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SEGMENTATION ON THE PITCH LEVEL: THE FRONTIER OF THE PYTHAGOREAN SYSTEM

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Abstract

We can observe an effort of linguists, musicologists, and also natural scientists (M. Boroda, G. Altmann, G. Wimmer, Z. Martináková, R. Köhler) to find the basic units of the human speech and, specially, music.

The dominated power of the present western music can be denoted shortly with the word "clavier". Grubby spoken, music scores are still pressed into the frame of 12 different music degrees within the octave, the discrete choice of pitch frequencies, and the octave equivalence. The similar general picture holds also for rhythm. The exception proves the rule.

In fact, tunings are doubtless classical kinds of segmentation of the western music. The idea is not new: to consider as segments the all relative frequency intervals within a tone system as segments. *Every interval should be derived from a few basic intervals a the minimal set of all basic intervals must carry the whole information about the tone system.* Note that in general that basic intervals need not be the smallest intervals of a tone system. Tone intervals are the smallest tone groupings and they chain the tones in the composition into one entity. For other segmentation theories, cf. [1], [6], [8].

Using only the (12 tone) equal tempered tuning, the segmentation of the music on the pitch level is trivial and not interesting: there is an only semitone ($\sqrt[12]{2}$). What will happen when we deal with the Pythagorean tuning which has also a relatively simple structure (comparing e.g. with other historical tunings)? *Is there also a finite number of basic intervals, or the set of all basic intervals is infinite?* This question was not solved in the literature and the answer to this question seems not to be trivial. E.g., it is known that the Pythagorean system is a dense subset of the real halfline. So, there is infinitely many different intervals in this tone system. There are sequences of intervals tending to 1, to ∞, or of to each number which you wish (say, to the perfect fifth, 3/2).

We solved the problem for 12 qualitative degree within the octave (for $n \neq 12$, the answer is open in general). *In this case, there are 23 semitone couples which can form the octave and perfect fifth in real numbers.* One couple is rational (the well-known semitones limma and apotome) and 22 irrational. There are 29 different semitones creating these couples. The main result of this paper, interesting from the philosophical viewpoint, consists of that there exists no semitone couple in transfinite numbers which yields any Pythagorean system. Technically, the paper is a continuation of the paper [3].

Keywords. Pythagorean system, Multi-valued systems

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1 Reflections to the segmentation of music

Look for some analogies about segmentations described in the natural sciences with the purpose to apply then these reflections to music.

Firstly, there are *segmentations on various levels which are ordered hierarchically.* Some examples. Molecules are segments of matter on the chemical level, they are the chemically smallest parts. However, the atomic level is a more tiny segmentation of the matter than molecules. Each atom has also its segments (electron, positron, neutron, etc.) which are composed into structures. But these subatomic units are segmented, too! A higher segmentation level than molecule is based on cells. A social community (not necessarily the human one) has its segmentation given by individuals. There is also another segmentation sequence over the molecule level: the earth ball, the solar system, the galaxy, etc. So, we see that the segmentation hierarchies have not a linear order.

The second reflection is that *time, respectively an evolution, is not used for segmentations.* A segmentation takes into consideration only the final state.

The third observation is a claim that we need a microscope or a telescope, a specially developed and powerful tool. When observing a cell, we need a optical microscope; for depicting of atoms, we need an electronic microscope; for describing of the quantum world, we need appropriate instruments – they interact with the observed object (it is very close to the psychological interaction: interpreter – listener). What is a common and important quality of all "microscopes and telescopes"? All they *transform the images.* We see and hear only images of objects which are zoomed, colored, get larger, get smaller, turned, viewed from inside, upside down, etc. No atom looks like that we see in an electronic microscope. The question is not to obtain a true image (what is it?). We tend to obtain such an image which shows explicitly the segmentation and structure of the entity. Finally, our senses are also only special transform tools (which differs from senses of insects, for instance).

