyset) = 0$;
2. if $S_1, S_2 \in \mathfrak{S}$, $S_1 \subset S_2$, then $\sigma(S_1) \leq \sigma(S_2)$.

In the book (Klir and Wierman, 1997) we can find also the review of the known uncertainty-based measures. Our construction of uncertainty measures is based on distance measuring between two sets. In Example 4 and Example 5, there are measured distances between two crisp sets: $L = \{5/4, 3/2, 7/4\}$ and E_N, the N-tone equal temperament, $N \in \mathcal{N}$. In Subsection 2.3, Equation (3), there is used an asymmetric set distance between E_{12} (crisp) and S (fuzzy). A bimeasure construction based on a distance between two uncertain sets we can find in Subsection 2.4, Equation (12).

Traditional quantities comparing musical temperatures are based on *harmonic* means of temperatures. We construct uncertainty measures using (generalized) harmonic mean in their construction as following.

Definition 1 *Let A be a finite set of positive numbers. Let $p \in \mathcal{N}$. Let $\mathfrak{S}$ be the class of all tone systems. A set function $\sigma_p : \mathfrak{S} \to [0, \infty]$ is called the temperament measure, if $\sigma_p(\emptyset) = 0$ and if $S \neq \emptyset$, then*

$$\sigma_p(S) = \frac{1}{\sum_{\phi \in A} w_\phi \mu_\phi^p(S)},$$

where $S \in \mathfrak{S}$ and $w_\phi \geq 0$, $\sum_{\phi \in A} w_a = 1$, are weights of mistunings, i.e.

$$\mu_\phi(S) = \min_{f \in ker(S)} \left| 1 - \frac{f}{\phi} \right|, \phi \in A.$$

Since we will deal only with temperament measures in this paper, we will often omit the adjective "temperament" for the sake of simplicity.

The proof of the following lemma is easy and omitted.

Lemma 1 *Let $S \in \mathfrak{S}$. Let $p \in \mathcal{N}$. Let σ'_p and σ''_p are two temperament measures on $S \in \mathfrak{S}$. Put*

$$\sigma^*_p(S) = (\sigma'_p \oplus \sigma''_p)(S) = \frac{1}{w_1/\sigma'_p(S) + w_2/\sigma''_p(S)}, S \in \mathfrak{S}, \tag{1}$$

where $w_1 + w_2 = 1$, $w_1 > 0, w_2 > 0$. Then σ^*_p is a temperament measure as well.

2.2 Increasing Number of Keys per Octave

In Table 1, there are twelve powers of $\sqrt[12]{2}^z$, $z = 0, 1, 2, \ldots, 11$, in the first column. In the second one, there are just intonation values, and in the third column, we can find the temperature of these tone systems expressed in cents. Notice how remarkably close most of them are to the ratios of small whole numbers. If the distribution were random we would expect an average temperature of 25 cents. Instead the average temperature is only cca 10 cents (for the chosen set of rationals). There arise a question: *is there something really special about E_{12}?*

The tone system E_{12}, the 12-tone equal temperament, is a commonly well-known theoretical tone system. It will serve us as a starting point when considering other tone systems. It is also a common element of the most of modern tone systems or tone systems families. An easy generalization of E_{12} is the following

Definition 2 *The N-equal tempered tone system*

$$E_N = \{f_z \in \mathcal{R}; f_z = \sqrt[N]{2}^z, z \in \mathcal{Z}\}, \ N \in \mathcal{N}.$$

The following theorem is known, cf. e.g. (Barbour, 1948).

E_{12} ratio ϕ	Just ratio f	$\tau_f = \phi/f$
1.000	$1/1 = 1.000$	0
1.059	$16/15 \approx 1.066$	-12
1.122	$9/8 = 1.125$	-4
1.189	$6/5 = 1.200$	-16
1.256	$5/4 = 1.250$	+14
1.335	$4/3 \approx 1.333$	+2
1.414	$7/5 = 1.400$	+17
1.498	$3/2 = 1.500$	-2
1.587	$8/5 = 1.600$	-14
1.682	$5/3 \approx 1.667$	+16
1.782	$16/9 \approx 1.778$	+4
(1.782	$7/4 = 1.750$	+31)
1.888	$15/8 = 1.875$	+12

Table 1: Temperature 12-ET/JI (in cents)

Theorem 1 *Denote by* $\{E_{N_k}\}_{N_k=1}^{\infty}$ *the sequence of equal tempered tone systems such that*

$$|\mu_{3/2}(E_{N_k})|_{N_k=1}^{\infty} \text{ strictly decreases to } 0 \text{ as } N_k \to \infty,$$

where $\{\mu_{3/2}(E_{N_k})\}_{N_k=1}^{\infty}$ *is a subsequence of the sequence of all fifth mistunings* $\{\mu_{3/2}(N)\}_{N=1}^{\infty}$ *of the equal tempered tone systems* $\{E_N\}_{N=1}^{\infty}$*. Then*

$$\{E_{N_k}\}_{N_k=1}^{\infty} = \{E_1, E_2, E_5, E_{12}, E_{41}, E_{53}, E_{306}, E_{665}, \ldots\}.$$

Proof. Consider the approximations

$$\frac{3}{2} \approx 2^{x/N_k},$$

where N_k denotes the number of equal steps per octave and x the order number of the tempered fifth, respectively. We have:

$$\frac{x}{N_k} \approx \log_2 \frac{3}{2} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{2 + \cfrac{1}{10} \cdots}}}}}}}}$$

This continuous fraction yields the following sequence:

$$\{a(N_k)\}_{N_k=1}^{\infty} = \{1, \frac{1}{2}, \frac{3}{5}, \frac{7}{12}, \frac{24}{41}, \frac{31}{53}, \frac{179}{306}, \frac{389}{665}, \ldots\},$$

$$\lim_{N_k \to \infty} a(N_k) = \log_2 \frac{3}{2}.$$

The corresponding fifth temperature sequence $\{\tau_{3/2}(E_{N_k})\}_{N_k=1}^{\infty}$ is as follows:

$$\begin{aligned} \{\tau_{3/2}(E_{N_k})\}_{N_k=1}^{\infty} = \{ \quad & 1 + 0.01048, \\ & 1 - 0.001135, \\ & 1 + 0.0002789, \\ & 1 - 0.0000394, \\ & 1 + 0.00000318, \\ & 1 - 0.00000007, \ldots\}. \end{aligned}$$

We see that $|\tau_{3/2}(E_{N_k})| \downarrow 0$ as $N_k \to \infty$.

We can verify that the third temperatures $\tau_{5/4}(E_N)$ are greater than $K = 81/80$ in the case $\tau_{3/2}(E_N) > 1$. For $\tau_{3/2}(E_{N_k}) < 1$, the N_k-tone equal tempered tone systems are then $E_{12}, E_{53}, E_{665}, \ldots$. The E_{12} is well known, the E_{665} is not very interesting since the number 665 is too large from the practical viewpoint. The E_{53} can be considered as a temperament of a very interesting Petzval's cyclic Tone system of the second type, see Example 11.

We compared E_N for various $N \in \mathcal{N}$ taking into the account only one interval, 3/2, the perfect fifth. A following trivial measure can be used

$$\sigma(S) = \frac{1}{\mu_{3/2}(S)}, \tag{2}$$

where $\mu_{3/2}(S) = \min_{f \in ker(S)} \left|1 - \frac{f}{3/2}\right|$, $S \in \mathfrak{S}$.

Analogously to the proof of Theorem 1, we can compute continuous fractions for other musical intervals, e.g. $\log_2 6/5, \log_2 5/4, \log_2 7/4$, and obtain separately decreasing sequences of mistunings and corresponding E_{N_k}. However, we would like to have measures that let us compare $E_N, n \in \mathcal{N}$, in a more complex view.

The general musicological and mathematical background and statistical methods for the definition of melodic and rhythmic weights (factors) in relevant quantities for tone systems with 12 tones per octave, were considered in (Beran and Mazzola, 1999). The following three examples give us a grubby imagination about the behavior of weighted measures like σ_p in Definition 1.

