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Equal Temperament and Pythagorean Tuning: a geometrical interpretation in the plane

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Abstract

We show the following: Pythagorean Tuning (17 valued) and Equal Temperament (12 valued) can be canonically represented as discrete sets of the plane and there is a natural parallel projection of Pythagorean Tuning to the points of Equal Temperament. This fact implies particularly that when performing a composition on two instruments simultaneously in both Pythagorean Tuning and Equal Temperament, then the Garbuzov zones (supports of tone fuzzy sets) can be considered as segments on parallel lines in the plane.

Keywords: Fuzzy sets of tones, Relation between Pythagorean Tuning and Equal Temperament, Many valued coding of information, Application to Mathematical theory of music.

1 Garbuzov zones

Some musical acousticians assert that the twelve-degree musical scales appeared not as merely artificial convention but, rather, that the natural development of musical culture led to aural selection of qualitative degrees, interval zones, each of them having its own, particular degree of individuality. In other words, there is a finite number of zones of music intervals in an octave. It seems that N. A. Garbuzov was the first who applied the fuzzy theory (in a naive form) when considering interval zones with regard to tuning in music, see [2], Table 4 and Table 5 (= Table 1 in this paper; Garbuzov zones of melodic musical intervals). These tables are statistics of hundreds measurements.

unison	(-12, 12)
minor second	(48, 124)
major second	(160, 230)
minor third	(272, 330)
major third	(372, 430)
fourth	(472, 530)
tritone	(566, 630)
fifth	(672, 730)
minor sixth	(766, 830)
major sixth	(866, 930)
minor seventh	(966, 1024)
major seventh	(1066, 1136)

Table 1: Garbuzov zones [in cents]

In this paper, we deal with a fuzziness of tunings. To each note (= a fuzzy set) in a score, there is a membership function of tones defined on the Garbuzov zone (= the support of the fuzzy set). The values of this membership function may be chosen from a discrete set (the theoretical case of tunings) or there is a continuous function (the typical situation; every assemble play).

The tone membership functions may take values derived from one or more theoretical tunings. On the other hand, theoretical tunings often uses two or more values for one qualitative degree. If there is only one non-zero value in each degree zone, we have no fuzziness. But this situation is only a theoretical possibility which never occur in real live music.

The choice of tunings should be made on the basis of acoustical, psychological, and mathematical principles and relations within a given composition. Indeed, there are known many types of theoretical tunings in literature, cf. [9].

Psychologically (i.e. after the sound was produced), the tuning in music is an example of the fact that the human perceptive mechanism uses various systems of fuzzy coding of information, see Section 2. Particularly, 17-tone tuning systems for the 12 qualitative different music interval degrees (e.g. Pythagorean Tuning and Just Intonation, [7], [12]) are well-known. The main requirement is that the subject's perceptive system would be able to encode the informational content unambiguously.

We show the following connection between the two most used (theoretical) tunings, Pythagorean Tuning (17 valued) and Equal Temperament (12 valued). These two systems can be canonically represented as discrete sets of the plane and there is a natural parallel projection of Pythagorean Tuning to the points of Equal Temperament, (see Figure 3). This connection was firstly observed in [4]. **This fact implies particularly that when performing a composition on two instruments simultaneously in both Pythagorean Tuning and Equal Temperament, then the Garbuzov zones (supports of tone fuzzy sets) can be considered as segments on parallel lines in the plane.**

2 Examples

We demonstrate that **fuzziness with regard to tuning is relevant because there are present ever two or more different tone systems under consideration (sequentially or simultaneously)**. E. g., in the case of the piano and the 12-degree 12-tone Equal Tempered Scale, there is an interference among the played tones and their overtone systems. These tone systems are mutually incommensurable (except of octaves) and the interference modifies valuable the sound. These modifications depend on the concrete composition, player, instrument, set of hall acoustics, etc.

It is known that the sensitivity of human aural perception is 5–6 cents. The psychophysical boundary [to what listeners or players tend to expect] is 1–2 cents.