The fourth observation is that there are only *finite number of all basic segments and also of all segments* (atoms, molecules, cells, men, stars, etc.) together on each segmentation level. Some segments we identify as equivalent ones and therefore there is a question to find the "Mendelejev table" of basic segments for a given segmentation level, a basis. Elements of the basis should generate every segment and the basis should contain the minimal number of elements. As we will see in the case of 12 qualitative degrees, the role of basic segments in the Pythagorean System play 29 semitones (which can be grouped specially into 23 couples).

Music abstracts both the real world and soul. Therefore the very *music segmentations are surely at least as simple as segmentations of the material world.*

2 Pythagorean System

In general, under the *Pythagorean System* is understand the set

$$\mathcal{P} = \{2^\alpha 3^\beta; \alpha, \beta \in \mathcal{Z}\}$$

where $\mathcal{Z} = \{\cdots, \text{-2, -1, 0, 1, 2, } \cdots\}$. Since the set $\mathcal{P}$ is a dense subset of the real halfline, the Pythagorean System has rather no practical sense as a tuning system for music. It defines, in fact, a free tuning. The Pythagorean system is usually restricted with some further conditions which are important for music. Such conditions are e.g.: (1) the number of qualitative musical degrees in the octave (e.g. 12 degrees: the unison [octave], minor second, major second, minor third, major third, pure fourth, tritone, pure fifth, minor sixth, major sixth, minor seventh, major seventh); (2) some harmonic structures; (3) number of keys within the octave on the keyboard; etc.

As a concentrated result of these restrictions are *Pythagorean Scales* which are traditionally subsets of $\mathcal{P}$ defined as follows:

$$Q_{[\beta_1,\beta_2]} = \{2^\alpha 3^\beta; \beta_1 \le \beta \le \beta_2,\ \alpha, \beta, \beta_1, \beta_2 \in \mathcal{Z}\}.$$

In [2], [3], [4], [7], there are Pythagorean Scales: P_{12}, P_{17}, P_{22}, P_{27}, P_{31} (here, the index denotes the number of values within the interval $[1;2)$). If for 12 qualitative musical degrees there are used 12, 17, 22, or more different values of tone frequencies within $[0;2)$, **the Pythagorean Scales are many valued coding systems of information in music**. For example, here are P_{17} and P_{31} (listed are values within the octave):

$P_{17} = Q_{[-6;10]} = \langle$[1/1], [256/243, 2 187/2 048], [9/8], [32/27, 19 683/16 384], [81/64], [4/3], [1 024/729, 729/512], [3/2], [128/81, 6 561/4 096], [27/16], [16/9, 59 049/32 768], [243/128]$\rangle$,

$P_{31} = Q_{[-13;17]} = \langle$[1/1, 531 441/524 288], [256/243, 2 187/2 048], [1 162 261 467/1 073 741 824, 9/8, 4 782 969/4 194 304], [32/27, 19 683/16 384], [$3^{21}/2^{33}$, 8 192/6 561, 81/64], [43 046 721/33 554 432, 4/3, 177 147/131 072], [1 024/729, 729/512], [387 420 489/268 435 456, 3/2, 1 594 323/1 048 576], [128/81, 6 561/4 096], [$3^{20}/2^{31}$, 27/16, 14 348 907/8 388 608], [16/9, 59 049/32 768], [$3^{22}/2^{34}$, 4 096/2 187, 243/128], [129 140 163/67 108 864]$\rangle$.

The values belonging to qualitative musical degrees are clustered with [...]. E.g., in the cluster $[32/27, 19\ 683/16\ 384]$ are two values for the minor third. In P_{31}, the values $1/1, 531\ 441/524\ 288$ and $129\ 140\ 163/67\ 108\ 864$ belong to one cluster. The using of individual values chosen from a cluster depends on the musical context.

For a better illumination, what the Pythagorean System is, let us consider the following construction of geometrical nets, [3].

Let $\mathcal{L}$ be the multiplicative group $((0,\infty), \cdot, 1, \le)$, equipped with the usual order $\le$. Recall that a *net* with values in $\mathcal{L}$ is a function from I to $\mathcal{L}$, where I is a directed partially ordered set, cf. [5].