Example 4 *Try*

$$\sigma(S) = \frac{\frac{1}{9} + \frac{1}{25} + \frac{1}{25} + \frac{1}{49}}{\frac{\mu^2_{3/2}(S)}{9} + \frac{\mu^2_{5/4}(S)}{25} + \frac{\mu^2_{6/5}(S)}{25} + \frac{\mu^2_{7/4}(S)}{49}}.$$

For illumination, execute the practical search for $S = E_N$, $N =$ 5,6, $\ldots$, 60 (we stop the evaluations on the number 60 since each frequency would fall in a ± 10 cent range from *some frequency $f \in E_{60}$). If there was no disadvantage in having more notes per octave, then E_{12} is only slightly special with regard to harmonies measured by σ. Several tunings with less than 60 notes do better, cf. Figure 2.*

Example 5 *Of course we expect the temperatures to get smaller as the number of divisions increases and hence their size decreases. We modify the harmonic mean in Example 4 as follows:*

$$\sigma_N^{(q)}(S) = \frac{1}{N^q} \cdot \sigma(S), q \in \mathcal{N}.$$

For $q = 1$, the only ones equal to or better than E_{12} on this criterion are E_{31}, E_{41} and E_{53}, cf. Figure 3.

Example 6 *For $q = 2$, then the situation is graphed in Figure 4.*

2.3 Possibility of Free Transpositions

The system E_{12} is considered by some musicians as the definitive ideal with respect to free transpositions in music. The possibility of free transpositions within a tone system S can be expressed then as a small distance between sets E_{12} and $S \in \mathfrak{S}$ (the denominator in the formula (3)).

Definition 3 *Let $\kappa > 0$. Let $p \in \mathcal{N}$. We say that the tone system S with the octave equivalence enables $(\kappa, E_{12}, \sigma'_p)$-free transpositions if*

$$\sigma'_p(S) = \frac{1}{\sum_{\phi \in E_{12} \cap [1;2)} w_\phi \mu_\phi^p(S)} > \kappa, \tag{3}$$

where $w_\phi > 0$, $\sum_{\phi=1}^{12} w_\phi = 1$, $\mu_\phi(S) = \min_{f \in ker(S) \cap [1,2)} \left|1 - \frac{f}{\phi}\right|$, $\phi \in E_{12}$.

2.4 Basic Musical Intervals Sound as Pure

Similarly to the construction of the measure σ'_p, we may understand what means if we say that basic acoustic intervals "sound as pure" in a tone system S.

Definition 4 *Let $\kappa > 0$. Let $p \in \mathcal{N}$. Let $L = \{6/5,\ 5/4,\ 3/2,\ 7/4\}$. We say that intervals 6/5, 5/4, 3/2, 7/4 sound (κ, L, σ''_p)-pure in the tone system $S \in \mathfrak{S}$ if*

$$\sigma''_p(S) = \frac{1}{\sum_{\phi \in L} w_\phi \mu_\phi^p(S)} > \kappa, \tag{4}$$

where $w_\phi > 0$, $\sum_{\phi=1}^{4} w_\phi = 1$, $\mu_\phi(S) = \min_{f \in ker(S)} \left|1 - \frac{f}{\phi}\right|$, $\phi \in L$.

Remark 1 Since superparticular ratios are mentioned by some theorists as important subject for "pure" harmony, for a curiosity, there is a list of the superparticular ratios used in this paper:

2/1, 3/2, 5/4, 6/5, 64/63, 81/80,
235/234, 240/239, 288/287, 336/335, 364/363, 455/454, 886/885,
1 230/1 229, 1 293/1 292, 2 013/2 012, 11 561/11 560, 25 382/25 381.

2.5 We Cannot Avoid Uncertainty

Grubby spoken (the precised definition will be given in Section 3.4), well tempered tone systems are those tone systems which enables, in the same time, both free transpositions and basic musical intervals sound as pure.

Let us open the question about the suitability of the chosen constructions of uncertainty measures σ'_p and σ''_p in Definitions 3 and 4. For instance, instead mistuning we may consider temperature with values expressed in cents or consider special weights, see Section 2.2.

The sets E_{12}, L, and $ker(S)$ are crisp and uncertainty is present in the formulas (3), (1), and (12), because of ambiguity given by $ker(S)$ (possible more kernel values for one fuzzy number). Of course, we may construct also measures σ'_p, σ''_p which are "more soft", including in addition fuzziness-type uncertainty in their constructions. But, we can also avoid uncertainty in all in these formulas by dealing with unambiguous and crisp sets.

However, there is a moment which should be underlined in connection with the uncertainty of tone systems: since $L \cap E_N = \emptyset$ for every $N \in \mathcal{N}$ (the uncomensurability of rationals and irrationals), the value

$$\frac{1}{\sigma'_p(S)} + \frac{1}{\sigma''_p(S)} > 0$$

strictly for every crisp tone system $S \in \mathfrak{S}$. Therefore, in crisp tone systems it is impossible to freely transpose and in the same time to sound the basic intervals as pure. We *must* make any appropriate compromise. Therefore, it is "more honest" to deal with tone systems as objects including uncertainty from the beginning (as we do in the paper). Thus, considering well tempered tone systems, we observe the principal using and sense of uncertainty based information.

For our purposes, there is reasonable to consider the following quantity:

$$(\sigma'_p \oplus \sigma''_p)(S) = \frac{1}{w_1/\sigma'_p(S) + w_2/\sigma''_p(S)}, S \in \mathfrak{S}, \tag{5}$$

where $p \in \mathcal{N}$ and $w_1 + w_2 = 1,\ w_1 > 0, w_2 > 0$. Tone system S is then the "better" well tempered the bigger is the value of $(\sigma'_p \oplus \sigma''_p)(S)$. However, this value is ever finite (we can normalize it with respect to 1).

We can see the corollary of Lemma 1, that the function

$$\sigma^*_p = \sigma'_p \oplus \sigma''_p \tag{6}$$

is a temperament measure on $S \in \mathcal{S}$, where σ'_p and σ''_p are defined as in Definitions 3 and 4, respectively.

3 WELL TEMPERED TONE SYSTEMS

3.1 Symmetry and Octave Equivalence

The notion of the tone system symmetry corresponds to the notion of even function in real analysis. For our purposes, it is enough

Definition 5 *A tone system $S \in \mathfrak{S}$ is said to be symmetric if*

$$f \in ker(S) \Rightarrow 1/f \in ker(S).$$

The octave (ratio 2/1) plays a role of "unit" in the theory of tone systems (as number 1 in the group of all real numbers with the operation of addition).

Definition 6 *We say that the relation of octave equivalence $\equiv$ is given on a tone system $S \in \mathfrak{S}$, if for every $f_1, f_2 \in ker(S)$,*

$$f_1 \equiv f_2 \Leftrightarrow (\exists z \in \mathcal{Z}; f_1 = f_2 \cdot 2^z).$$

For $f \in ker(S)$, denote by $[f]$ the octave equivalence class such that $f \in [f]$.

Lemma 2 *Let S, Σ be two symmetrical tone systems with octave equivalence. Then*
(a) if $f \in ker(S)$, then $2/f \in ker(S)$;
(b) if τ_f is a temperature, then $\tau_f = \tau_{2f} = 1/\tau_{2/f}$.

Proof.
(a) If $k \in ker(S)$, then $1/f \in ker(S)$ according to the symmetry. By the octave equivalence, $2^z/f \in ker(S)$, $z \in \mathcal{Z}$.
(b) If $f \in ker(S)$, $f \cdot \tau_f = \phi \in ker(\Sigma)$, then $2^z f \cdot \tau_{2^z f} = 2^z \phi \in ker(\Sigma)$. Thus, $\tau_f = \tau_{2^z f}$, $z \in \mathcal{Z}$.
We have: $(1/f)/(\tau_{1/f}) = (1/\phi)$. Thus, $\tau_f = 1/\tau_{1/f} = 1/\tau_{2^z/f}$, $z \in \mathcal{Z}$.

3.2 The Tempered Fifth Approximations

The classical tuning algorithms based on the so called *spiral of fifths* uses the fifth as "yard stick". The idea is as follows. If we consider the tone system $S \in \mathfrak{S}$ with $ker(S) = \{\Theta^z_{3/2,\tau_{3/2}} \cdot 2^c; c, z \in \mathcal{Z}\}$, then for some $\Theta_{3/2,\tau_{3/2}}$ the set $ker(S) \cap [1,2]$ is finite (e.g., for $\Theta_{3/2,\tau_{3/2}} = \sqrt[12]{2^7}$, we have E_{12}) or for other ones it will be infinite and dense in $[1,2]$ (Kronecker's lemma, cf. (Kuipers and Niederreiter, 1974); e.g., for $\Theta_{3/2,\tau_{3/2}} = 3/2$, the Pythagorean Tuning, cf. (Haluška, 2000). The tone systems of the first kind we will call to be *cyclic* and the second ones – *open*, respectively. More precisely,

Definition 7 A tone system $S \in \mathfrak{S}$ with $ker(S) = \{\Theta^z_{3/2,\tau_{3/2}} \cdot 2^c; c, z \in \mathcal{Z}\}$ is *cyclic* if there exist natural numbers n, N such that

$$2^n = \Theta^N_{3/2,\tau_{3/2}}. \tag{7}$$

Every *equal tempered* tone system is cyclic (for some $z_0 \in \mathcal{Z}$, put $\Theta_{3/2,\tau_{3/2}} = 2^{z_0/N}$. Then $2^{z_0} = \Theta_{3/2,\tau_{3/2}}^N$). There are cyclic tone systems which are not equally tempered, cf. Subsection 4.1 and Subsection 4.2.