2.1 Sequential fuzziness

In [2], we can find the numerical data of the following experiment. The first 12 measures of Air from J. S. Bach's Suite in D ("On the G-String") were interpreted by three famous Russian violinists of that time: Oistrach, Elman, and Cimbalist. The piece of Bach's Air was chosen because (1) the tempo is Slow (*Lento*, M.M. $1/4 = 52$); (2) it consists of great number of pitches of various duration; (3) the piece has two parts, the second one is a repetition of the first one; (4) the piece is well-known, often recorded and interpreted; (5) it is played violin (not fix tuned instrument).

The recordings of Oistrach, Elman, and Cimbalist were numerical analysed with the accuracy of 5 cents.

The analysis scheme is in Figure 1 (ET denotes intervals which can be identified as Equal tempered, PT – Pythagorean, JI – pure [Just Intoned], and UI – unidentified, do not belong to the considered tunings Equal Temperament, Pythagorean Tuning, or Just Intonation), where (1) Oistrach produced 65 intervals (25 ET, 19 PT, 15 JI, 30 UI); (2) Elman produced 66 intervals (27 ET, 26 PT, 14 JI, 27 UI); (3) Cimbalist produced 70 intervals (18 ET, 13 PT, 11 JI, 49 UI). [1]

It is clear that we may consider also another set of theoretical tunings (e.g. Praetorius, Werckmeister, Neindhardt, Agricola, etc.) to have no "unidentified" intervals or to have a less number of the mentioned theoretical tunings covering the set of all produced intervals. However, we may also say that there are three (Oistrach's, Elman's, and Cimbalist's) unique tunings for Bach's Aria.

[1] The various number of intervals produced by three violinists depended only on the various manners how they played the trill in the second (fourteenth) measure. This fact is not important for the analysis.

2.2 Simultaneous fuzziness

It is known, that some string orchestra tend to play in Pythagorean Tuning, some in Just Intonation, and the large symphonic orchestras in Equal Temperament. It is also clear that there is no problem for two or more professional players (say Oistrach and Cimbalist) to play the Bach's Air together.

The effect of fuzziness of tuning (without any psychological interaction of players) was used specially when tuning some historical organs with two consoles which were tuned each to a different tuning.

In modern pipe organs, some special two-row registers (called double voices) are constructed on the physical principle of beats, the interference of sound waves. Such are e.g. the sound timbres named *Vox coelestis, Unda maris, Voce umana, Gemshornschwebung, Schwebend Harf*, cf. [1].

3 Two tunings

In the following sections we stress on the interplay between two theoretical tunings – Pythagorean and Equal Temperament. We will consider neither any concrete composition nor any psychological interactions of players.

Since the Pythagorean and Equal Tempered Tunings are well-known, a detailed algorithmic/theoretical explanation is not necessary. We recall briefly only the basic facts.

For the sake of tuning, it is reasonable to identify the tone pitch with its relative frequency to the frequency of a fundamental, fixed tone (conventionally, such a tone is usually taken $a^1 = 440$ Hz in the experience; we take $c = 1$ for simplicity). So, in fact, we deal only with relative frequencies of music intervals.

Pythagorean Tuning, cf. [12], Table 0.1, was created as a sequence of numbers of the form 2^p3^q, where p, q are integers. This tuning was established about five hundred years B. C. and used in the Western music up to the 14th century. It is often assumed that Pythagorean Tuning (especially with regard to the differentiation of " enharmonics") is/was used in performance. At present this tuning is mostly used when interpreting Gregorian chants.

Equal Tempered Scale (simply, Equal Temperament; known already to Andreas Werckmeister – this is obvious from his book "Erweiterte und verbesserte Orgel-Probe", 1698), has been widely used since the appearance of the collection of compositions "Das wohltemperierte Klavier" (1721, by Johann Sebastian Bach), and is commonly used in the present days. The sequence of real numbers (the ratios of pitch frequencies relative to the frequency of the first tone in the scale) $W = \{\sqrt[12]{2}^i;\ i = 0, 1, \ldots, 12\}$ defines fully this tuning.

4 Scales

Let us denote by $\mathcal{Z}, \mathcal{Q}, \mathcal{R}, \mathcal{C}$ the sets of all integer, rational, real, and complex numbers, respectively. If $\mathcal{L} = ((0, \infty), \cdot, 1, \leq)$ is the usual multiplicative group with the usual order on the real line and $a \leq b$, $a, b \in (0, \infty)$, then b/a is an $\mathcal{L}$-length of the interval (a, b). Since this terminology is not obvious, we borrow the usual musical terminology, i.e. we simply say that b/a is an interval. This inaccuracy does not lead to any misunderstanding because the term "interval" is used only in this sense in this paper.