We consider directed partially ordered sets $I = \{(m_i, n_i)\} \subset \mathcal{Z}^2$ ($(a,b) \le (c,d)$ if and only if $a \le c, b \le d$, where $(a,b), (c,d) \in \mathcal{Z}^2$) which are lattices with the property

$$m_i + n_i = i,\ m_i \le m_{i+1}, n_i \le n_{i+1},\ i \in \mathcal{Z}.$$

Let $X > 0, Y > 0$. By the **geometrical net** (the abbreviation: **GN**) we mean the net

$$\{\Gamma_{m_i,n_i} = X^{m_i} Y^{n_i}; (m_i, n_i) \in I \subset \mathcal{Z}^2\}.$$

For example, in Figure 1, there are the lattices I for nets P_{17} (circles) and P_{31} (both circles and squares). If we take $X = 253/243$ (limma) and $Y = 2187/2048$ (apotome), we obtain exactly values of P_{17} and P_{31}, e.g. $f = X^3Y^2 = 4/3$, etc.

It is easy to see that the notion of GN generalizes the elementary notion of the *geometrical progression* and the notion of GN can be easy generalized to finite number of $X, \ldots, Y$. Non elementary examples of GNs are tone systems in music. Specially, Pythagorean Scales P_{12}, P_{17}, P_{22}, P_{28}, P_{31} are GNs satisfying (1) and (2), cf. [3].

Every tone system can be expressed as a GN.