Lemma 3 *A cyclic tone system $S \in \mathfrak{S}$ with $ker(S) = \{\Theta^z_{3/2,\tau_{3/2}} \cdot 2^c; c, z \in \mathcal{Z}\}$ is equal tempered if and only if the numbers n and N in Definition 7 are relatively prime.*

Proof. Our assertion is equivalent to the following:

Let N, n be two relatively prime natural numbers.

Let $k_i = i \cdot n (\text{mod } N)$ for every $i = 0,\ 1,\ \ldots, N-1$. If $\{p_0, p_1, \ldots, p_{N-1}\}$ is a permutation of the set $\{k_0, k_1, \ldots, k_{N-1}\}$ such that $k_i \leq k_{i+1}\}$ for every $i = 0, 1, \ldots, N-2$, then $p_{i+1} - p_i = 1$.

Indeed, suppose that $i, j \in \{0, 1, \ldots, N-1\}$ and $k_i = k_j$. Then $i \cdot n = j \cdot n (\text{mod } N)$. Therefore $n \cdot (i-j) = 0(\text{ mod } N)$. Since N, n are relatively prime, we obtain $i - j = 0(\text{ mod } N)$. We conclude $i = j$. That means that the set $\{p_0, p_1, \ldots, p_{N-1}\}$ is equal to the set $\{0, 1, \ldots, N-1\}$.

3.3 Basic Law of Tempering

The equation

$$\tau_{6/5} \cdot \tau_{5/4} = \tau_{3/2}, \tag{8}$$

is called the *basic law of tempering*. This equation holds in the most of the historical well tempered tone systems.

Considering the tone systems constructed as the pure fifth approximations, we obtain the following helpful lemma.

Lemma 4 *Let $ker(S) = \{(3/2)^z \cdot 2^c; c, z \in \mathcal{Z}\}$. Let $n, m \in \mathcal{Z}$ be such that $(3/2)^n 2^\alpha = \tau_{6/5} \cdot 6/5, (3/2)^m 2^\beta = \tau_{5/4} \cdot 5/4$ (powers of the fifth temperatures of the major and minor thirds) for some $\alpha \in \mathcal{Z}$ and $\beta \in \mathcal{Z}$. Let $\tau_{6/5} \cdot \tau_{5/4} = \tau_{3/2}$. Then $n + m = 1$.*

The proof is trivial and we omit it.

Since there is a possibility of many good approximations of 6/5, 5/4, 3/2 in the same tone system $S \in \mathfrak{S}$, we will understand that the basic law of tempering is satisfied if there exist trinities of elements of S with temperatures $\tau_{6/5}, \tau_{5/4}, \tau_{3/2}$ such that (8).

3.4 A Formalization of Well Tempered Tone Systems

We integrate the essential ideas of various constructions of *well tempered tone systems* based on the tempered fifths in

Definition 8 *Let $K > 1$. Let $\tau_{3/2} \in \mathcal{R}$, $1/K < \tau_{3/2} < K$. Let $\mathcal{Z}_1 \subset \mathcal{Z}$. Let $\kappa > 0$. Let $p \in \mathcal{N}$. Let σ_p^* be defined by (6). We say that a tone system $S \in \mathfrak{S}$, is a well tempered (more precisely, $(K, \kappa, \sigma_p^*, \tau_{3/2}, \mathcal{Z}_1)$-well tempered tone system) if*

1.
$$ker(S) = \bigcup_{z\in\mathcal{Z}_1} [\Theta^z_{3/2,\tau_{3/2}}],$$

2.
$$f \in ker(S) \Rightarrow 1/f \in ker(S);$$

3.
$$\exists\Theta_{6/5,\tau_{6/5}} \in ker(S), \exists\Theta_{5/4,\tau_{5/4}} \in ker(S); \tau_{6/5}\cdot\tau_{5/4} = \tau_{3/2};$$

4.
$$\sigma^*_p(S) > \kappa.$$

In what follows, we will suppose $p = 2$ for temperament measures.

4 THE PETZVAL'S TONE SYSTEMS

Consider tone systems $S \in \mathfrak{S}$ with $ker(S) = \bigcup_{z\in\mathcal{Z}_1\subset\mathcal{Z}}[\Theta^z_{3/2,\tau_{3/2}}]$, see Definition 8, 1. (which are, of course, not equally tempered in general). Besides the fifth virtual mistuning $\tau_{3/2}$ we ask also for appropriate mistuning of $\{3/2, 5/4, 6/5, 7/4\}$.

Taking $\Theta_{3/2,\tau_{3/2}} = 3/2$ (exactly, $\tau_{3/2} = 1$), we have the following approximating sequence $\{K^*_n\}^\infty_{n=1}$ of the tempered major thirds such that $\lim_{n\to\infty} K^*_n = 5/4$ and with the increasing absolute value of the power of $\Theta_{3/2,\tau_{3/2}}$ (we do not describe the obvious algorithm how to obtain this sequence):

$$\begin{aligned} K^*_n &= \Theta^4_{3/2,\tau_{3/2}}, \Theta^{-8}_{3/2,\tau_{3/2}}, \Theta^{45}_{3/2,\tau_{3/2}}, \Theta^{-314}_{3/2,\tau_{3/2}}, \Theta^{351}_{3/2,\tau_{3/2}}, \ldots \\ &\approx 1.265625, 1.2486, 1.251205, 1.249832, 1.24996, \ldots. \end{aligned}$$

The analogous approximating sequence $\{F^*_n\}^\infty_{n=1}$ of the tempered minor thirds is as follows:

$$\begin{aligned} F^*_n &= \Theta^{-3}_{3/2,\tau_{3/2}}, \Theta^{9}_{3/2,\tau_{3/2}}, \Theta^{-44}_{3/2,\tau_{3/2}}, \Theta^{315}_{3/2,\tau_{3/2}}, \Theta^{-350}_{3/2,\tau_{3/2}}, \ldots \\ &\approx 1.185185, 1.201355, 1.198849, 1.200128, 1.200075, \ldots. \end{aligned}$$

The analogous approximating sequence $\{G^*_n\}^\infty_{n=1}$ of the tempered natural sevenths is as follows:

$$\begin{aligned} G^*_n &= \Theta^{10}_{3/2,\tau_{3/2}}, \Theta^{-14}_{3/2,\tau_{3/2}}, \Theta^{39}_{3/2,\tau_{3/2}}, \Theta^{-67}_{3/2,\tau_{3/2}}, \Theta^{239}_{3/2,\tau_{3/2}} \ldots \\ &\approx 1.77778, 1.80203, 1.75384, 1.75752, 1.75018, 1.74040, \ldots. \end{aligned}$$

Now, let $\Theta_{3/2,\tau_{3/2}}$ be a *tempered* fifth and consider sequences

$$\{K_n\}^\infty_{n=1}, \{F_n\}^\infty_{n=1}, \{G_n\}^\infty_{n=1},$$

where

$$\begin{aligned} K_n = \;& \Theta^4_{3/2,\tau_{3/2}}, \Theta^{-8}_{3/2,\tau_{3/2}}, \Theta^{45}_{3/2,\tau_{3/2}}, \Theta^{-314}_{3/2,\tau_{3/2}}, \Theta^{351}_{3/2,\tau_{3/2}}, \ldots, \\ F_n = \;& \Theta^{-3}_{3/2,\tau_{3/2}}, \Theta^{9}_{3/2,\tau_{3/2}}, \Theta^{-44}_{3/2,\tau_{3/2}}, \Theta^{315}_{3/2,\tau_{3/2}}, \Theta^{-350}_{3/2,\tau_{3/2}}, \ldots, \\ G_n = \;& \Theta^{10}_{3/2,\tau_{3/2}}, \Theta^{-14}_{3/2,\tau_{3/2}}, \Theta^{39}_{3/2,\tau_{3/2}}, \Theta^{-67}_{3/2,\tau_{3/2}}, \Theta^{239}_{3/2,\tau_{3/2}}, \ldots. \end{aligned}$$

Definition 9 *The well tempered tone system S_n, $n = 1, 2, 3, \ldots$, is said to be the Petzval's tone system of the n-th type if*

$$\{K_n, F_n, G_n\} \subset ker(S_n).$$

Especially,

$$\{\Theta^{-3}_{3/2,\tau_{3/2}}, \Theta^{4}_{3/2,\tau_{3/2}}, \Theta^{10}_{3/2,\tau_{3/2}}\} \subset ker(S_1)$$

and

$$\{\Theta^{9}_{3/2,\tau_{3/2}}, \Theta^{-8}_{3/2,\tau_{3/2}}, \Theta^{-14}_{3/2,\tau_{3/2}}\} \subset ker(S_2)$$

characterize the *Petzval's tone systems of the first type* and *Petzval's tone systems of the second type*, respectively. Systems S_n, $n \geq 3$, are not used in practice.