The following definition is important for a visual interpretation of an interesting coherence between the Pythagorean and Equal Tempered Tunings.

Equal Tempered Scale can be generalized as follows, [5], [6].

Definition 1 Let $\{x, y, \ldots, z \in \mathcal{R}; 1 < x < y < \ldots < z < 9/8\}$ be a set of n numbers. Let $p_0, p_1, \ldots, p_m$, $q_0, q_1, \ldots, q_m$, ..., $r_0, r_1, \ldots, r_m$ be $m \times n$ nonnegative integers such that

$$\begin{gathered} 0 = p_0 \leq p_1 \leq \ldots \leq p_m \\ 0 = q_0 \leq q_1 \leq \ldots \leq q_m \\ \vdots \\ 0 = r_0 \leq r_1 \leq \ldots \leq r_m, \end{gathered}$$

and

$$p_j + q_j + \ldots + r_j = j,\ j = 0, 1, \ldots, m.$$

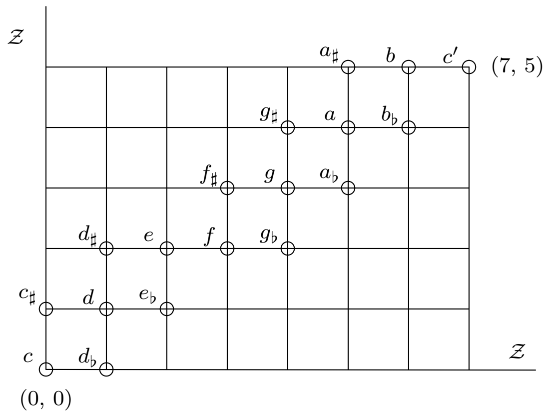

Figure 2: Image of Pythagorean Tuning in θ

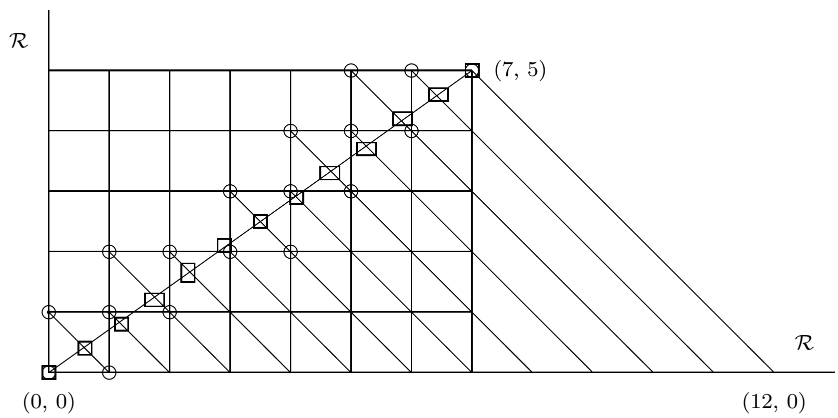

Figure 3: Images of Equal Temperament in the plane

Let

$$a_1, a_2, \ldots, a_k \in \mathcal{R}$$

be fixed numbers and $k \leq n$. Then the set

$$S = \{u_0 = x^{p_0} y^{q_0} \ldots z^{r_0},$$
$$u_1 = x^{p_1} y^{q_1} \ldots z^{r_1}, \ldots,$$
$$u_m = x^{p_m} y^{q_m} \ldots z^{r_m}\}$$

is said to be an
m-degree n-interval $(a_1, a_2, \ldots, a_k)$*-scale*
if there exist

$$i_1, i_2, \ldots, i_k \in \{0, 1, \ldots, m\},$$

such that

$$a_j = u_{i_j}, \ j = 1, 2, \ldots, k,$$

By this definition Equal Temperament is a 12-degree 1-interval (2/1)-scale which is not a 12-*degree* n-*interval* $(2/1, 3/2)$-*scale*, $n \geq 2$.