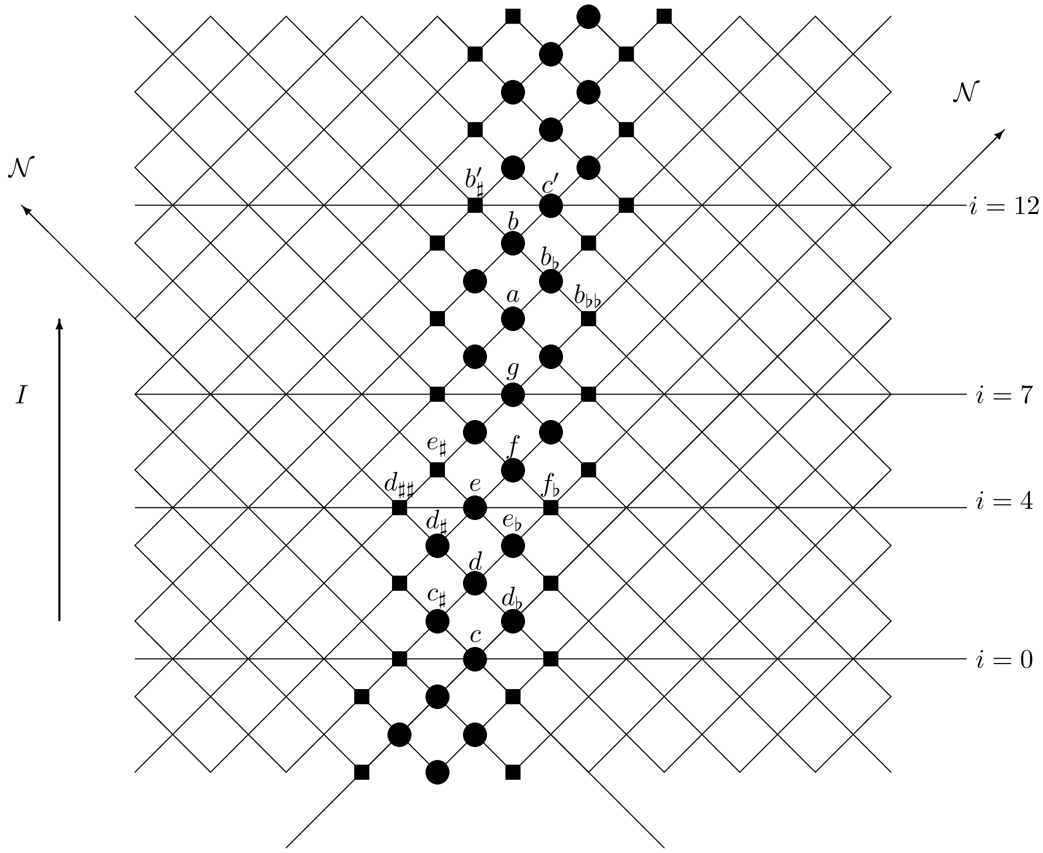

Figure 1: Directed sets I for P_{17} and P_{31}, $P_{17} \subset P_{31}$

3 Searching in transcendental numbers

Problem A. To find all couples (X, Y) in **algebraic** numbers such that

$$X^{m_{12}} Y^{n_{12}} = 2, \tag{1}$$

$$X^{m_7} Y^{n_7} = 3/2, \tag{2}$$

$$m_i + n_i = i, \ i = 0, 7, 12, \ 0 \le m_7 \le m_{12}, 0 \le n_7 \le n_{12}. \tag{3}$$

for some m_i, n_i nonnegative integer numbers. [1]

The solution of Problem A., see [3], in the explicit form is following:
$(X_1, Y_1) = (2^8/3^5, 3^7/2^{11}) = (256/243, 2187/2048)$,
$(X_2, Y_2) = (\sqrt[2]{2^{16}/3^{10}}, \sqrt[2]{3^2/2^3}) = (256/243, 3/\sqrt[2]{8})$,
$(X_3, Y_3) = (\sqrt[3]{2^5/3^3}, \sqrt[3]{3^9/2^{14}}) = (\sqrt[3]{32}/3, 27/\sqrt[3]{16384})$,
$(X_4, Y_4) = (\sqrt[4]{2^{13}/3^8}, \sqrt[4]{3^4/2^6}) = (\sqrt[4]{8192}/9, 3/\sqrt[2]{8})$,
$(X_5, Y_5) = (\sqrt[5]{2^2/3^1}, \sqrt[5]{3^{11}/2^{17}}) = (\sqrt[5]{4/3}, \sqrt[5]{177147/131072})$,
$(X_6, Y_6) = (\sqrt[6]{2^{10}/3^6}, \sqrt[6]{3^6/2^9}) = (\sqrt[3]{32}/3, 3/\sqrt[2]{8})$,
$(X_7, Y_7) = (\sqrt[7]{2^{18}/3^{11}}, \sqrt[7]{3^1/2^1}) = (\sqrt[7]{262144/177147}, \sqrt[7]{3/2})$,
$(X_8, Y_8) = (\sqrt[8]{2^7/3^4}, \sqrt[8]{3^8/2^{12}}) = (\sqrt[8]{128/81}, 3/\sqrt[2]{8})$,
$(X_9, Y_9) = (\sqrt[9]{2^{15}/3^9}, \sqrt[9]{3^3/2^4}) = (\sqrt[3]{32}/3, \sqrt[9]{27/16})$,
$(X_{10}, Y_{10}) = (\sqrt[10]{2^4/3^2}, \sqrt[10]{3^{10}/2^{15}}) = (\sqrt[5]{4/3}, 3/\sqrt[2]{8})$,
$(X_{11}, Y_{11}) = (\sqrt[11]{2^{12}/3^7}, \sqrt[11]{3^5/2^7}) = (\sqrt[11]{4096/2187}, \sqrt[11]{243/128})$,
$(X_{13}, Y_{13}) = (\sqrt[13]{2^9/3^5}, \sqrt[13]{3^7/2^{10}}) = (\sqrt[13]{512/243}, \sqrt[13]{2187/1024})$,
$(X_{14}, Y_{14}) = (\sqrt[14]{2^{17}/3^{10}}, \sqrt[14]{3^2/2^2}) = (\sqrt[14]{131072/59049}, \sqrt[7]{3/2})$,
$(X_{15}, Y_{15}) = (\sqrt[15]{2^7/3^3}, \sqrt[15]{3^9/2^{13}}) = (\sqrt[15]{128/27}, \sqrt[15]{19687/8192})$,
$(X_{16}, Y_{16}) = (\sqrt[16]{2^{14}/3^8}, \sqrt[16]{3^4/2^5}) = (\sqrt[8]{128}/3, \sqrt[16]{81/32})$,
$(X_{18}, Y_{18}) = (\sqrt[18]{2^{11}/3^6}, \sqrt[18]{3^6/2^8}) = (\sqrt[18]{2048/729}, \sqrt[9]{27/16})$,
$(X_{20}, Y_{20}) = (\sqrt[20]{2^8/3^4}, \sqrt[20]{3^8/2^{11}}) = (\sqrt[5]{4/3}, \sqrt[20]{6561/2048})$,
$(X_{21}, Y_{21}) = (\sqrt[21]{2^{16}/3^9}, \sqrt[21]{3^3/2^3}) = (\sqrt[21]{65536/19683}, \sqrt[7]{3/2})$,
$(X_{23}, Y_{23}) = (\sqrt[23]{2^{13}/3^7}, \sqrt[23]{3^5/2^6}) = (\sqrt[23]{8192/2187}, \sqrt[23]{243/64})$,
$(X_{25}, Y_{25}) = (\sqrt[25]{2^{10}/3^5}, \sqrt[25]{3^7/2^9}) = (\sqrt[5]{4/3}, \sqrt[25]{2187/512})$,
$(X_{28}, Y_{28}) = (\sqrt[28]{2^{15}/3^8}, \sqrt[28]{3^4/2^4}) = (\sqrt[28]{32768/6561}, \sqrt[7]{3/2})$,
$(X_{30}, Y_{30}) = (\sqrt[30]{2^{12}/3^6}, \sqrt[30]{3^6/2^7}) = (\sqrt[5]{4/3}, \sqrt[30]{729/128})$,
$(X_{35}, Y_{35}) = (\sqrt[35]{2^{14}/3^7}, \sqrt[35]{3^5/2^5}) = (\sqrt[5]{4/3}, \sqrt[7]{3/2})$.