Remark 2 *Note that $\Theta^{10}_{3/2,\tau_{3/2}}$ is a better approximation of 7/4 than $\Theta^{-2}_{3/2,\tau_{3/2}}$ when $\tau_{3/2} < 1$, cf. Remark 4. Therefore, we start from $\Theta^{10}_{3/2,\tau_{3/2}}$ when constructing the Petzval's tone systems sequence and omit $\Theta^{-2}_{3/2,\tau_{3/2}}$ at all.*

Remark 3 *J. M. Petzval introduced and considered these tone systems in the crisp case. He used a naive approach (from pre present viewpoint, not using any measure theory) when measuring tone systems and verified his evaluations with musical and physical experience; being himself a fine mechanician, he constructed and played a reed organ equipped with the keyboard with 53 steps per octave. The corresponding cyclic 53-tone system is now known also as* Petzval tuning, *cf. Example 11. We do not add any adjective like "fuzzy" or "generalized" to the term "Petzval's tone system" in this paper.*

4.1 Tone Systems I

Properties For the Petzval's tone systems of the first Type we have the following expressions:

$$\Theta^{-3}_{3/2,\tau_{3/2}} = {\tau_{3/2}}^{-3} \cdot \frac{2^5}{3^3}, \Theta^{4}_{3/2,\tau_{3/2}} = {\tau_{3/2}}^{4} \cdot \frac{3^4}{2^6}, \Theta^{10}_{3/2,\tau_{3/2}} = {\tau_{3/2}}^{10} \cdot \frac{3^{10}}{2^{15}}.$$

These equations imply:

$$\tau_{6/5} \cdot \frac{6}{5} = {\tau_{3/2}}^{-3} \cdot \frac{2^5}{3^3}, \tau_{5/4} \cdot \frac{5}{4} = {\tau_{3/2}}^{4} \cdot \frac{3^4}{2^6}, \tau_{7/4} \cdot \frac{7}{4} = {\tau_{3/2}}^{10} \cdot \frac{3^{10}}{2^{15}}.$$

Thus,

$$\begin{aligned} \tau_{6/5} &= {\tau_{3/2}}^{-3} \cdot \tfrac{80}{81}, \\ \tau_{5/4} &= {\tau_{3/2}}^{4} \cdot \tfrac{81}{80}, \\ \tau_{7/4} &= {\tau_{3/2}}^{10} \cdot \tfrac{59049}{57344}. \end{aligned} \tag{9}$$

The consequences from the formulas (9) we collect into the following

Theorem 2 *For every Petzval tone system $S_1 \in \mathfrak{S}$ of the first type,*

- $\tau_{6/5} \cdot \tau_{5/4} = \tau_{3/2}$;

- *the major scale (based on the value $\Theta^i_{3/2,\tau_{3/2}}$):*

$$D_i^{(major)} = \{\Theta^i_{3/2,\tau_{3/2}}, \Theta^{i+2}_{3/2,\tau_{3/2}}, \Theta^{i+4}_{3/2,\tau_{3/2}}, \Theta^{i-1}_{3/2,\tau_{3/2}}, \Theta^{i+1}_{3/2,\tau_{3/2}}, \Theta^{i+3}_{3/2,\tau_{3/2}}, \Theta^{i+5}_{3/2,\tau_{3/2}}, \Theta^{i}_{3/2,\tau_{3/2}}\};$$

- *the minor scale(based on the value $\Theta^i_{3/2,\tau_{3/2}}$):*

$$D_i^{(minor)} = \Theta^i_{3/2,\tau_{3/2}}, \Theta^{i+2}_{3/2,\tau_{3/2}}, \Theta^{i-3}_{3/2,\tau_{3/2}}, \Theta^{i-1}_{3/2,\tau_{3/2}}, \Theta^{i+1}_{3/2,\tau_{3/2}}, \Theta^{i-4}_{3/2,\tau_{3/2}}, \Theta^{i-2}_{3/2,\tau_{3/2}}, \Theta^{i}_{3/2,\tau_{3/2}}\};$$

- *there is only one whole tone interval, $W_1 = {\tau_{3/2}}^2 \cdot 9/8$;*
- *there are two semitones: $\Delta_1 = {\tau_{3/2}}^{-5} \cdot 2^8/3^5$ (the minor semitone) and $\Delta_2 = {\tau_{3/2}}^7 \cdot 3^7/2^{11}$ (the major semitone), and $W_1 = \Delta_1 \cdot \Delta_2$;*
- *$\Delta_1 = \Delta_2$ is equivalent to $\tau_{3/2} = \sqrt[12]{2^{19}/3^{12}}$ (the temperature of the E_{12} fifth);*
- *the major and minor scales based on tones $\Theta^i_{3/2,\tau_{3/2}}$ and $\Theta^{i+3}_{3/2,\tau_{3/2}}$ are related: if we take the basic value $\Theta^i_{3/2,\tau_{3/2}}$ for the major scale and $\Theta^{i+3}_{3/2,\tau_{3/2}}$ for minor scale, then both these scales consist of the same numbers;*
- *the major scale is a union of two equal tetrachords:*

$$\{\Theta^{i+0}_{3/2,\tau_{3/2}}, \Theta^{i+2}_{3/2,\tau_{3/2}}, \Theta^{i+4}_{3/2,\tau_{3/2}}, \Theta^{i-1}_{3/2,\tau_{3/2}}\}$$

and

$$\{\Theta^{(i+0)+1}_{3/2,\tau_{3/2}}, \Theta^{(i+2)+1}_{3/2,\tau_{3/2}}, \Theta^{(i+4)+1}_{3/2,\tau_{3/2}}, \Theta^{(i-1)+1}_{3/2,\tau_{3/2}}\}.$$

Remark 4 We see also that the reasonable are only the tone systems of the first type with $\tau_{3/2} < 1$ (since if $\tau_{3/2} \geq 1$, then $\tau_{5/4} > 81/80$, analogously $\tau_{6/5}$. Also, $\tau_{7/4} > 59049/57344$ is a big mistuning of the sevenths). Also, here is the reason why we choose $\Theta^{10}_{3/2,\tau_{3/2}}$ and not $\Theta^{-2}_{3/2,\tau_{3/2}}$: the corresponding $\tau_{7/4} = 64/(63{\tau_{3/2}}^2)$ which is not appropriate for $\tau_{3/2} < 1$.

Example 7 $\tau_{3/2} = 1, \tau_{5/4} = 81/80, \tau_{6/5} = 80/81, \tau_{7/4} = 64/63$. *The Pythagorean tone system.*

Example 8 $\tau_{3/2} = 1 - 1/886, \tau_{5/4} = 1 + 1/126, \tau_{6/5} = 1 - 1/111, \tau_{7/4} = 1 + 1/5$. A rational approximation of E_{12}.