5 Semitones

In this section we describe the structure of Pythagorean Tuning
$P = \{1,\ 256/243,\ 2187/2048,\ 9/8,\ 32/27,\ 19683/16384,\ 81/64,\ 4/3,\ 1024/729,\ 729/512,\ 3/2,\ 128/81,\ 6561/4096,\ 27/16,\ 16/9,\ 59049/32768,\ 243/128,\ 2\}$
in the sense of Definition 1. Let

$$n = 12, a_{12} = 2/1, a_7 = 3/2.$$

Then we look for rational numbers $x, y, \ldots, z$ such that

$$\begin{array}{c} p_0 + q_0 + \ldots + r_0 = 0 \\ p_1 + q_1 + \ldots + r_1 = 1 \\ \vdots \\ p_7 + q_7 + \ldots + r_7 = 7 \\ \vdots \\ p_{12} + q_{12} + \ldots + r_{12} = 12, \end{array}$$

$$0 = p_0 \leq p_1 \leq \ldots \leq p_{12},$$
$$0 = q_0 \leq q_1 \leq \ldots \leq q_{12}, \ldots,$$
$$0 = r_0 \leq r_1 \leq \ldots \leq r_{12}.$$

It is easy to see, that $x, y, \ldots, z$ are of the form

$$2^\alpha 3^\beta, \ \alpha, \beta \in \mathcal{Z}.$$

The following theorem can be proved, see [4].

Theorem 1 *The unique (up to the order of* x, y*) pair of rational intervals for 12-degree 2-interval (2/1, 3/2)-scales is*

$$(x, y) = (256/243, 2187/2048).$$

The concrete values of x, y can be obtained from Definition 1, cf. [4] (and they are well-known as the *minor and major Pythagorean semitones*, respectively).

Possible numbers $p_i, q_i, \ i = 0, 1, \ldots, 12$, are:

$$\begin{array}{rcr} 0+0 &=& 0, \\ 1+0 &=& 1, \\ 1+1 &=& 2, \\ 2+1 &=& 3, \\ 2+2 &=& 4, \\ 3+2 &=& 5, \\ 4+2 &=& 6, \\ 4+3 &=& 7, \\ 5+3 &=& 8, \\ 5+4 &=& 9, \\ 6+4 &=& 10, \\ 6+5 &=& 11, \\ 7+5 &=& 12. \end{array}$$

Now we express the numbers of P (in the sense of S) and denote them by $c = 1, d_\flat = x, c_\sharp = y, d = xy, e_\flat = x^2 y, d_\sharp = xy^2, e = x^2 y^2, f = x^3 y^2, g_\flat = x^4 y^2, f_\sharp = x^3 y^3, g = x^4 y^3, a_\flat = x^5 y^3, g_\sharp = x^4 y^4, a = x^5 y^4, b_\flat = x^6 y^4, a_\sharp = x^5 y^5, b = x^6 y^5 c' = x^7 y^5$.

It is evident that the sum of exponents $1+0 = 1$ for $d_\flat$ is the same as $0+1 = 1$ for $c_\sharp$, likewise $2+1 = 3$ for $e_\flat$ as $1+2 = 3$ for $d_\sharp$, $4+2 = 6$ for $g_\flat$ as $3+3 = 6$ for $f_\sharp$, $5+3 = 8$ for as $a_\flat$ $4+4 = 8$ for $g_\sharp$, and $6+4 = 10$ for $b_\flat$ as $5+5 = 10$ for $a_\sharp$. Thus we have obtained the following theorem:

Theorem 2 *Let*

$$P_{r,s,t,u,v} = \{c, r, d, s, e, f, t, g, u, a, v, b, c'\},$$

where $r = c_\sharp, d_\flat; s = d_\sharp, e_\flat; t = f_\sharp, g_\flat; u = g_\sharp, a_\flat; v = a_\sharp, b_\flat$. *Then*