Problem T. To find all couples (X, Y) in **real** numbers such that (1), (2), (3) for some m_i, n_i nonnegative integer numbers.

Theorem 1 *Let $m_{12}, n_{12}, m_7, n_7 \in \mathcal{Z}$ be numbers such that (1), (2), (3) for some real numbers $X > 0, Y > 0$. Denote by $d = \begin{vmatrix} m_{12} & n_{12} \\ m_7 & n_7 \end{vmatrix}$.*

Then X, Y are algebraic numbers and

$$X = \sqrt[d]{\frac{2^{n_7+n_{12}}}{3^{n_{12}}}}, Y = \sqrt[d]{\frac{3^{m_{12}}}{2^{m_{12}+m_7}}}.$$

[1] The condition (1) shows that we consider 12 qualitative degrees (clusters) within $[1; 2)$. The condition (2) asserts that the seventh degree is the perfect fifth $(3/2)$.

Proof.

The trivial cases $d = 0$ and $m_7 m_{12} = 0$ contradict (1) or (2). Without loss of generality suppose $d \neq 0$ and $m_7 m_{12} \neq 0$.

Consider the following bijection $\Lambda : Z \mapsto \lambda$, where

$$Z = 2^{8-19\lambda} 3^{-5+12\lambda}, \ \lambda \in (-\infty, +\infty), Z \in (0, +\infty). \tag{4}$$

Denote by λ_X, λ_Y the corresponding values of λ to X, Y in bijection Λ, respectively. So,

$$X = 2^{8-19\lambda_X} 3^{-5+12\lambda_X}, Y = 2^{8-19\lambda_Y} 3^{-5+12\lambda_Y}.$$