Example 9 $\tau_{3/2} = \tau_{5/4} = \sqrt[3]{80/81}, \tau_{6/5} = 1$. *The open tone system with the exact minor third.*

Example 10 *The Opelt's system, cf. (Helmholtz, 1877), the approximation of the previous tone system, a cyclic system, $N = 19, x = 11$:*

$$\tau_{3/2} \approx 239/240, \tau_{5/4} \approx 234/235, \tau_{6/5} \approx 11561/11560, \tau_{7/4} \approx 80/81.$$

$$\begin{tikzcd}
\mathcal{P}_{\Delta_1,\Delta_2} \arrow[r, "\vartheta"] \arrow[d, hook, "\zeta"'] & \mathcal{Z}^2 \arrow[d, hook, "\eta"] \\
\mathcal{R}_{\Delta_1,\Delta_2} & \mathcal{C} \arrow[l, "\vartheta_*"']
\end{tikzcd}$$

Figure 1: The embedding ζ

Images in the complex plane For Δ_1, Δ_2, the minor and major semitones, denote by

$$\mathcal{P}_{\Delta_1,\Delta_1} = \{\Delta_1^\alpha \Delta_2^\beta; \alpha, \beta \in \mathcal{Z}\}.$$

Consider the map $\vartheta : \mathcal{P}_{\Delta_1,\Delta_2} \to \mathcal{Z}^2, \vartheta(\Delta_1^\alpha \Delta_2^\beta) = (\alpha, \beta)$. Then

$$\vartheta((\Delta_1^\alpha \Delta_2^\beta)(\Delta_1^\gamma \Delta_2^\delta)) = (\alpha + \gamma, \beta + \delta),$$
$$\vartheta((\Delta_1^\alpha \Delta_2^\beta)^\gamma) = (\gamma\alpha, \gamma\beta), \alpha, \beta, \gamma, \delta \in \mathcal{Z}.$$

Embed $\mathcal{Z}^2$ identically into the complex plane $\mathcal{C}$. Denote this injection by η. Denote by ϑ_*^{-1} the extended map $\vartheta_*^{-1} : \mathcal{C} \to \mathcal{R}_{\Delta_1,\Delta_2}$, where $\mathcal{R}_{\Delta_1,\Delta_2} = \{\Delta_1^\alpha \Delta_2^\beta; \alpha, \beta \in \mathcal{R}\}$. So we have the following commutative diagram, cf. Figure 1, which defines the embedding $\zeta : \mathcal{P}_{\Delta_1,\Delta_2} \to \mathcal{R}_{\Delta_1,\Delta_2}$.

In Figure 5, we see the images of major and minor scales of tone systems of the first type in the map ϑ.

Remark 5 From this image it is clear, cf. (Haluška, 2000), that the Petzval's tone system of the first type is a direct generalization of the Pythagorean Tuning structure.

Cyclic tone systems of the first type

Lemma 5 *Let D_p be the union of $p+1$ major and $p+1$ minor scales over the set:*

$$\{\Theta^0_{3/2,\tau_{3/2}}, \Theta^1_{3/2,\tau_{3/2}}, \cdots, \Theta^p_{3/2,\tau_{3/2}}\}. \tag{10}$$

Let S be the tone system of the first type. Then the set D_p consists of $(p+10)$ values.

Proof. For the construction of $(p+1)$, major and $(p+1)$ minor scales over the set (10) ("the spiral of $p+1$ generalized fifths"), there is used the following sequence of $(p+10)$

values:

$$\begin{array}{rl} D_p = & \bigcup_{i=0}^{p}(D_i^{(major)} \cup D_i^{(minor)}) \\ = & \{\Theta_{3/2,\tau_{3/2}}^{-4}, \Theta_{3/2,\tau_{3/2}}^{-3}, \Theta_{3/2,\tau_{3/2}}^{-2}, \Theta_{3/2,\tau_{3/2}}^{-1}, \\ & \overbrace{\Theta_{3/2,\tau_{3/2}}^{0}, \Theta_{3/2,\tau_{3/2}}^{1}, \cdots, \Theta_{3/2,\tau_{3/2}}^{p}}^{p+1}, \\ & \Theta_{3/2,\tau_{3/2}}^{p+1}, \Theta_{3/2,\tau_{3/2}}^{p+2}, \Theta_{3/2,\tau_{3/2}}^{p+3}, \Theta_{3/2,\tau_{3/2}}^{p+4}, \Theta_{3/2,\tau_{3/2}}^{p+5}\}. \end{array}$$

The main advantage of cyclic tone systems of the first type consists of the following:

Lemma 6 *Let S_1 be a Petzval's cyclic tone system of the first type. Let $D_p \subset S_1$ be the union of p major and p minor scales over the set*

$$\Theta_{3/2,\tau_{3/2}}^{0}, \Theta_{3/2,\tau_{3/2}}^{1}, \cdots, \Theta_{3/2,\tau_{3/2}}^{p}.$$

Then the set D_p consists only of $(p+1)$ values.

Proof.

Since the tone system is cyclic, there exist n, m such that (7). Put $p = m - 1$. Then

$$\begin{array}{rl} 2^n\Theta_{3/2,\tau_{3/2}}^{-4} & = \Theta_{3/2,\tau_{3/2}}^{p-3}, \\ 2^n\Theta_{3/2,\tau_{3/2}}^{-3} & = \Theta_{3/2,\tau_{3/2}}^{p-2}, \\ 2^n\Theta_{3/2,\tau_{3/2}}^{-2} & = \Theta_{3/2,\tau_{3/2}}^{p-1}, \\ 2^n\Theta_{3/2,\tau_{3/2}}^{-1} & = \Theta_{3/2,\tau_{3/2}}^{p}, \\ 2^n\Theta_{3/2,\tau_{3/2}}^{0} & = \Theta_{3/2,\tau_{3/2}}^{p+1}, \\ 2^n\Theta_{3/2,\tau_{3/2}}^{1} & = \Theta_{3/2,\tau_{3/2}}^{p+2}, \\ 2^n\Theta_{3/2,\tau_{3/2}}^{2} & = \Theta_{3/2,\tau_{3/2}}^{p+3}, \\ 2^n\Theta_{3/2,\tau_{3/2}}^{3} & = \Theta_{3/2,\tau_{3/2}}^{p+4}, \\ 2^n\Theta_{3/2,\tau_{3/2}}^{4} & = \Theta_{3/2,\tau_{3/2}}^{p+5}. \end{array}$$

We have:

$$\begin{array}{rl} D_p = & \{\Theta_{3/2,\tau_{3/2}}^{-4}, \Theta_{3/2,\tau_{3/2}}^{-3}, \ldots \Theta_{3/2,\tau_{3/2}}^{p+3}, \Theta_{3/2,\tau_{3/2}}^{p+4}, \Theta_{3/2,\tau_{3/2}}^{p+5}\} \\ \equiv & \{\Theta_{3/2,\tau_{3/2}}^{0}, \Theta_{3/2,\tau_{3/2}}^{1}, \cdots, \Theta_{3/2,\tau_{3/2}}^{p}\} \end{array},$$

where $\equiv$ denotes the octave equivalence.

Let the octave be decomposed into $p+1$ smaller intervals, segments (semitones in the case of 12 degree tone systems). These segments are of two types – minor and major. Let the minor segment has m equal elementary intervals and the major segment has $m+n, m > n$ equal elementary intervals. Then the major or minor scales have $5(m+2n) + 2(m+n) = p+1$, i.e.

$$7m + 12n = p + 1$$

elementary intervals.

In the following theorem there are equal tempered tone systems of the first type. The proof is trivial and omitted.

Theorem 3 *If $m = 0$, then*

$$p + 1 = 12, 24, 36, 48, 60, 72, \ldots, 612, \ldots.$$

So, we obtained specially the 12-tone equal temperature, 24-tone (1/4-tone) equal temperature, 36-tone (1/6-tone) equal temperature, 72-tone (1/12-tone) equal temperature (considered in 1920-ties by Alois Hába, cf. (Hába, 1922), 612-tone system was considered by Josef Sumec in 1917.

Theorem 4 *If $m = 1$, then*

$$p + 1 = 19, 31, 43, 55, 67, \ldots.$$

The corresponding temperatures are:

$$0.99584, 0.997012, 0.997530, 0.997822, 0.998009, \ldots$$

Proof. $1.0824{\tau_{3/2}}^{19} = 1$, $1.0972{\tau_{3/2}}^{31} = 1$, $1.1122{\tau_{3/2}}^{43} = 1, \ldots.$

These systems are also well known in the literature in the present days. An article which characterizes the very special properties of 12-, 19- and 31-tone scales is (Balzano, 1980).

If $m = 2$, then $p + 1 = 38$, 50, 62, ... Etc. for $m = 3, 4, \ldots$. However, for $m \geq 2$ the fifth temperatures $\tau_{{3/2}_m}$ are greater than those in the cases $m = 0$ or 1.

Note that some tone systems for higher m are supersets of those for lower m. E.g., the 38-tone system contains the 19-tone system, 62-tone contains the 31-tone system as their proper subset.