1. *$P_{r,s,t,u,v}$ are 12-degree 2-interval (2/1, 3/2)-scales,*

2. $$P = \bigcup_{r,s,t,u,v} P_{r,s,t,u,v}.$$

Theorem 3

$$P = \{u : \exists k \in \{0, 1, 2, \ldots, 16\},$$

$$u = (x^4y^3)^k(x^4y^2)(\mathrm{mod}\ \ x^7y^5)\}.$$

Proof. Evidently, $x^7y^5 = 2/1 = 2, x^4y^3 = 3/2$. Further, it is
$x^4y^2 = g_\flat, (x^4y^2) \cdot (x^4y^3) \cdot (x^7y^5)^{-1} = x = d_\flat, x \cdot (x^4y^3) = x^5y^3 = a_\flat, (x^5y^3) \cdot (x^4y^3) \cdot (x^7y^5)^{-1} = x^2y = e_\flat, (x^2y) \cdot (x^4y^3) = x^6y^4 = b_\flat, (x^6y^4) \cdot (x^4y^3) \cdot (x^7y^5)^{-1} = x^3y^2 = f, (x^3y^2) \cdot (x^4y^3) \cdot (x^7y^5)^{-1} = 1 = c, 1 \cdot (x^4y^3) = g, (x^4y^3) \cdot (x^4y^3) \cdot (x^7y^5)^{-1} = xy = d, (xy) \cdot (x^4y^3) = x^5y^4 = a, (x^5y^4) \cdot (x^4y^3) \cdot (x^7y^5)^{-1} = x^2y^2 = e, (x^2y^2) \cdot (x^4y^3) = x^6y^5 = b, (x^6y^5) \cdot (x^4y^3) \cdot (x^7y^5)^{-1} = x^3y^3 = f_\sharp, (x^3y^3) \cdot (x^4y^3) \cdot (x^7y^5)^{-1} = y = c_\sharp, y \cdot (x^4y^3) = x^4y^4 = g_\sharp, (x^4y^4) \cdot (x^4y^3) \cdot (x^7y^5)^{-1} = xy^2 = d_\sharp, (xy^2) \cdot (x^4y^3) = x^5y^5 = a_\sharp$.

Corollary 1 *Pythagorean Tuning is a geometrical progression of 17 rationals with the quotient 3/2 considered modulo 2 (in the multiplicative group $\mathcal{L}$). Combine the results of tis section and consequently collect Table 2 (in the fifth column, there are values in cents, i.e. in the isomorphism $\Gamma_i \mapsto 1200 \cdot \log_2 \Gamma_i$; in the sixth column, there is a musical denotation).*

6 Images in the plane

Denote by $\mathcal{Q}_{x,y} = \{x^\alpha y^\beta; \alpha, \beta \in \mathcal{Z}\}$. Consider the map $\theta : \mathcal{Q}_{x,y} \to \mathcal{Z}^2, \theta(x^\alpha y^\beta) = (\alpha, \beta)$. Then

$$\theta((x^\alpha y^\beta)(x^\gamma y^\delta)) = (\alpha + \gamma, \beta + \delta),$$

$$\theta((x^\alpha y^\beta)^\gamma) = (\gamma\alpha, \gamma\beta), \alpha, \beta, \gamma, \delta \in \mathcal{Z}.$$

Lemma 1 *The map $\theta : \mathcal{Q}_{x,y} \to \mathcal{Z}^2$ is an isomorphism.*

Proof. It is sufficient to show that $\theta(a) = \theta(b)$ implies $a = b$. Indeed, then

$$\theta(a) = \theta(x^{\alpha_1}y^{\beta_1}) = \theta(x^{\alpha_2}y^{\beta_2}) = \theta(b),$$

$$(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$$

which implies

$$\alpha_1 = \alpha_2, \beta_1 = \beta_2, x^{\alpha_1}y^{\beta_1} = x^{\alpha_2}y^{\beta_2}.$$

Embed $\mathcal{Z}^2$ identically into the complex plane $\mathcal{C}$. Denote this injection by η. Denote by θ_*^{-1} the extended map $\theta_*^{-1} : \mathcal{C} \to \mathcal{R}_{x,y}$, where $\mathcal{R}_{x,y} = \{x^\alpha y^\beta; \alpha, \beta \in \mathcal{R}\}$. So we have the following commutative diagram, cf. Figure 4, which defines the embedding $\varphi : \mathcal{Q}_{x,y} \to \mathcal{R}_{x,y}$.

In Figure 2, we see the image of Pythagorean Tuning in the isomorphism θ.