By (1) and (2),

$$\begin{array}{lcr} X^{m_7} \cdot (2^{8-19\lambda_Y} 3^{-5+12\lambda_Y})^{7-m_7} & = & 3/2, \\ X^{m_{12}} \cdot (2^{8-19\lambda_Y} 3^{-5+12\lambda_Y})^{12-m_{12}} & = & 2. \end{array} \tag{5}$$

By (5),

$$X = \frac{2^{1/m_{12}}}{(2^{8-19\lambda_Y} 3^{-5+12\lambda_Y})^{n_{12}/m_{12}}} = \frac{(3/2)^{1/m_7}}{(2^{8-19\lambda_Y} 3^{-5+12\lambda_Y})^{n_7/m_7}}.$$

This implies

$$\frac{1}{m_{12}} \log_K 2 - \frac{n_{12}}{m_{12}} R = \frac{1}{m_7} \log_K (3/2) - \frac{n_7}{m_7} R,$$

where $R = \log_K Y$ for some $K \neq 1, K > 0$. We have:

$$\frac{1}{m_{12}} \log_K 2 - \frac{1}{m_7} \log_K (3/2) = R \left(\frac{n_{12}}{m_{12}} - \frac{n_7}{m_7} \right) = -R \frac{d}{m_{12} m_7}.$$

Consequently,

$$\frac{-m_7 \log_K 2 + m_{12} \log_K (3/2)}{d} = R = (8 - 19\lambda_Y) \log_K 2 + (-5 + 12\lambda_Y) \log_K 3.$$

Now,

$$\lambda_Y (-19 \log_K 2 + 12 \log_K 3) = \frac{-m_7 \log_K 2 + m_{12} \log_K (3/2)}{d} - 8 \log_K 2 + 5 \log_K 3.$$

Put $K = 3^{12}/2^{19} = 531441/524288$ (i.e., the *Pythagorean comma*) and $U = 2^8/3^5 = 256/243$ (i.e., the *minor Pythagorean semitone*). Then $-19 \log_K 2 + 12 \log_K 3 = \log_K K = 1$ and $8 \log_K 2 + 5 \log_K 3 = \log_K U$ and

$$\lambda_Y = \frac{\begin{vmatrix} m_{12} & \log_K 2 \\ m_7 & \log_K 3/2 \end{vmatrix}}{d} - \log_K U.$$

Symmetrically,

$$\lambda_X = \frac{\begin{vmatrix} n_{12} & \log_K 2 \\ n_7 & \log_K 3/2 \end{vmatrix}}{\begin{vmatrix} \log_K n_{12} & m_{12} \\ \log_K n_7 & m_7 \end{vmatrix}} - \log_K U = \frac{\begin{vmatrix} \log_K 2 & n_{12} \\ \log_K 3/2 & n_7 \end{vmatrix}}{d} - \log_K U.$$

(4) implies (is equivalent to) $\lambda_X = \log_K \frac{X}{U}$. Then $\log_K X = \lambda_X + \log_K U$. Analogously for Y. We have:

$$X = K^{\frac{\begin{vmatrix} \log_K 2 & n_{12} \\ \log_K 3/2 & n_7 \end{vmatrix}}{d}}, Y = K^{\frac{\begin{vmatrix} m_{12} & \log_K 2 \\ m_7 & \log_K 3/2 \end{vmatrix}}{d}}.$$

Finally,

$$X = K^{\frac{\begin{vmatrix} \log_K 2 & n_{12} \\ \log_K 3/2 & n_7 \end{vmatrix}}{d}} = K^{\log_K \frac{2^{n_7/d}}{(3/2)^{n_{12}/d}}} = \sqrt[d]{\frac{2^{n_7+n_{12}}}{3^{n_{12}}}}.$$

Analogously,

$$Y = \sqrt[d]{\frac{3^{m_{12}}}{2^{m_{12}+m_7}}}.$$

So, both $X = X(d), Y = Y(d)$ are algebraic numbers. $\square$

Corollary 1 *The solution of Problem T coincides with the solution of Problem A.*

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