4.2 Tone Systems II

Properties For the Petzval's tone systems of the second Type we have the following expressions:

$$\Theta^{9}_{3/2,\tau_{3/2}} = {\tau_{3/2}}^{9} \cdot \frac{3^9}{2^{14}}, \Theta^{-8}_{3/2,\tau_{3/2}} = {\tau_{3/2}}^{-8} \cdot \frac{2^{13}}{3^8}, \Theta^{-14}_{3/2,\tau_{3/2}} = {\tau_{3/2}}^{-11} \cdot \frac{2^{23}}{3^{11}}.$$

These equations imply:

$$\tau_{6/5} \cdot \frac{6}{5} = {\tau_{3/2}}^{9} \cdot \frac{3^9}{2^{14}}, \tau_{5/4} \cdot \frac{5}{4} = {\tau_{3/2}}^{-8} \cdot \frac{2^{13}}{3^8}, \tau_{7/4} \cdot \frac{7}{4} = {\tau_{3/2}}^{-11} \cdot \frac{2^{23}}{3^{11}}.$$

Thus,

$$\begin{aligned} \tau_{6/5} &= {\tau_{3/2}}^{9} \cdot \tfrac{5\cdot 3^8}{2^{15}}, \\ \tau_{5/4} &= {\tau_{3/2}}^{-8} \cdot \tfrac{2^{15}}{5\cdot 3^8}, \\ \tau_{7/4} &= {\tau_{3/2}}^{-14} \cdot \tfrac{2^{25}}{7\cdot 3^{14}}. \end{aligned} \tag{11}$$

The consequences from the formulas (11) we collect into the following

Theorem 5 *For every Petzval tone system S_2 of the second type,*

- $\tau_{5/4} \cdot \tau_{6/5} = \tau_{3/2}$;
- *the major scale (based on the value* $\Theta^i_{\tau_{3/2}}$*):*

$$\begin{aligned} D_i^{(major)} = & \{\Theta^i_{3/2,\tau_{3/2}}, \Theta^{i+2}_{3/2,\tau_{3/2}}, \Theta^{i-8}_{3/2,\tau_{3/2}}, \Theta^{i-1}_{3/2,\tau_{3/2}}, \\ & \Theta^{i+1}_{3/2,\tau_{3/2}}, \Theta^{i-9}_{3/2,\tau_{3/2}}, \Theta^{i-7}_{3/2,\tau_{3/2}}, \Theta^{i}_{3/2,\tau_{3/2}}\}; \end{aligned}$$

- *the minor scale (based on the value* $\Theta^i_{\tau_{3/2}}$*):*

$$\begin{aligned} D_i^{(minor)} = & \{\Theta^i_{3/2,\tau_{3/2}}, \Theta^{i+2}_{3/2,\tau_{3/2}}, \Theta^{i+9}_{3/2,\tau_{3/2}}, \Theta^{i-1}_{3/2,\tau_{3/2}}, \\ & \Theta^{i+1}_{3/2,\tau_{3/2}}, \Theta^{i+8}_{3/2,\tau_{3/2}}, \Theta^{i+10}_{3/2,\tau_{3/2}}, \Theta^{i}_{3/2,\tau_{3/2}}\}; \end{aligned}$$

- *the major and minor scales based on tones* $\Theta^i_{3/2,\tau_{3/2}}$ *and* $\Theta^{i-9}_{3/2,\tau_{3/2}}$ *are related: if we take the basic value* $\Theta^i_{3/2,\tau_{3/2}}$ *for the major scale and* $\Theta^{i-9}_{3/2,\tau_{3/2}}$ *for minor scale, then both these scales consist of the same numbers;*
- *there are two whole tone intervals:* $W_1 = {\tau_{3/2}}^2 \cdot 9/8$ *(the major whole tone) and* $W_2 = {\tau_{3/2}}^{-10} \cdot 2^{16}/3^{10}$ *(the minor whole tone);*
- *there are two semitones:* $\Delta_2 = {\tau_{3/2}}^7 \cdot 3^7/2^{11}$ *(the major semitone) and* $\Delta_3 = {\tau_{3/2}}^{17} \cdot 2^{27}/3^{17}$ *(the "cross" semitone);*
- *the following equations holds:*

$$\frac{\Delta_2}{\Delta_3} = \left({\tau_{3/2}}^{12} \cdot \frac{3^{12}}{2^{19}}\right) = \left(\frac{W_1}{W_2}\right)^2, \Delta_3 = \frac{\Delta_1^2}{\Delta_2}.$$

Remark 6 The condition $\tau_{3/2} < 1$ is superfluous in this case, we can choose $\tau_{3/2} < 1, \tau_{3/2} > 1$, or $\tau_{3/2} = 1$. For $\tau_{3/2} = 1$, we have: $\tau_{5/4} = 885/886, \tau_{6/5} = 886/885, \Theta_{7/4} = 455/454$. For $\tau_{3/2} < 1$, we obtain tone systems with purer thirds. For $\tau_{3/2} > 1$, we obtain tone systems with purer sevenths.

There in no tetrachordal structure of the major scale.

Images in the complex plane In Figure 6, we see the images of major and minor scales of Tone systems of the second type in the map ϑ.

Remark 7 From this image it is clear, cf. (Haluška, 1998), that the Petzval's tone system of the second type is a direct generalization of Just Intonation.

Cyclic tone systems of the second type, Petzval tunig Let the octave be decomposed into $p+1$ smaller intervals, segments (semitones in the case of 12 degree tone systems). The smallest segment is the interval between the major and minor whole tones, let it contains m disjoint equal elementary intervals. The minor semitone let has $n, n > m$, elementary intervals. Then the major semitone has $n+2m$ elementary intervals. Then the major and minor scales contain totally $3(2n+3m) + 2(2n+2m) + 2(n+2m)$ elementary intervals, i.e.

$$12n + 17m = p + 1.$$

In the following theorem there are equal tempered tone systems of the second type. The proof is trivial and omitted.

Theorem 6 *If $m = 0$, then*

$$p + 1 = 12, 24, 36, 48, 60, 72, \ldots, 612, \ldots.$$

We see that the equal tempered Petzval's tone systems of the first and second types coincide.

Theorem 7 *If $m = 1$, then*

$$p + 1 = 41, 53, 65, 77, \ldots.$$

The corresponding temperatures are:

$$0.983512, 0.999961, 0.999758, 0.999621, \ldots$$

Example 11 *The 53-tone cyclic system, "Petzval tuning". Here $m = 1$, the minor semitone ($\sqrt[53]{2^8} \approx 26/25$) contains 3 elementary intervals, the major semitone ($\sqrt[53]{2^9} \approx 1.067577$) has 5 ones, the minor whole tone ($\approx$1.110295) has 8, and the major whole tone ($\approx$1.124911) has 9 elementary intervals,*

$$\tau_{3/2} = -25381/25382, \tau_{5/4} = 1229/1230,$$
$$\tau_{6/5} = 1293/1292, \tau_{7/4} = 364/363.$$

Theorem 8 *If $m = 2$, then*

$$p + 1 = 70, 82, 94, 106, \ldots.$$

We can consider $m = 3, 4, \ldots$. However, these systems seem to bring no new or interesting quality from the acoustic viewpoint.

5 BIMESURES AND THE LAST SQUARE METHOD

What about the interplay between two temperaments of a system S? We need to construct a measure for the distance between two uncertain sets.

Let $\Sigma^{(a)}$ and $\Sigma^{(m)}$, be the psychoacoustic and musical temperaments of an (unknown) tone system $S \in \mathfrak{S}$ which we would like to be well tempered.

There are questions: given musical temperament or its structure, find concrete S with optimal psychoacoustic temperament in some sense. Specially, to find S with the optimal psychoacoustic temperament when the musical mistuning is arbitrary.

The following uncertainty bimeasure is motivated with the method of last squares:

$$\lambda(\Sigma^{(a)}, \Sigma^{(m)}) = \frac{1}{\sum_{\phi \in L} w_\phi [\mu_\phi^{(m)} - \mu_\phi^{(a)}]^2}, \tag{12}$$

where

$$\mu_\phi^{(m)} = \min_{f \in ker(\Sigma^{(m)}) \cap [1;2)} \left| 1 - \frac{f}{\phi} \right| \cdot sgn\left(1 - \frac{f}{\phi}\right),$$

$$\mu_\phi^{(a)} = \min_{f \in ker(\Sigma^{(a)}) \cap [1;2)} \left|1 - \frac{f}{\phi}\right| \cdot sgn\left(1 - \frac{f}{\phi}\right),$$

and $\phi \in \{6/5, 5/4, 3/2, 7/4\} = L$, $\sum_{\phi \in L}, w_\phi = 1, w_\phi > 0$.

Note that Definition 8 works for the case of crisp situation up to this place and the uncertainty context is a direct generalization of the crisp case. In this section we will demonstrate results essentially using the uncertainty based information theory and such that the reduction to crisp case is trivial and pure.

The following Subsection 5.1 concerns the Petzval's tone systems of the first type.

5.1 Optimal Temperatures I, Huygens – Fokker Tuning

Let us search a tone system $S = \Sigma^{(a)}$ among the Petzval tone systems of the first type. Substitute (9) into the function $\lambda(\Sigma^{(a)}, \Sigma^{(m)})$, see (12). For the sake of simplicity, let $w_{6/5} = w_{5/4} = w_{3/2} = w_{7/4}$ in Equation (12).