Theorem 4 *Let $\pi : \mathcal{C} \to p$ be the projection of the complex plane $\mathcal{C} = \{(\alpha, \beta) \in \mathcal{R}^2\}$ into the line*

$$p : 5\alpha - 7\beta = 0$$

along the line

$$q : \alpha + \beta = 1.$$

Then

$$W = \theta_*^{-1}(\pi(\eta(\theta(P)))).$$

Proof. See Figure 3. Denote the individual points of q by

$$\begin{array}{rcr} w_0 & = & \frac{0}{12} \cdot (7,5), \\ w_1 & = & \left(\frac{7}{12}, \frac{5}{12}\right) = \frac{1}{12} \cdot (7,5), \\ w_2 & = & 2 \cdot \left(\frac{7}{12}, \frac{5}{12}\right) = \frac{2}{12} \cdot (7,5), \\ & \vdots & \\ w_{12} & = & 12 \cdot \left(\frac{7}{12}, \frac{5}{12}\right) = \frac{12}{12} \cdot (7,5). \end{array}$$

We have:

$$\begin{array}{rcl} w_0 & = & \pi(\eta(\theta(c))), \\ w_1 & = & \pi(\eta(\theta(D_\sharp))) = \pi(\eta(\theta(d_\flat))), \\ w_2 & = & \pi(\eta(\theta(d))), \\ & \vdots & \\ w_{12} & = & \pi(\eta(\theta(c'))), \end{array}$$

x^0y^0	2^03^0	1	1.0	0.00	c
x^1y^0	2^83^{-5}	256/243	1.053497942	90.22	$d_\flat$
x^0y^1	$2^{-11}3^7$	2187/2048	1.067871094	113.68	$c_\sharp$
x^1y^1	$2^{-3}3^2$	9/8	1.125	203.91	d
x^2y^1	2^53^{-3}	32/27	1.185185185	294.14	$e_\flat$
x^1y^2	$2^{-14}3^9$	19683/16384	1.201354981	317.60	$d_\sharp$
x^2y^2	$2^{-6}3^4$	81/64	1.265625	407.82	e
x^3y^2	2^23^{-1}	4/3	1.333333333	498.04	f
x^4y^2	$2^{10}3^{-6}$	1024/729	1.404663923	588.27	$g_\flat$
x^3y^3	$2^{-9}3^6$	729/512	1.423828125	611.73	$f_\sharp$
x^4y^3	$2^{-1}3^1$	3/2	1.5	701.96	g
x^5y^3	2^73^{-4}	128/81	1.580246914	792.18	$a_\flat$
x^4y^4	$2^{-12}3^8$	6561/4096	1.601806641	815.90	$g_\sharp$
x^5y^4	$2^{-4}3^3$	27/16	1.6875	905.86	a
x^6y^4	2^43^{-2}	16/9	1.777777777	996.09	$b_\flat$
x^5y^5	$2^{-15}3^{10}$	59049/32768	1.802032473	1019.55	$a_\sharp$
x^6y^5	$2^{-7}3^5$	243/128	1.898437528	1109.78	b
x^7y^5	2^1	2	2.0	1200.00	c'

Table 2: Pythagorean Tuning

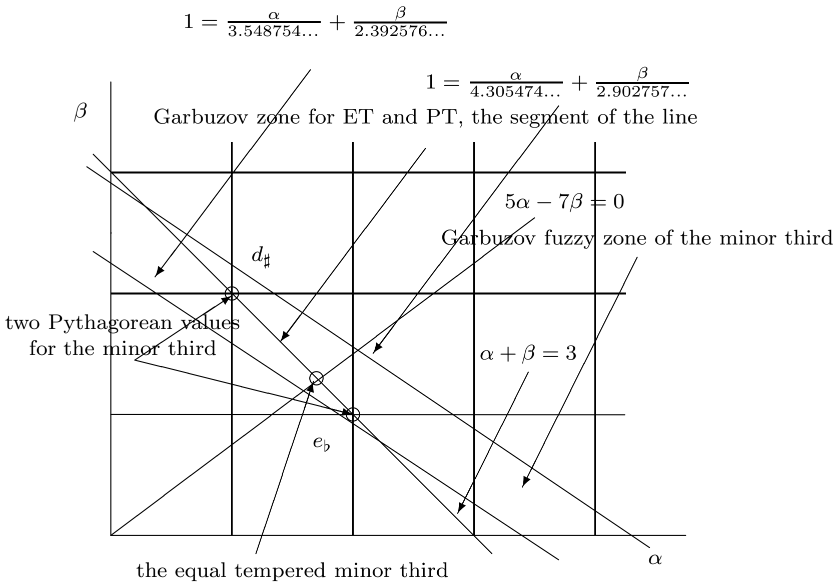

Figure 5: The Garbuzov zone of the minor third

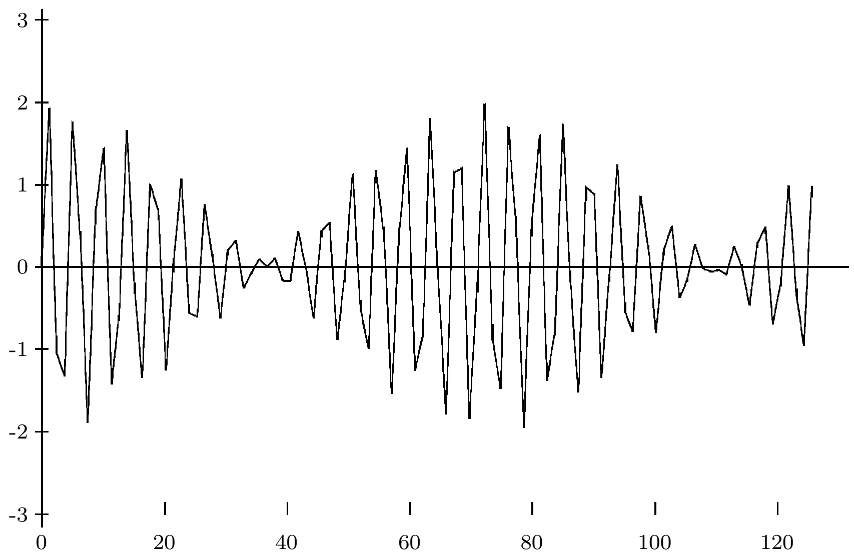

Figure 6: $\sin(\alpha \cdot 3/2) + \sin(\alpha \cdot \sqrt{2})$

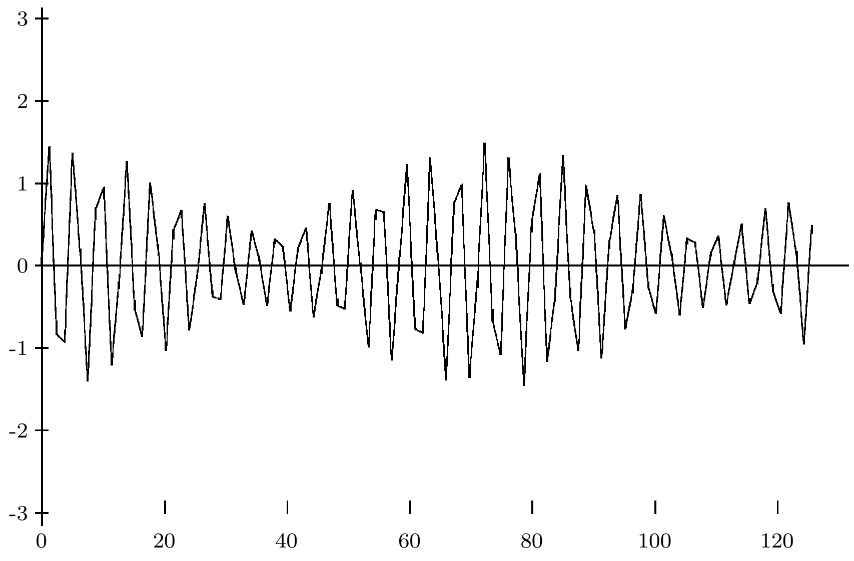

Figure 7: $\sin(\alpha \cdot 3/2) + (1/2) \cdot \sin(\alpha \cdot \sqrt{2})$

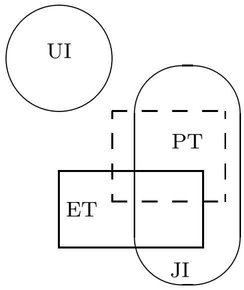

Figure 1: Oistrach, Elman, Cimbalist

$$
\begin{tikzcd}
\mathcal{Q}_{x,y} \arrow[r, "\theta"] \arrow[d, hook, "\varphi"'] & \mathcal{Z}^2 \arrow[d, hook, "\eta"] \\
\mathcal{R}_{x,y} & \mathcal{C} \arrow[l, "\theta_*^{-1}"']
\end{tikzcd}
$$