The function $\lambda(\Sigma^{(a)}, \Sigma^{(m)})$ is then a function of the variable $\tau_{3/2}$. We evaluate the maximum of the function $\lambda(\Sigma^{(a)}, \Sigma^{(m)})$ applying the well-known method from the mathematical analysis (the first derivative put to zero, the second derivative should be negative in the solution points, etc.). We obtain the following algebraic equation of the 26th degree with respect to psychoacoustic temperature $\tau_{3/2} = {\tau_{3/2}}^{(a)}$:

$$\begin{aligned} -2(\tau_{3/2}^{(m)} - \tau_{3/2}) - 8 \cdot \tfrac{81 \cdot \tau_{3/2}^3}{80} \left(\tau_{5/4}^{(m)} - \tfrac{81 \cdot \tau_{3/2}^4}{80}\right) \\ +6 \cdot \tfrac{80}{81 \cdot \tau_{3/2}^4} \left(\tau_{6/5}^{(m)} - \tfrac{80}{81 \cdot \tau_{3/2}^3}\right) + 20 \cdot \tau_{3/2}^9 \left(\tau_{7/4}^{(m)} - \tfrac{59049 \cdot \tau_{3/2}^{10}}{57344}\right) = 0. \end{aligned} \tag{13}$$

Since $\mu_{3/2}^{(m)}, \mu_{5/4}^{(m)}, \mu_{6/5}^{(m)}, \mu_{7/4}^{(m)}$ we can choose arbitrary within reasonable boundaries, let $\mu_{3/2}^{(m)} = 0, \mu_{5/4}^{(m)} = 0, \mu_{6/5}^{(m)} = 0, \mu_{7/4}^{(m)} = 0$ (pure intervals). As the result, we obtained the following "ideal" temperature of the searched tone system $S \in \mathfrak{S}$:

$$\tau_{3/2} \approx 0.997224, \tau_{5/4} \approx 1.001305, \tau_{6/5} \approx 0.995925, \tau_{7/4} \approx 1.001505.$$

In the following two examples we will suppose that we have some *appriori* knowledge about the structure of the set $\{\mu_{3/2}^{(m)}, \mu_{5/4}^{(m)}, \mu_{6/5}^{(m)}, \mu_{7/4}^{(m)}\}$.

Example 12 *(Huygens – Fokker tuning)*
Let $\tau_{3/2}^{(m)} = \tau_{5/4}^{(m)} = \tau_{6/5}^{(m)} = \tau_{7/4}^{(m)}$. *The equation (13) implies* $\mu_{3/2} \approx 1/336$ *and*

$$\tau_{3/2} \approx 335/336, \tau_{5/4} \approx 2013/2012, \tau_{6/5} \approx 287/288, \tau_{7/4} \approx 335/336.$$

The closest cyclic tone system is given with $x/N = 18/31$*, i.e. a 31-tone tone system with the tempered fifth on the* 18*-th step. This tone system is known as the Huygens – Fokker tuning, cf. (Fokker, 1955).*

Example 13 *(A 43-tone system)*

Let $\tau_{3/2}^{(m)} = \tau_{5/4}^{(m)}/5 = \tau_{6/5}^{(m)}/5 = \tau_{7/4}^{(m)}/5$. *The equation (13) implies*

$$\mu_{3/2} \approx 1/397, \tau_{3/2} \approx 396/397.$$

The closest cyclic system is for $x/N = 25/43$, *i.e. a 43-tone system with the tempered fifth on the* 25*-th step.*

5.2 Optimal Temperatures II

Not bringing the boring computations of the expression (13), for the case of the Petzval's tone system of the second type with the minimal mistuning we have the following results:

$$\tau_{3/2} \approx 1.000034, \tau_{5/4} = 0.9986, \tau_{6/5} \approx 1.001436, \tau_{7/4} \approx 1.001722.$$

We see that all basic intervals are mistuned a tiny amount, so we hear them as pure. The Petzvals tone systems of the second type are excellent. However, there is a cost to be paid with the relatively large number of steps per octave.

Acknowledgement. The author thanks to G. J. Klir for his kind gift of the book Klir and Wierman (1997). He is also grateful to J. Beran for the discussion about statistical investigations of weight values in uncertainty measures for concrete compositions.

REFERENCES

Beran, J. and Mazzola, G. (1999), "Visualising the relationship between time series by hierarchical smoothing." *Journal of computational and graphical statistics* **8**, pp. 213 – 238.

Balzano, G. J. (1980), "The group-theoretic description of twelvefold and micro-tonal pitch systems." *Computer music journal* **5**, pp. 66-84.

Barbour, J. M. (1948), "Music and Ternary continued fractions." *Amer. Math. Monthly*, **55**, p. 545.

Barbour, J. M. (1951), *Tuning and Temperament: A Historical survey*, Michigan State College Press, East Lancing. Reprint: Da Capo, New York, 1973.

Benade, A. (1976), *Fundamentals of Musical Acoustics*, Oxford University Press. Reprint: Dover, New York, 1990.

Boomsliter, P. and Creel, W. (1961), "The long pattern hypothesis in harmony and hearing." *Journal of Music Theory* **5**, pp. 2 – 30.

Erményi, L. (1904), *Petzvals Theorie der Tonsysteme*, B. G. Teubner, Leipzig.

Feichtinger, H. and Dörfler, M. (eds.) (1999), *Diderot forum on mathematics and music*, Österreichische Computer Gesellschaft, Vienna.

Fokker, A. (1955), "Equal temperament and the 31-keyed organ." *Scientific Monthly* **81**, pp. 161 – 166.

Garbuzov, N. A. (1948), *The zonal nature of the human aural perception* (in Russian), Izdatelstvo Akademii Nauk SSSR, Moscow – Leningrad.

Hába, A. (1922), *Harmonic foundation of the 1/4-tone system* (in Czech), Hudební Matice Umělecké besedy, Prague.

Haluška, J. (1998), “Comma 32 805/32 768.” *The Int. Journal of Uncertainty, Fuzziness and Knowledge - Based Systems*, **6**, pp. 295 – 305.

Haluška, J. (2000), “Equal Temperament and Pythagorean tuning: a geometrical interpretation in the plane.” *Fuzzy Sets and Systems*, **114**(2), pp. 87 – 95.

Hellegouarch, Y. (1982), “ Scales.” *C. R. Math. Rep. Acad. Sci. Canada* **4**, no. 5, 277 – 281.

Helmholtz, H. (1877), *On the sensation of tone*, translated by A. J. Ellis (1954), Dover Publication, New York.

Jorgenson, O. H. (1991), *Tuning: Containing the Perfection of 18th-century Temperament; the Lost Art of 19th-century Temperament; and the Science of Equal Temperament*, East Lancing, Michigan State University Press.

Klir, G. J. and Wierman, M. J. (1997), *Uncertainty-Based Information, Elements of Generalized Information Theory*, Creighton University, Omaha.

Kuipers L. and Niederreiter H.(1974), *Uniform distribution of sequences*, Wiley, London.

Mazzola, G. (1990), *Geometrie der Töne*, Birkhäuser Verlag, Basel – Boston – Berlin.

Neuwirth, E. (1997), *Musical Temperaments*, Springer Verlag, Berlin – New York.

Romanowska, A. (1979), ”On some algebras connected with the theory of harmony”, *Colloquium Mathematicum*, **41**, pp. 181 – 185.

Sethares, W. (1998), *Tuning, Timbre, Spectrum, Scale*, Springer Verlag, Berlin – New York.

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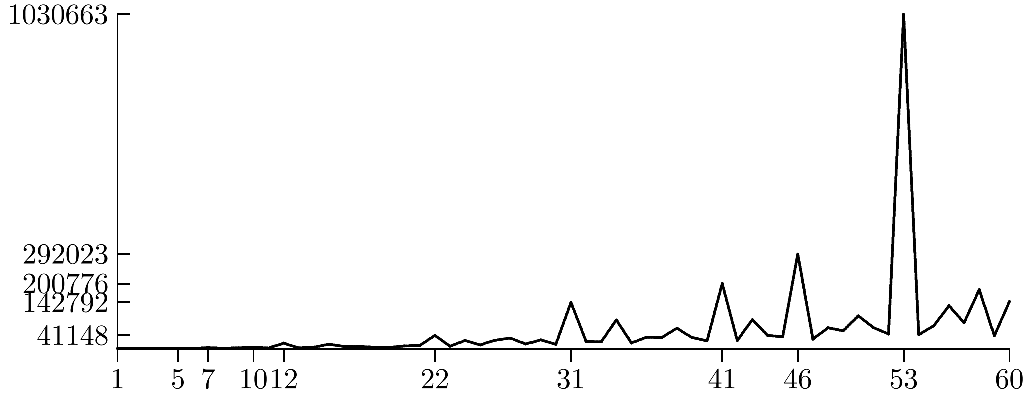

Figure 2: $\sigma(E_N)$, $N = 5, 6, \ldots, 60$.