Figure 4: The embedding φ

and

$$
\begin{array}{lcll}
\theta_*^{-1}(w_0) & = & (x^7y^5)^0 & = 1, \\
\theta_*^{-1}(w_1) & = & (x^7y^5)^{\frac{1}{12}} & = \sqrt[12]{2}, \\
\theta_*^{-1}(w_2) & = & (x^7y^5)^{\frac{2}{12}} & = (\sqrt[12]{2})^2, \\
 & \vdots & & \\
\theta_*^{-1}(w_{12}) & = & (x^7y^5)^{\frac{12}{12}} & = 2.
\end{array}
$$

Corollary 2 *The image of Equal Temperament in Theorem 4 can be obtained also with the projection of the images of the whole-tone (9/8) scales:*
$(c, d, e, f_\sharp, g_\sharp, a_\sharp)$ *and* $(d_\flat, e_\flat, f, g, a, b)$
into the line
$p : 7\alpha - 5\beta = 0$
along the line
$q : \alpha + \beta = 1.$

The proof of the following theorem is easy. Here values of the boundaries of the Garbuzov zones are understand in relative frequencies.

Theorem 5 *(Fuzzy) boundaries of the Garbuzov zones in the map θ_* (in the complex plane with coordinates (α, β)) are the following lines:*

$$1 = \frac{\alpha}{\log_x R} + \frac{\beta}{\log_y R}, \; 1 = \frac{\alpha}{\log_x Q} + \frac{\beta}{\log_y Q},$$

where (R, Q) are the Garbuzov zones, see Table 1.

In Figure 5, we see the Garbuzov zone for the minor third (see Table 1) in the map θ_*.

7 Fuzziness and beats

Suppose that a musical piece (e.g. Bach's Air) is played simultaneously by two violinists both in Pythagorean and Equal Tempered Tunings and we are interesting in the result of such an operation. Moreover, we simplify the situations as follows: we consider only the theoretical tuning values of basic tones and do not mention overtones which arise automatically.

In Figure 6, we see such the interference of the PT and ET fifths both with the equal intensity of the sound (in Figure 7, the intensity rate of the sounds is 1:2). The resulting sound has an altering intensity and also pitch. For some frequency rates the kind of altering will get stabile after a short time (rational rates, harmonic tone intervals, e.g. 5:4), for some not (irrational rates, inharmonic tone intervals, e.g. $3/2{:}\sqrt{2}$).

For the proof of the following theorem, see [10], [11].

Theorem 6 *Denote by K the set of all solutions of the equation*

$$\sin(\alpha \cdot 3/2) + \sin\left(\alpha \cdot \sqrt{2}\right) = 0.$$

Denote by Δ the set of all distances between the neighbour points in $K \subset \mathcal{L}$.

Then Δ is a uniform distributed dense subset of the segment $I = [\min \Delta, \max \Delta]$.

Combining Theorem 4, Theorem 5, Theorem 6, and taking into account that the interference sound is a continuous nonperiodical function of time, we obtain the following

Theorem 7 *The interference intensity of the sound of the PT and ET fifths is a continuous nonperiodical (multi) function defined on an segment on the line $\alpha+\beta = 7$ in the map θ_*.*

The qualitative similar results we obtain when considering the other 11 musical zones. We have, see Figure 5:

Theorem 8 *Garbuzov zones of the interference of PT and ET considered in the map θ_* are segments on the lines $\alpha + \beta = n,\ n \in \mathcal{N}$.*

8 Remarks, problems

In [7], there is studied the relation between the Pythagorean Tuning and Just Intonation (Pure Tuning). In [6], there is studied the relation between Pythagorean and Praetorius (= middle tone) Tunings. In [8], there shown that the set of all tetrachords is naturally structured as a lattice of fuzzy sets. In [5], there is explained a structure of the set of diatonic scales.

The following question connected with the fuzzy sets is suggested to consider as an open problem. Define and describe operations on the set of all (theoretical) m-degree n-interval scales. Fuzziness can be considered sequentially or simultaneously (melodically or harmonically).

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