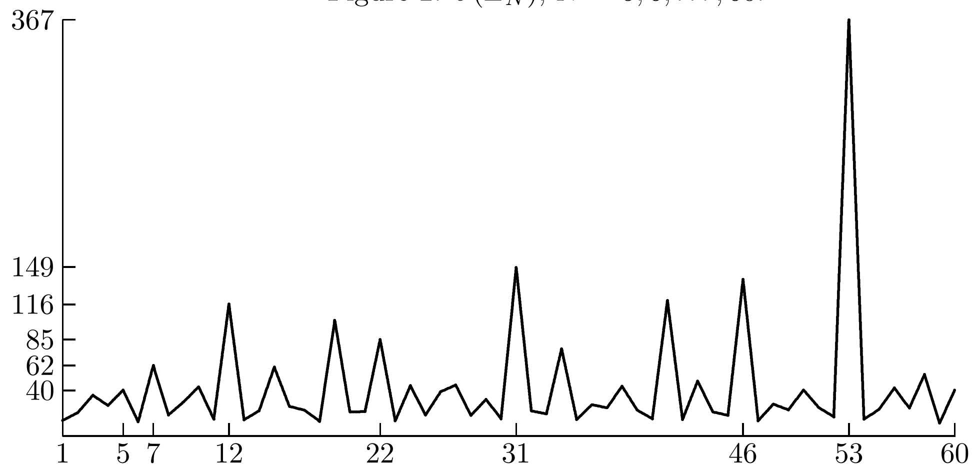

Figure 3: $\sigma_N^{(1)}(E_N)$, $N = 5, 6, \ldots, 60$.

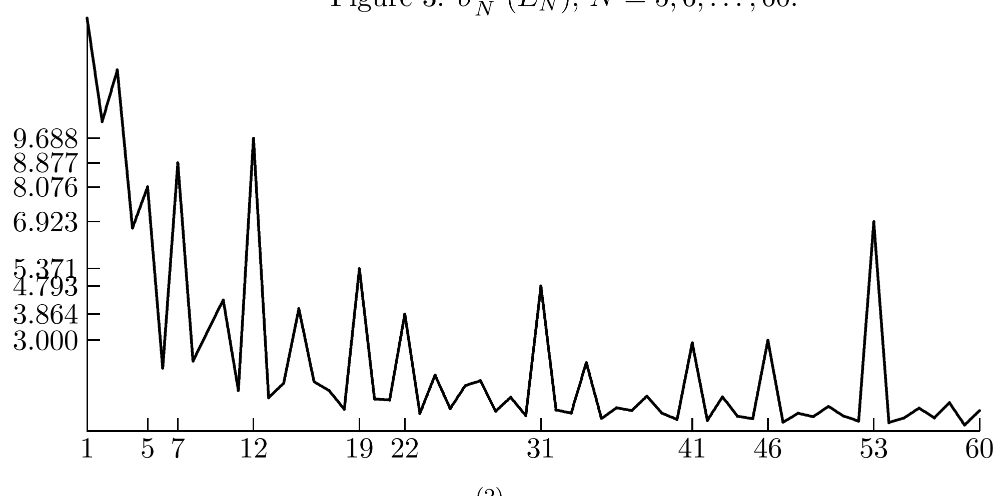

Figure 4: $\sigma_N^{(2)}(E_N)$, $N = 5, 6, \ldots, 60$.

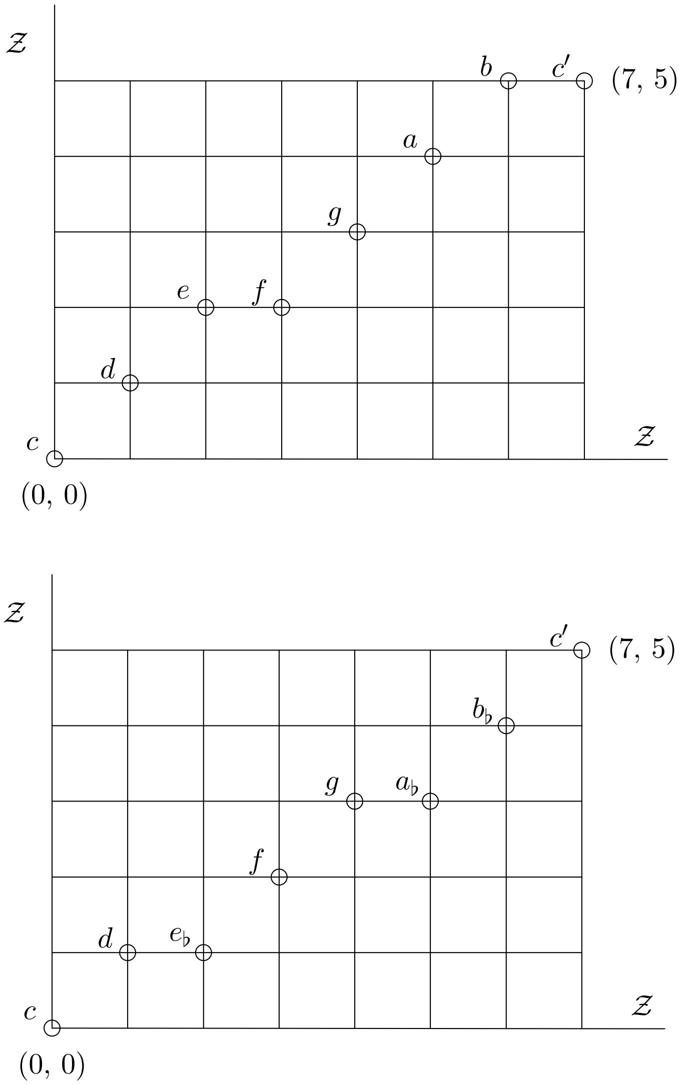

Figure 5: Major and minor scales of Tone systems of the first type in ϑ

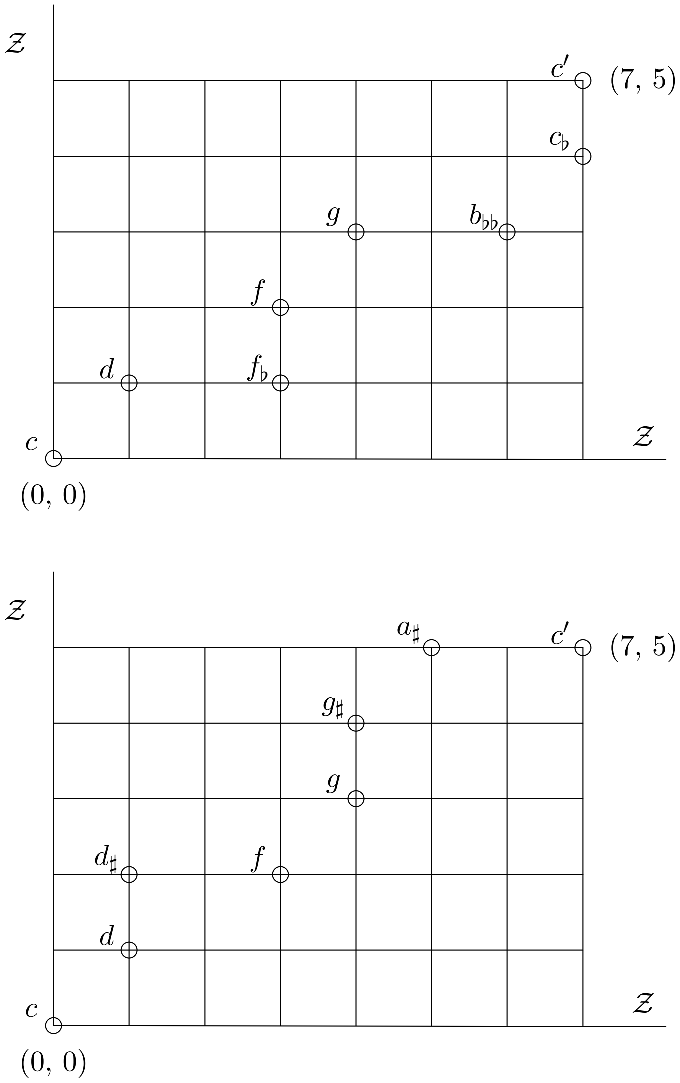

Figure 6: Major and minor scales of Tone systems of the second type in ϑ